# Nonlinear state of parametric interaction between waves in an active medium

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The conditions for saturation of an unstable mode at the expense of its parametric decay into damped perturbations are investigated. It is shown that the instability cannot be stabilized if the instability increment exceeds the maximum damping. The dynamics of the multimode regime that arises is considered in the case of strong Langmuir turbulence. The maximum field amplitude is determined for this case.

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A characteristic feature of an active medium is the possibility of the excitation in it of unstable oscillations. Principal interest is attached here to the nonlinear dynamics and the mechanism of saturation of the instability. One such mechanism can be the parametric decay of a linearly unstable wave into damped perturbations. The quasi-stationary states that are generated here were investigated in Refs. 1–3.

In the present work, the nonlinear dynamics of the parametric interactions of waves in an active medium are considered by analytic and numerical methods. It is shown that under conditions in which the damping of the perturbations is relatively small, the quasi-stationary state is not achieved in a system of three waves and the subsequent dynamics are determined by the interaction of many waves. The behavior of the system in this case is studied through the example of the interaction of Langmuir and ion-sound waves in a nonisothermal plasma. Such a situation is realized, in particular, in experiments on the interaction of an electron beam with the plasma.

1. In the approximation linear in the perturbations, we get the following from the system of equations which describes the decay interaction of three waves $\omega_0=\omega_1+\omega_2$, $\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2$,[4] in the presence of a linear oscillation buildup $\gamma$ and damping $\nu_1$, $\nu_2$ in a homogeneous medium:

$$C_0(t)=C_0(0)e^{\gamma t},$$
$$\frac{d^2C_k}{dt^2}-\Gamma\frac{dC_k}{dt}-[\nu_k(\Gamma+\nu_k)+V^2|C_0(0)|^2e^{2\gamma t}]C_k=0. \tag{1}$$

Here $C_j$ is the complex amplitude of the $j$-th wave, normalized so that $\omega_j|C_j|^2$ is the energy density of the wave, $V$ is the coupling coefficient which can be regarded as real if we assume that the dissipation and buildup are both sufficiently small, $\Gamma=\gamma-\nu_1-\nu_2$; $k=1,2$.

By the change of variables

$$C_k=C_k(\xi),\quad \xi=V|C_0(t)|/\gamma=V|C_0(0)|e^{\gamma t}/\gamma$$

Eq. (1) is transformed into a modified Bessel function. The increasing solution of Eq. (1) of interest to us is of the form[5]

$$C_k(\xi)=\text{const}\cdot\xi^{\Gamma/2\gamma}I_\mu(\xi),\quad \mu=-\frac{\Gamma+2\nu_k}{\gamma}, \tag{2}$$

where $I(\xi)$ is the modified Bessel function. It follows from the solution (2) that the behavior of the perturbation is completely determined by the value of the pumping. To begin with, while $V|C_0(t)|\ll\gamma$, the perturbations do not build up and $C_k\sim\exp(-\nu_k t)$. The parametric interaction is effectively turned on when $V|C_0(t)|\gtrsim\gamma$, and, thanks to the continuing growth of $C_0(t)$, the increase in the perturbations takes place with acceleration; at $V|C_0(t)|\gg\gamma$, we have $C_k(t)\sim\exp(V|C_0(0)|e^{\gamma t})$ and the perturbations rapidly overtake the pump wave in amplitude. The linear stage of the parametric instability is concluded at this point and further evolution of the system requires a nonlinear treatment.

For an investigation of the nonlinear dynamics of the wave, it is convenient to transform to real quantities. Setting $C_j=|C_j|\exp(-i\alpha_j)$ ($\operatorname{Im}\alpha_j=0$) and separating the real and imaginary parts of the equations, we get, after simple transformations,

$$\left(\frac{d}{dt}-2\gamma\right)s_0=\left(\frac{d}{dt}+2\nu_1\right)s_1=\left(\frac{d}{dt}+2\nu_2\right)s_2=2VA(t)\,\mathrm{tg}\,\psi, \tag{3}$$
$$s_0s_1s_2\cos^2\psi=A^2(0)e^{2\Gamma t}=A^2(t); \tag{4}$$
$$s_j=|C_j|^2,\quad \psi=\alpha_1+\alpha_2-\alpha_0.$$

