Nonlinear frequency shift of whistler waves (helicons) in a uniform collisionless plasma

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We develop a consistent theory for the nonlinear frequency shift of monochromatic whistler waves, which is caused by their resonance interaction with the particles in a collisionless uniform plasma. Equations (4.5) to (4.7) give the main results.

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1. INTRODUCTION

The intensive experimental study of monochromatic whistler signals in the magnetosphere (see, e.g., [1–4]) has led to the discovery of a number of peculiar phenomena which are apparently manifestations of nonlinear effects. Several of these effects are at the present time completely accessible also for a study under laboratory conditions.

At not too large amplitudes the main nonlinear mechanism is the resonance wave-particle interaction. Amongst the effects of this interaction which have so far not yet been studied is the nonlinear frequency shift of whistler waves caused by it; the present paper is devoted to a study of this effect. In addition to being of interest because of the principles involved, this problem is also important for an understanding of the self-modulation and the self-focusing of whistler waves, effects determined by the nonlinear dependence of the frequency on the amplitude (these problems go beyond the scope of the present paper).

It is well known that whistlers and electrostatic Langmuir waves have much in common as far as the resonance interaction with particles is concerned. In particular, one can similarly introduce for whistlers, the concept of trapped and untrapped particles, the nonlinear period of the oscillations of the resonance particles for whistlers being proportional to the square root of the amplitude, just as for Langmuir waves (for a rather extensive literature about this problem, see, e.g., the review[5]). However, this analogy is true only up to terms of first order in the parameter $v_R/kv_\perp^2\tau$, where $v_R=(\omega-\omega_c)/k$ is the resonance velocity, i.e., the velocity of the electrons that are in exact resonance with the whistler wave, $v_\perp$ is the transverse electron velocity, k the wavenumber, ω_c the electron gyro-frequency, and τ the nonlinear time of the oscillations of the resonance particles. On the other hand, in order to obtain the frequency shift caused by the resonance interaction between the wave and the particles, it is necessary to take terms of second order in the parameter $v_R/kv_\perp^2\tau$ into account in the expansion of the distribution function, and in that approximation the analogy between whistlers and Langmuir waves is no longer true. Because of this it is impossible to simply transfer the results of the calculation of the frequency shift, obtained in[6,7] for Langmuir waves (as was done, e.g., in[8]) to the whistler case. It is necessary to start from more exact equations of motion for the resonance particles in the field of the whistler wave, retaining higher-order terms in $v_R/kv_\perp^2\tau$. Moreover, it is important that in Refs. 6 and 7 a problem with initial conditions was considered. However, to obtain results which might be compared with experiments, it is often necessary to consider another statement of the problem which takes into account the form of the leading wave front (for whistlers this problem may differ from the problem with initial conditions not only in the kinematics, but also qualitatively by another form of the distribution function).

The present paper is devoted to evaluating the nonlinear frequency shift $\delta\omega$ of whistlers, taking into account the above-mentioned peculiarities of these waves.

2. BASIC EQUATIONS

We write the equation for the electric field of a whistler propagating along the magnetic field (z-axis) in the form

$$\mathscr{E}_x=A(z,t)\cos[kz-\omega t+\varphi(z,t)],\qquad \mathscr{E}_y=-A(z,t)\sin[kz-\omega t+\varphi(z,t)],\tag{2.1}$$

where $\varphi(z,t)$ is the correction to the phase caused by the nonlinear interaction of the resonance electrons with the wave, while k is connected with ω by the dispersion equation

$$k^2c^2/\omega^2=N^2(\omega,\vartheta=0)=\omega_p{}^2/\omega(\omega_c-\omega).\tag{2.2}$$