According to the linear consideration given above (see also Refs. 2, 3), the parametric coupling of the waves is effective when the characteristic growth rate of the decay instability significantly exceeds the linear buildup and decay. Moreover, in accord with (4), at $\Gamma>0$, the amplitudes of the waves increase exponentially, which also leads to a relative increase in the nonlinear components in (3). Therefore, the stated problem can be solved by perturbation theory. In the zeroth approximation corresponding to the conservative case $\gamma=\nu_1=\nu_2=0$, as is well known,[2,4] the system (3), (4) has the solution

$$m_1=s_0+s_1=\text{const},\quad m_2=s_0+s_2=\text{const},$$
$$s_0s_1s_2\cos^2\psi=A^2=\text{const}, \tag{5}$$
$$s_0=s_a+(s_b-s_a)\,\mathrm{sn}^2[V(s_c-s_a)^{1/2}(t-t_0),\ \varkappa^2],$$

where $\varkappa^2=(s_b-s_a)/(s_c-s_a)$ and the quantities $s_a<s_b<s_c$ are roots of the equation

$$s(s-m_1)(s-m_2)=A^2. \tag{6}$$

Acting in the spirit of the method of Van der Pol, we shall assume that the form of the solution does not change when account is taken of small linear components, and the zeroth-approximation integrals become slowly varying functions of time (in comparison with the zeroth-approximation characteristic period). Equations describing these slow changes are obtained from (3)–(5) by averaging over the rapid oscillations of the amplitudes:

$$\frac{dm_k}{dt} = -2\nu_k m_k + 2(\gamma + \nu_k)\langle s_0\rangle, \qquad k = 1, 2,$$
$$\langle s_0\rangle = s_a + (s_b - s_a)\frac{K(\varkappa) - E(\varkappa)}{\varkappa^2 K(\varkappa)}. \tag{7}$$

Here $K(\varkappa)$ and $E(\varkappa)$ are the complete elliptic integrals of the first and second kind, respectively.

The change of the integral $A$ with time is determined by the relation (4). We assume for simplicity that the amplitude of one of the perturbations at the initial instant is sufficiently small that we can assume $A^2(t) \ll m_k^3(t)$.

Then we find from (6) (setting $m_1 \gg m_2$ for definiteness):

$$s_c \approx m_1(t), \quad s_b \approx m_2(t), \quad s_a \approx A^2(t)/m_1 m_2 \ll s_b,\ s_c. \tag{8}$$

Substituting (8) in (7), we obtain an equation for the determination of the modulus of the elliptic integrals $\varkappa^2(t) \approx m_2/m_1$, which determines the change with time of the period ($T \sim 4K(\varkappa)$) of fast oscillations of the wave amplitudes:

$$\frac{1}{2}\frac{d\ln\varkappa^2}{dt} = \mu_1 - \mu_2 - [\mu_1\varkappa^2 - \mu_2]\frac{K(\varkappa) - E(\varkappa)}{\varkappa^2 K(\varkappa)}. \tag{9}$$

Here we have introduced the notation $\mu_k = \gamma + \nu_k$ for symmetry.

In the region $0 < \varkappa^2 < 1$, using the expansion of the elliptic integrals in a series,[5] we transform Eq. (9) to the form

$$\frac{d\varkappa^2}{dt} = (2\mu_1 - \mu_2)\varkappa^2 - \left(\mu_1 - \frac{\mu_2}{8}\right)\varkappa^4. \tag{10}$$

From (10) we find

$$\varkappa^2(t) = \varkappa_*^2 \frac{\varkappa^2(0)e^{2\gamma t}}{\varkappa_*^2 + \varkappa^2(0)(e^{2\alpha t} - 1)}, \tag{11}$$

$$\varkappa^2(0) \equiv \varkappa^2(t=0), \qquad \alpha = \mu_1 - \frac{\mu_2}{2}, \qquad \varkappa_*^2 = \frac{16\mu_1 - 8\mu_2}{8\mu_1 - \mu_2}. \tag{12}$$

Here $\varkappa^2$ is the stable stationary value of $\varkappa^2(t)$, which is achieved in the case $0 < \varkappa^2(0) < 1$, and $\frac{1}{2} < \mu_1/\mu_2 < \frac{7}{8}$.

In the narrow region $\frac{7}{8} \lesssim \mu_1/\mu_2 \lesssim 1$, the use of the expansion to obtain (10) is no longer correct. In this case, the solution of Eq. (9) is found in the region $1 - \varkappa^2 \equiv \varkappa'^2 \ll 1$ and leads to the stationary value $\varkappa^2 \sim 1$, determined by the equation

$$\varkappa'^2 \ln\frac{4}{\varkappa'} = \frac{\mu_2}{\mu_1} - 1. \tag{13}$$

If $0 < \mu_1/\mu_2 < \frac{1}{2}$, then $\varkappa^2(t) \to 0$. In this case, $\Gamma < 0$ and the system results in a quasistationary regime, studied by Rabinovich.[3] Finally, in the case $\mu_1 > \mu_2$, the assumption $m_1(t) > m_2(t)$ is violated. In this case, it suffices to redefine the modulus of the elliptic functions $\tilde{\varkappa}^2 = 1/\varkappa^2$, for which the given analysis is entirely valid.

We now determine the time variation of the wave amplitudes under the conditions in which $\varkappa^2(t)$ reaches a stationary value. It is evident that in this case $m_1(t)$ and $m_2(t)$ behave in a similar manner ($m_2(t)/m_1(t) = \text{const}$). From Eqs. (7), (8), we find

$$m_1(t) = m_1(0)e^{2\beta t}, \quad m_2(t) = \varkappa_*^2 m_1(t),$$
$$\beta = \gamma - \frac{\mu_1\mu_2(1 - \varkappa_*^2)}{\mu_2 - \varkappa_*^2\mu_1}, \tag{14}$$

where $\varkappa_*^2$ is determined from (12) or (13). It follows from (12) and (14) that $\beta \approx \frac{1}{2}(\gamma - \nu_2)$, i.e., the wave amplitudes increase when the linear increment predominates over the larger damping. If $\varkappa_*^2 \to 1$, then $\beta \approx \gamma$, and at $\varkappa_*^2 \to 0$, as was to be expected, $\beta \approx -\nu_1 < 0$. When $\varkappa^2 = \varkappa_*^2$, the period of fast oscillations (see (5), (8)) decreases with passage of time:

$$T \approx 2K(\varkappa_*)/m_1^{1/4}(t) \sim e^{-\beta t}. \tag{15}$$

Equations (3) and (4) were also solved by numerical methods on a computer. Figure 1 shows the results of such a calculation for $\gamma = 1$, $\nu_1 = 0.1$, $\nu_2 = 0.15$. It is seen that the period of the nonlinear oscillations of the amplitudes decreases, in accord with (15), and their envelope increases exponentially with a characteristic increment of $\beta = 0.8$. We obtain $\beta \approx 0.83$ from (13), (14) for the given example. It is seen that the obtained analytic solution is in complete accord with the results of the numerical calculation.

Thus, when the increment of the linear instability is sufficiently great ($\beta > 0$), the decay interaction in the system of three waves does not stabilize the instability—the amplitudes of the waves increase exponentially, and the characteristic period of exchange of energy among the waves increases exponentially with time.

2. If one of the interacting waves has a low frequency, as for example, in Mandel'shtam–Brillouin scattering, then the increment of the parametric instability can, with increase in the amplitude of the pump wave, become greater than this frequency (modified decay). In this case, the interaction of the waves is described by the set of equations ($\omega_2 \ll \omega_0, \omega_1$; $\delta = \omega_1 - \omega_0$)[4]:

$$i\left(\frac{d}{dt} - \gamma\right)C_0 = VC_1C_2e^{-i\delta t}, \quad i\left(\frac{d}{dt} + \nu_1\right)C_1 = VC_0C_2^*e^{i\delta t},$$
$$\frac{d^2C_2}{dt^2} + 2\nu_2\frac{dC_2}{dt} + \omega_2^2C_2 = -2\omega_2VC_0C_1^*e^{-i\delta t}. \tag{16}$$

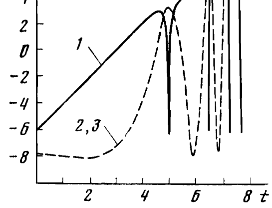


FIG. 1. Dependence of the squares of the amplitudes of the waves on the time: curves 1, 2, 3,—are $\ln r_0$, $\ln s_1$ and $\ln s_2$, respectively.