We assume here that $A(z,t)$ is a slowly varying function, $\partial\varphi/\partial t\ll\omega$, $\partial\varphi/\partial z\ll k$. Using these conditions we can write the equations determining the evolution of the wave in the form[9]

$$\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial z}\right)\varphi+\frac{v_g'}{2}\left(\frac{\partial\varphi}{\partial z}\right)^2-\frac{v_g'}{2A}\frac{\partial^2A}{\partial z^2}=-\delta\omega,\tag{2.3}$$

$$\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial z}\right)U+v_g'\frac{\partial}{\partial z}\left(U\frac{\partial\varphi}{\partial z}\right)=2\gamma U,\tag{2.4}$$

where U is the energy density of the wave:

$$U=\frac{1}{16\pi}\left[\frac{\partial}{\partial\omega}(\omega\varepsilon_{\alpha\beta})E_\alpha E_\beta+H^2\right]=\frac{c^2(\omega_c-\omega)}{2\pi v_g{}^2\omega_c}A^2,\tag{2.5}$$

v_g is the group velocity, $v_g'=\partial v_g/\partial k$, while γ and $\delta\omega$ are the nonlinear growth rate and the correction to the frequency which are determined by the expressions

$$\delta\omega=-2\pi e\frac{v_g\omega}{kc^2A}\int d\mathbf{v}\,v_\perp(\delta f-\delta f_L)\cos 2\xi, \quad (2.6)$$

$$\gamma=2\pi e\frac{v_g\omega}{kc^2A}\int d\mathbf{v}\,v_\perp\delta f\sin 2\xi. \quad (2.7)$$

Here $\delta f=f-f_0$ is the correction to the unperturbed electron distribution function due to the wave-particle interaction, while δf_L is the corresponding correction in the linear approximation. The quantity ξ is defined by the expression

$$2\xi=kz-\omega t+\varphi(z,t)+\psi+\pi/2. \quad (2.8)$$

where ψ is the phase of the electron velocity rotation:

$$v_x=v_\perp\cos\psi,\quad v_y=v_\perp\sin\psi.$$

The calculation of δf reduces to solving the equations of motion for the resonance electrons; we turn to that now. We shall then consider not too strong waves which are such that we can when solving the equations of motion neglect in first approximation the quantity $\varphi(z,t)$ in the Eqs. (2.1) for the wave field and in (2.8), i.e., we can neglect the nonlinear correction to the phase, evaluating it only in the next approximation. The condition for such an approximation will be given at the end of Sec. 4 after the evaluation of $\delta\omega$.

We denote by u the deviation of the z-component of the velocity from its resonance value:

$$u=v_z-v_R,\quad v_R=(\omega-\omega_c)/k, \quad (2.9)$$

and we consider first the case when the wave amplitude is constant. In a reference frame that moves with the phase velocity ω/k of the wave, the longitudinal velocity component is then equal to $v_z-\omega/k=u-\omega_c/k$. In this frame the electric field of the wave vanishes and, hence, the kinetic energy of the particle

$$T=[(v_z-\omega/k)^2+v_\perp^2]m/2$$

is conserved. Therefore, u and $v_\perp$ change in such a way that the quantity

$$(u-\omega_c/k)^2+v_\perp^2=2T/m \quad (2.10)$$

is an integral of motion. As to the equations for u and the phase, we can write them in the Hamiltonian form studied in[10]:

$$\frac{d\xi}{dt}=\frac{\partial\mathscr{H}}{\partial u},\quad \frac{du}{dt}=-\frac{\partial\mathscr{H}}{\partial\xi}, \quad (2.11)$$

$$\mathscr{H}=\frac{ku^2}{4}-\frac{1}{2}\omega_c h\cos 2\xi\left[\frac{2T}{m}-\left(u-\frac{\omega_c}{k}\right)^2\right]^{1/2}, \quad (2.12)$$

where h is the amplitude of the magnetic field of the wave divided by the external magnetic field:

$$h=H/B_0. \quad (2.13)$$

The quantity $\mathscr{H}$ is another integral of motion which determines the motion of particles in the field of a monochromatic wave and an external magnetic field. For resonance particles which determine the effects considered here $u\ll v_R\sim\omega_c/k$ and $u\ll v_{\perp T}$ ($v_{\perp T}$ is the transverse electron thermal velocity). The term containing the radical in (2.12) can thus be expanded in powers of u. Restricting ourselves to the first two terms we have

$$\mathscr{H}=\frac{ku^2}{4}-\frac{1}{2}\omega_c hW+\frac{\sin^2\xi}{k\tau^2}-u\frac{\omega_c\cos 2\xi}{2k^2\tau^2W^2}, \quad (2.14)$$