In the absence of a linear buildup and damping, Eqs. (16) were investigated in Refs. 6, 7. The analytic solution in dimensionless coordinates takes the form

$$m_0=s_0+s_1=\text{const},\qquad f_0=-2(s_0s_1)^{1/2}\cos\psi=\text{const},$$

$$s_0=\frac{m_0}{2}\left\{1+\left[1-\left(\frac{f_0}{m_0}\right)^2\right]^{1/2}\cos\left(2\int\rho dt\right)\right\},\qquad \rho=\frac{f_0}{\Omega^2}\sin^2\frac{\Omega t}{2}. \tag{17}$$

Here $s_0$, $s_1$, $\psi$ have the same meaning as in (3) and (4), $\rho$, $\Omega$ are the dimensionless amplitude and frequency of the wave $C_2$, and the solution (17) describes the nonlinear stage of the instability, when $\rho\gg 1$. Repeating the same considerations as in the previous section, we find that the amplitude of the high-frequency waves behaves in accord with (17), where the integrals of motion are now functions of time:

$$m(t)=m_0e^{\beta t},\qquad f(t)=f_0e^{\beta t},\qquad \beta=\gamma-\nu_1. \tag{18}$$

The solution for the perturbation of the density has the form

$$\rho\approx\frac{f(t)}{2\Omega^2}\{1-\exp[-(\beta+\nu_2)t]\cos\Omega t\}. \tag{19}$$

The obtained analytic solution agrees with the results of the numerical solution of Eqs. (16) on a computer. Figure 2 shows as an example the results of such a calculation for $\gamma=1$, $\nu_1=0.1$; $\nu_2=0.5$; $\delta=0.1$; $\Omega=0.2$. From the drawing, we have $\beta=0.9$, in complete agreement with (18).

3. We now consider the effect of a linear buildup and damping on the interaction of waves with random phases. For simplicity, we consider the packet of waves to be one-dimensional and sufficiently narrow that the resonance conditions $\omega_0=\omega_1+\omega_2$, $k_0=k_1+k_2$ are satisfied only for three waves. In this case, the behavior of the squares of the amplitudes of the waves is described by the set of equations[1,4]

$$-\varepsilon_{12}\left(\frac{d}{dt}-2\gamma\right)N_0=\varepsilon_{20}\left(\frac{d}{dt}+2\nu_1\right)N_1=\varepsilon_{01}\left(\frac{d}{dt}+2\nu_2\right)N_2$$
$$=2V^2(N_0N_1+N_0N_2-N_1N_2). \tag{20}$$

Here $\varepsilon_{\alpha\beta}=|\partial\omega/\partial k_\alpha-\partial\omega/\partial k_\beta|$, $V$ is the matrix element of the interaction. The solution of the considered problem is worked out in the book of Tsytovich in the absence of linear buildup and decay.[1] Several special stationary solutions of Eq. (20) are also considered

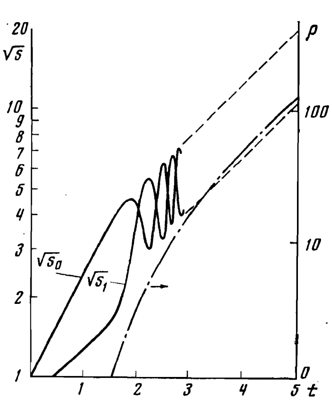


FIG. 2.