where

$$W^2=\frac{2T}{m}-\left(\frac{\omega_c}{k}\right)^2=v_\perp^2-2u\frac{\omega_c}{k}+u^2=\text{const}, \quad (2.15)$$

$$\tau=(h\omega_c kW)^{-1/2}. \quad (2.16)$$

If we neglect the last term in (2.14), that expression takes the form of the Hamiltonian of the mathematical pendulum in a gravitational field or the Hamiltonian of a particle moving in the field of an electrostatic Langmuir wave. It is just in this lowest approximation that the above mentioned analogy between whistlers and Langmuir wave holds. The quantity τ^{-1} then determines the angular frequency of the velocity oscillations for trapped and untrapped particles. The characteristic width of the resonance region is equal in velocity space to (for a particle with a fixed value of W)

$$(\Delta v_z)_R\sim 1/k\tau. \quad (2.17)$$

We can thus assume that $u\lesssim 1/k\tau$. The ratio of the fourth term in (2.14) to any of the first three terms is then of the order of magnitude of

$$b=\frac{\omega_c}{2k^2W^2\tau}\sim\left(\frac{v_R}{v_\perp}\right)^2\frac{1}{\omega_c\tau}, \quad (2.18)$$

which we shall assume to be a small parameter: $b\ll 1$. The "mathematical pendulum approximation" is obtained if we neglect in the dimensionless Hamiltonian $k\tau^2\mathscr{H}$ terms $\propto b$.

Taking terms up to and including the first order in b into account we can write instead of (2.14)

$$\mathscr{H}=\frac{k}{4}\left(u-\frac{2b}{k\tau}\cos 2\xi\right)^2+\frac{\sin^2\theta}{k\tau^2}-\frac{1}{2k\tau^2} \quad (2.19)$$

(this quantity differs from (2.14) by terms $\propto b^2$). The first of the equations of motion (2.11) can then be written in the form

$$\frac{d\xi}{dt}=\frac{1}{\tau\varkappa}(1-\varkappa^2\sin^2\xi)^{1/2}, \quad (2.20)$$

where we introduced instead of $\mathscr{H}$ the dimensionless quantity $\varkappa$:

$$\varkappa^2=(k\tau^2\mathscr{H}+1/2)^{-1}, \quad (2.21)$$

which does not change when an electron moves in the wave field (2.1) with a constant amplitude. It follows from (2.19) and (2.21) that

$$u=\frac{2}{k\tau\varkappa}[(1-\varkappa^2\sin^2\xi)^{1/2}+b\varkappa\cos 2\xi]. \qquad (2.22)$$

It is clear from (2.20) that for trapped particles $|\varkappa|>1$ and

$$-\arcsin|1/\varkappa|\leqslant\xi\leqslant\arcsin|1/\varkappa|, \qquad (2.23)$$

while for untrapped particles ξ can be arbitrary and $|\varkappa|<1$. The line $|\varkappa|=1$ corresponds to the phase plane separatrix which separates the trapped and the untrapped particle regions.

Integrating (2.20) with the condition $\xi=\xi_0$ $(t=0)$, τ = const, we get

$$F(\xi,\varkappa)-F(\xi_0,\varkappa)=t/\tau\varkappa, \qquad (2.24)$$

where $F(\xi,\varkappa)$ is an incomplete elliptical integral of the first kind with modulus $\varkappa$. The relations (2.22) and (2.24) give the solution of the equation of motion for τ = const both for the untrapped and for the trapped particles in the approximation studied here.

Using (2.24) and (2.22) we express the initial values ξ_0, u_0 in terms of ξ, $\varkappa$, and t:

$$\xi_0=\operatorname{am}(F(\xi,\varkappa)-t/\tau\varkappa,\varkappa), \qquad (2.25)$$

$$\frac{k\tau u_0}{2}=\frac{1}{\varkappa}\operatorname{dn}\left(F(\xi,\varkappa)-\frac{t}{\tau\varkappa},\varkappa\right)-2b\operatorname{sn}^2\left(F(\xi,\varkappa)-\frac{t}{\tau\varkappa},\varkappa\right)+b. \qquad (2.26)$$