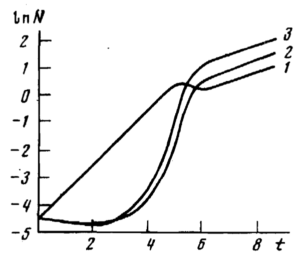


FIG. 3. Time dependence of the squares of the amplitudes of waves with random phases; curves 1, 2, 3 correspond to $\ln N_0$, $\ln N_1$ and $\ln N_2$, respectively.

there. However, in the general case, the stationary solutions of Eqs. (20) turn out to be unstable. As in the previous discussion (Sec. 1), we can easily show that in the given case, under certain conditions (see below), the linear effects are small on the nonlinear portion of the interaction of the waves ($\tau_{\text{nonl}}\ll\tau_{\text{lin}}$) and the system rapidly reaches a state with quasiequilibrium distribution of the quanta, in which

$$N_1N_0+N_2N_0-N_1N_2\ll N_0^2.$$

An approximate solution of the nonstationary nonlinear problem (20) can also be obtained under the assumption (which is confirmed by numerical calculation) that in the quasiequilibrium state the nonlinear interaction of the waves, which converts the energy fed into the system from the pump wave, is proportional to the number of pump quanta $N_0$:

$$N_1N_0+N_2N_0-N_1N_2=Q(t)N_0. \tag{21}$$

It then follows from (20) that we can assume

$$N_{1,2}=q_{1,2}(t)N_0, \tag{22}$$

where $Q$ and $q_{1,2}$ are slowly changing functions and $Q/N_0\ll 1$ (we note that at $\gamma=\nu_1=\nu_2=0$, the equilibrium state corresponds to $Q=0$).

From (20)–(22), we obtain

$$N_0(t)\sim N_1(t)\sim N_2(t)\sim e^{\beta t},\qquad \beta=\frac{\varepsilon_{12}\gamma-\varepsilon_{20}\nu_1-\varepsilon_{10}\nu_2}{\varepsilon_{01}+\varepsilon_{02}+\varepsilon_{12}}. \tag{23}$$

It follows from (23) that under the conditions in which $\beta>0$ the amplitudes of the waves on the nonlinear portion grow exponentially. Here the characteristic time of the nonlinear interaction $\tau_{\text{nonl}}\sim 1/N_0\sim e^{-\beta t}$ decreases, improving the conditions of applicability of the obtained solution. Correspondingly, the condition $\beta>0$ determines the region of instability of the stationary states investigated by Tsytovich.[1]

Figure 3 shows the numerical solution of Eqs. (20) for $\gamma=1$; $\nu_1=0.1$; $\nu_2=0.2$; $2V^2=1$; $\varepsilon_{01}=\varepsilon_{12}=2\varepsilon_{02}=1$. It is seen that, after the linear stages, the system, in accord with the solution (23) transforms into a state in which the amplitudes of the waves increase exponentially with increment $\beta\approx 0.33$. It follows from (23) that $\beta=0.3$ in this case.

Thus, it follows from the analysis given above that the nonlinear dynamics of the interaction of the waves

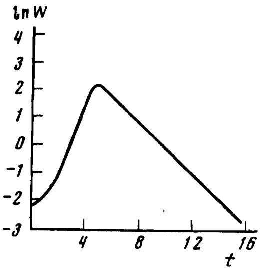


FIG. 4. Change in the energy density of waves with time

$$W=\frac{1}{L}\int|E|^2dx.$$

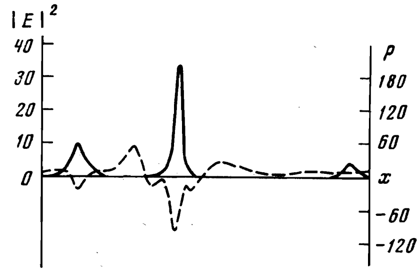


FIG. 5. Spatial distribution for $\rho(x)$, $E(x)$.

depends essentially on the linear parameters ($\gamma$, $\nu_k$) of the system. The relation between these parameters determines the final state that is realized in the system. Thus, in the case of waves with fixed phases, the linearly unstable wave can be stabilized only when its increment does not exceed the largest damping decrement of the perturbations. A similar condition can occur also for waves with random phases. If this condition is not satisfied, then the decay of the unstable wave only slows the linear instability, without stopping it. Here the perturbations parametrically connected with the pump wave turn out to be unstable. The physical explanation of the obtained result follows from the energy balance and consists of the fact that in the case $\gamma>\nu_k$ (or $\Gamma=\gamma-\nu_1-\nu_2>0$) the absorption can compensate the pump only when the amplitude of the perturbation exceeds the amplitude of the pump wave. But in this case, as is well known,[2,4] the interaction of the waves is virtually nonexistent.