We need also in what follows the time-averages $\overline{u_0(\xi,\varkappa,t)}$ and $\overline{u_0^2(\xi,\varkappa,t)}$, which can be evaluated using (2.26):

$$\frac{k\tau\bar{u}_0}{2}=\begin{cases}b\left[\dfrac{2E(1/\varkappa)-K(1/\varkappa)}{K(1/\varkappa)}\right], & |\varkappa|\geqslant1\\[2ex] \dfrac{\pi}{2\varkappa K(\varkappa)}+b\left[\dfrac{\varkappa^2K(\varkappa)+2(E(\varkappa)-K(\varkappa))}{\varkappa^2K(\varkappa)}\right], & |\varkappa|<1\end{cases} \qquad (2.27)$$

(the sign of $\varkappa$ is the same as that of $d\xi/dt$), and

$$\frac{k^2\tau^2\overline{u_0^2}}{4}=\begin{cases}\dfrac{E(1/\varkappa)-K(1/\varkappa)(1-\varkappa^{-2})}{K(1/\varkappa)}, & |\varkappa|\geqslant1\\[2ex] \dfrac{E(\varkappa)}{\varkappa^2K(\varkappa)} & |\varkappa|<1\end{cases} \qquad (2.28)$$

These expressions differ from those obtained in[11] for electrostatic waves by terms of the order b.

3. DISTRIBUTION FUNCTION

We start from the general kinetic equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{R}}-\frac{e}{mc}\{c\mathbf{E}+\mathbf{v}\times(\mathbf{H}+\mathbf{B}_0)\}\frac{\partial f}{\partial \mathbf{v}}=0,$$

where the distribution function f is assumed to depend on t, Z, v_z, $v_\perp$, and ψ. If we change to new independent variables

$$t,\ z=v_gt-Z,\quad 2\xi=kZ-\omega t+\varphi+\psi+\pi/2,$$
$$u=v_z-(\omega-\omega_c)/k,\quad W^2=v_\perp^2-2u\omega_c/k+u^2, \qquad (3.1)$$

the kinetic equation for the resonance particles takes, with the degree of accuracy which we have adopted, the form (cf.[12,13])

$$\frac{\partial f}{\partial t}+(v_0-u)\frac{\partial f}{\partial z}+\left(\frac{ku}{2}-\frac{b}{\tau}\cos 2\xi\right)\frac{\partial f}{\partial \xi}-\frac{\sin 2\xi}{k\tau^2}(1+2b\tau ku)\frac{\partial f}{\partial u}=0, \qquad (3.2)$$

where all derivatives are taken for constant W,

$$v_0=v_g-v_R=v_g(1+\omega_c/2\omega);$$

v_0 is the resonance velocity in the frame of reference moving with the group velocity of the wave. (The z-axis is now in our case directed antiparallel to the wave-vector, i.e., along the direction of motion of the resonance particles.)

We consider the solution of the kinetic Eq. (3.2) for two cases.

A. The field has the form of an infinite plane wave (2.1) and is switched on instantaneously at $t=0$. In that case τ = const for $t>0$ and we can assume that the distribution function is independent of z so that the equations for the characteristics corresponding to Eq. (3.2) have the form (2.11) and (2.14), while their solutions have the form (2.24) and (2.22). At $t=0$ we may assume the distribution function to be the same as the unperturbed one. We assume that the latter depends solely on v_z and $v_\perp^2$, i.e., $f_0=f_0(v_z, v_\perp^2)$. We express it in terms of the variables u and W:

$$f(u,W)=f_0(v_R+u,\ W^2+2u\omega_c/k-u^2) \qquad (3.3)$$

and we expand in powers of $1/k\tau$ up to and including second-order terms

$$f(u,W)=f_0(v_R,W)+f'u+{}^1/_2f''u^2, \qquad (3.4)$$

where

$$f'=\left(\frac{\partial}{\partial v_z}+\frac{2\omega_c}{k}\frac{\partial}{\partial v_\perp^2}\right)f_0\Big|_{v_z=v_R,\ v_\perp^2=W^2}, \qquad (3.5)$$

$$f''=\left[\left(\frac{\partial}{\partial v_z}+\frac{2\omega_c}{k}\frac{\partial}{\partial v_\perp^2}\right)^2-2\frac{\partial}{\partial v_\perp^2}\right]f_0\Big|_{v_z=v_R,\ v_\perp^2=W^2}. \qquad (3.6)$$

According to Liouville's theorem we get the distribution function from (3.4) by replacing u by $u_0=u(\xi,\varkappa,t)$ where the latter function is determined by Eq. (2.26). Hence, the change of the distribution function in the resonance region, caused by the action of the field of the wave, has the form

$$\delta f(\xi,\varkappa,t)=f'[u_0(\xi,\varkappa,t)-u]+{}^1/_2f''[u_0^2(\xi,\varkappa,t)-u^2]. \qquad (3.7)$$

We assume here that u is expressed in terms of ξ and $\varkappa$ through Eq. (2.22). The change in the distribution function can, in the linear approximation in the resonance region, be written in the form

$$\delta f_L=\frac{1}{k^2\tau^2}(f'+f''u)\frac{[\cos 2(\xi-{}^1/_2kut)-\cos 2\xi]}{u}\frac{v_\perp}{W}, \qquad (3.8)$$

where

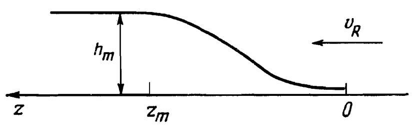

FIG. 1. Packet with a smooth leading front. h_m is the maximum magnetic field amplitude (divided by the external magnetic field). The arrow indicates the direction in which the resonance particles move; the packet moves in the opposite direction.

$$v_\perp/W\approx 1+\omega_c u/kW^2.$$

It is clear from (2.26) that expression (3.7) is a rapidly oscillating function of $\varkappa$ (for $t>\tau$) with a frequency which increases with t. The main contribution to the asymptotic value of $\delta\omega$ at large t (see (2.6)) will therefore come only from the average distribution function

$$\overline{\delta f}=f'(\bar{u}_0-u)+{}^1/_2 f''(\overline{u_0^2}-u^2), \tag{3.9}$$

where $\overline{u_0}$ and $\overline{u_0^2}$ are given by (2.27) and (2.28) while $u(\xi,\varkappa)$ is given by Eq. (2.22).

B. We now consider a second statement of the problem, which often corresponds to more realistic conditions (e.g., for magnetospheric experiments). We assume that the wave is a sufficiently long wavepacket (Fig. 1) with a smooth leading front with dimensions which are considerably larger than the nonlinear length $l_N=v_0\tau$. The resonance particles then move from the unperturbed region ($z=-\infty$), where the distribution function has the form (3.3), in the direction of the increasing field which we shall assume to depend merely on z (neglecting the change with time in the shape of the packet). The quantity $\varkappa$ is then no longer conserved, but we can find the distribution function using the conservation of the adiabatic invariants, as was done in[12]. To find the latter in the case considered we write down the equations of the characteristics for Eq. (3.2), assuming that $\tau=\tau(z)$ and $\partial f/\partial t=0$, in Hamiltonian form:

$$dp/dz=-\partial\mathscr{H}/\partial\xi,\quad d\xi/dz=\partial\mathscr{H}/\partial p, \tag{3.10}$$

$$p=u-u^2/2v_0, \tag{3.11}$$

where

$$\mathscr{H}=\frac{ku^2(p)}{4v_0}+\frac{\sin^2\xi}{kv_0\tau^2(z)}-\frac{u(p)\,b(z)}{v_0\tau(z)}\cos 2\xi, \tag{3.12}$$

while $u(p)$ is the function which is the inverse of (3.11). If we define

$$\varkappa^{-2}=kv_0\tau^2\mathscr{H}, \tag{3.13}$$

the function $u(\xi,\varkappa)$ is the same as (2.22). We can write the adiabatic invariant in the form

$$\frac{k}{4}\int p\,d\xi=\frac{k}{4}\int\left(u(\xi,\varkappa)-\frac{u^2(\xi,\varkappa)}{2v_0}\right)d\xi, \tag{3.14}$$