4. Thus, if the dissipation does not stabilize the instability, then the perturbations connected with the pump wave turn out to be unstable. The latter can also decay in the general case, generating a new pair of perturbations and so on. Since, in accord with the previous analysis, after each decay act, the increment of growth of the perturbations decreases, it follows that the exponential increase evidently ceases when stable perturbations arise as a result of the next decay, and a balance is established between the inflow of energy from the source of the instability and dissipation, to the extent that it is redistributed over the spectrum.

We now consider the nonlinear dynamics of waves in the field of a coherent pump in the example of the interaction of Langmuir waves and ion-sound waves in a nonisothermal plasma ($T_e\gg T_i$). The initial unstable Langmuir perturbation can build up, for example, because of the hydrodynamic instability of the cold beam of electrons in the plasma.[8] The initial system of dynamic equations in dimensionless variables has the form

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}+i\nu_e E=\rho E+i\gamma E_{k_0}e^{ik_0x}, \tag{24}$$

$$\frac{\partial^2\rho}{\partial t^2}-u^2\frac{\partial^2\rho}{\partial x^2}+2\nu_s\frac{\partial\rho}{\partial t}=\frac{\partial^2|E|^2}{\partial x^2}. \tag{25}$$

Here $E(x,t)$ is the amplitude of the Langmuir wave, $\rho(x,t)$ is the quasineutral perturbation of the plasma density, $u$ is the velocity of the ion sound, $\nu_e$ and $\nu_k$ are the damping decrements of the Langmuir and ion-sound waves, respectively, $\gamma$ is the increment of the linear instability of the Langmuir harmonic with $k=k_0$. In the general case, $\nu_e$ $\nu_s$, $\gamma$ are integral operators, but for simplicity, we shall consider them here to be positive numbers, which correspond to hydrodynamic instability and collision-dominated dissipation.

Equations (24) and (25) were solved numerically on a computer according to a difference scheme with periodic boundary conditions. As initial conditions, the Langmuir perturbations are specified with a Gaussian distribution and random phases, and in the absence of plasma density perturbations. The results of the calculation are shown in Figs. 4–6 ($L=8$, $u=0.8$, $k_0=2\pi/L$, $\gamma=1$, $\nu_e=0.3$). According to the numerical solution, initially, in correspondence with the results of Sec. 1, the unstable harmonic, which excites perturbations that grow with an increment which depends exponentially on the time, grows with the increment $\gamma$. At the end of the linear stage, the Langmuir field is formed into a wave packet of the soliton type, which creates a well in the density, and thanks to the high inertia of the density oscillations, the establishment of the balance of thermal and Fermi surface pressures has an oscillatory character. The time of termination of the exponential growth of the perturbations $t_*$ and the corresponding shape of the packet of Langmuir waves are determined by the condition ($L$ is the spatial scale of the system, $l$ the width of the packet)

$$\gamma\approx\nu_e L/l(t_*), \tag{26}$$

which corresponds to the compensation of the pump and absorption of the energy of the plasmons. At the same time, because of the absence of a balance of pressure of plasmons and thermal pressure of the plasma, the increase in the amplitude of the soliton is continued. In this case, the initial width of the wave packet rapidly contracts into a narrow soliton. In the time of contraction, the number of plasmons trapped in the density well does not change:

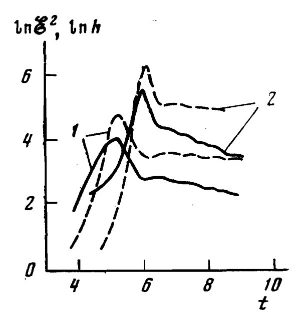


FIG. 6. Time dependence of the amplitude and well depth of the left (1) and central (2) soliton.

$$\mathscr{E}^2 l = a_1 \approx \text{const}, \tag{27}$$

and the formation of the soliton takes place automatically, as in the conservative case[9,10]:

$$\mathscr{E} \sim (t_0 - t)^{-1}, \quad l \sim (t_0 - t)^2. \tag{28}$$

Here $\mathscr{E}$ and $l$ are the amplitude and width, respectively, of the soliton.