where the integration is over the complete range over which ξ varies, corresponding to the period of the oscillations, i.e., for untrapped particles ($|\varkappa|\leqslant 1$) $-\pi/2\leqslant\xi\leqslant\pi/2$, while for trapped particles ($|\varkappa|>1$) ξ varies in the range $-\arcsin|1/\varkappa|<\xi<\arcsin|1/\varkappa|$ and in the opposite direction. As a result we obtain the equations for the conservation of the adiabatic invariants in the form

$$\frac{\mu(\varkappa,z)}{\tau(z)}=\text{const}\quad(|\varkappa|\leqslant 1),\qquad\frac{\nu(\varkappa)}{\tau(z)}=\text{const}\quad(|\varkappa|\geqslant 1), \tag{3.15}$$

where, with the accuracy which we are using,

$$\mu(\varkappa,z)=\frac{E(\varkappa)}{\varkappa}-\frac{\pi}{2kv_0\tau(z)}\left(\frac{1}{\varkappa^2}-\frac{1}{2}\right), \tag{3.16}$$

$$\nu(\varkappa)=E(1/\varkappa)-K(1/\varkappa)(1-\varkappa^{-2}). \tag{3.17}$$

Expression (3.16) contains an additional term $\propto 1/kv_0\tau$ as compared with the analogous quantity found in[12] (this is caused by the term $\propto u^2$ in (3.14)). For the effects which interest us here this term is important. We note that taking into account terms $\propto b$ in the basic equations did not lead to a difference between Eq. (3.16) and the corresponding expression obtained (by a different method) in[13] without taking terms $\propto b$ into account. Expression (3.17) is the same as the one obtained in[14] where terms $\propto b$ were also neglected.

We now find the distribution function. As in[12] it will be convenient for us to assume that the amplitude increases smoothly, starting not from zero but from a rather small value A_0 where the amplitude A_0 is "switched on" instantaneously at $z=0$. Moreover, at distances of the order of several times $v_0\tau_0$ ($\tau_0=\tau(0)$) an ergodic distribution function is established (see[11]) of the kind (3.9). As after that the amplitude changes slowly at distances much longer than $v_0\tau$, we may assume that the distribution function at $z=0$ has the form

$$f(\varkappa_0,0)=f_0(v_R,W)+f'\frac{2}{k\varkappa_0\tau_0}+\frac{1}{2}f''\left(\frac{2}{k\varkappa_0\tau_0}\right)^2 \tag{3.18}$$

(for $|\varkappa|\ll 1$). After this the discussion goes as in[12]. For untrapped particles the quantity $\varkappa$ changes from $|\varkappa|\ll 1$ to $\varkappa=\varkappa(z)$. In that case

$$\frac{\mu(\varkappa,z)}{\tau(z)}=\frac{\pi}{2}\left[\frac{1}{\varkappa_0\tau_0}-\frac{1}{kv_0\varkappa_0^2\tau_0^2}\right]. \tag{3.19}$$

Determining $\varkappa_0$ from this and substituting it into (3.18) we get the distribution function for the untrapped particles in the form

$$f_{ut}(\varkappa,z)=f_0(v_R,W)+f'\frac{2}{k\tau}\left[\frac{2E(\varkappa)}{\pi\varkappa}-\frac{1}{kv_0\tau}\left(\frac{1}{\varkappa^2}-\frac{1}{2}-\frac{4E^2(\varkappa)}{\pi^2\varkappa^2}\right)\right]+\frac{1}{2}f''\left(\frac{2}{k\tau}\right)^2\frac{4E^2(\varkappa)}{\pi^2\varkappa^2}. \tag{3.20}$$

If the particle is "trapped" by the wave at some z_1, $\varkappa$ at the point z is determined from the relations

$$\nu(\varkappa)/\tau(z)=1/\tau_1,$$

$$\frac{2}{\pi\tau_1}\frac{\varkappa_0}{|\varkappa_0|}-\frac{1}{2kv_0\tau_1^2}=\frac{1}{\varkappa_0\tau_0}-\frac{1}{kv_0\varkappa_0^2\tau_0^2}, \tag{3.21}$$

where $\tau_1=\tau(z_1)$. Eliminating τ_1 from this we get

$$\frac{1}{\varkappa_0\tau_0}=\frac{\varkappa_0}{|\varkappa_0|}\frac{2\nu(\varkappa)}{\pi\tau}-\frac{1}{kv_0}\frac{\nu^2(\varkappa)}{\tau^2}\left(\frac{1}{2}-\frac{4}{\pi^2}\right). \tag{3.22}$$