Since the self-simular mode is initiated immediately after the linear stage, we can then assume that at the beginning of this regime ($h$ is the depth of the well),

$$h(t_*) \sim \frac{|E_{k_0}(0)|^2}{u^2} \exp\frac{|E_{k_0}(t_*)|}{\gamma}, \tag{29}$$

$$a_1 \sim |E_{k_0}(t_*)|^2 L. \tag{30}$$

Here $h \sim l^{-2}$ in correspondence with the condition of existence of a localized solution of the Schrödinger equation (24). From (26), (27), (29), (30), we obtain

$$\mathscr{E}^2 l \sim \gamma^2 L \ln^2\left[\frac{\gamma}{v_e}\frac{u}{L|E_{k_0}(0)|}\right]. \tag{31}$$

Saturation of the amplitude of the soliton sets in when the thermal pressure of the plasma is equal to the pressure of the plasmons and, consequently,

$$\mathscr{E}^2_{max} \approx u^2 h. \tag{32}$$

The relations (31) and (32) permit us to estimate the maximum amplitude of the soliton:

$$\mathscr{E}_{max} \sim \frac{\gamma^2 L}{u} \ln^2\left[\frac{\gamma}{v_e}\frac{u}{L|E_{k_0}(0)|}\right], \tag{33}$$

or, in dimensional form,

$$\frac{|E|_{max}}{(4\pi NT)^{1/2}} \sim \left(\frac{\gamma}{\omega_{pi}}\right)^2 \frac{1}{k_0\lambda_D} \ln^2\left[\frac{\gamma}{v_e}\frac{k_0\lambda_D(4\pi NT)^{1/2}}{|E_{k_0}(0)|}\right]. \tag{34}$$

When the amplitude of the soliton reaches its maximum value (33), the depth of the density well, thanks to the inertia of the ions, continues to increase and becomes greater than the corresponding equilibrium value $h \sim \mathscr{E}^2_{max}/u^2$, as a consequence of which the thermal pressure of the plasma exceeds the pressure of the plasmons. The establishment of the equilibrium state is accompanied, on the one hand, by a decrease in the amplitude of the soliton, and, on the other, by sound radiation carrying away the excess energy of the ions. However, this equilibrium state is not stationary, since, as a consequence of the self-simulating contraction of the width of the soliton, the condition (26) is violated in the direction of an effective increase in the role of the dissipation. Due to the dissipation, the amplitude of the soliton falls off as $\sim \exp(-\nu t)$, correspondingly, its width increases. This process takes place up to the time when balance is no longer achieved between the pump and the energy dissipation. In the steady state,

$$l \sim v_e L/\gamma, \quad \mathscr{E}^2 \sim u^2 h. \tag{35}$$

It should be noted that the propagation of strong perturbations of the density, brought about by the fast self-similar increase in the amplitude of the soliton, can have a significant effect on the course of the final stage of evolution of the system. Since the maximum of the increment of the modulation instability corresponds to a perturbation of the density with $k \sim 2k_0$, where $k_0$ is the wave number of the pump, there will be $2k_0L$ solitons on the scale $L$, in agreement with the results of the numerical calculation and with the experimental data.[11] The density perturbations, colliding with the soliton, can destroy it.[12] Moreover, the strong modulation of the plasma density at distances less than the wavelength of the unstable harmonic can lead to an effective disconnection of the instability. It is clear that the effects just pointed out occur at not very large sound damping.

Analysis of the results of numerical calculation, carried out qualitatively, allow us to describe completely the dynamics of interaction of the waves. The picture described above can be realized, in particular, in the case of the interaction of a cold beam of electrons with the plasma. As estimates show, for sufficiently intense beams, the parametric interaction of the waves appears earlier than the capture of the particles by the wave. In this case, however, it follows from the estimates of (34) that $E^2_{max} \gtrsim 4\pi NT$ and consequently, the Landau damping begins to play an important role. This will in turn determine the stationary amplitude of the soliton. If the saturation of the instability is determined by the capture of particles by the wave, the amplitude of which significantly exceeds the threshold of decay of the instability, then the further dynamics of the parametric decay of this wave will also take place in a fashion similar to the picture considered above.

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