During the motion of the particle the phase volume

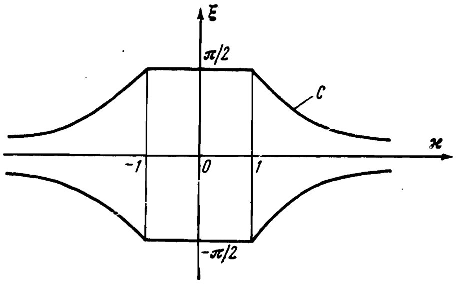

FIG. 2. Integration region for the integrals in (4.2). The equation of the curve C is: $\xi = \arcsin(1/\varkappa)$.

$$d\Omega = |d\varkappa| \int |\partial p(\xi, \varkappa)/d\varkappa| d\xi,$$

must be conserved; the integral is here over the complete range over which ξ varies, corresponding to the period of the oscillations. One checks easily that for untrapped particles $d\Omega_{ut} = 4|d_\mu|/k\tau$, while for trapped particles $d\Omega_t = 8|d\nu|/k\tau$. $d\Omega_{ut}$ and $d\Omega_t$ can then be expressed in terms of the initial value $\varkappa_0$ as follows:

$$d\Omega_{ut} = \frac{2\pi}{k}\frac{|d\varkappa_0|}{\tau_0\varkappa_0^2}\left[1 - \frac{2}{kv_0\tau_0\varkappa_0}\right],$$
$$d\Omega_t = \frac{4}{k\tau}\sum_\pm |d\nu^\pm| = \frac{2\pi}{k\tau_0}\left[\frac{|d\varkappa_0^+|}{(\varkappa_0^+)^2} + \frac{|d\varkappa_0^-|}{(\varkappa_0^-)^2}\right]. \quad (3.23)$$

We used in the last formula the fact that trapped particles with a given $\varkappa$ can, according to (3.22), have initial values $\varkappa_0$ of both signs ($\varkappa_0^+ > 0$, $\varkappa_0^- < 0$). It is immediately clear from (3.23) that (cf.[15])

$$d\Omega_t = d\Omega_{ut}^+ + d\Omega_{ut}^-. \quad (3.24)$$

If we now use the particle-number conservation law,

$$f_t(\varkappa, z)\, d\Omega_t = f_{0\,ut}^+ d\Omega_{ut}^+ + f_{0\,ut}^- d\Omega_{ut}^-, \quad (3.25)$$

where $f_{0ut}^\pm$ are the initial distribution functions for $\varkappa_0^\pm$, and Eqs. (3.22) to (3.25), we get for the trapped particle distribution function up to and including terms of second order of $1/\omega\tau$

$$f_t(\varkappa, z) = f_0(v_R, W) + f' \frac{2\nu^2(\varkappa)}{k^2 v_0 \tau^2}\left(\frac{4}{\pi^2} - \frac{3}{2}\right) + \frac{1}{2} f'' \left[\frac{4\nu(\varkappa)}{\pi k\tau}\right]^2. \quad (3.26)$$

Expressions (3.20) and (3.26) give the complete representation of the distribution function of the wavepacket in the region where[1)] $\partial h(z)/\partial z \geq 0$.

4. NONLINEAR FREQUENCY SHIFT

We shall be interested solely in the asymptotic value of the nonlinear frequency shift when the distribution function is ergodic. We can then write Eq. (2 6) in the form

$$\delta\omega = -4\frac{\omega_p^2}{k^2c^2} v_g \left[\int_0^\infty \frac{W^3 dW}{\tau(W)} J_1 + 2\frac{\omega_c}{k}\int_0^\infty \frac{W\,dW}{\tau(W)} J_2\right]; \quad (4.1)$$

$$J_1 = \frac{1}{n}\int_S d\varkappa \frac{d\xi \cos 2\xi}{\varkappa^2(1-\varkappa^2\sin^2\xi)^{1/2}}\left[f' a_1 + f''\left(a_2 - \frac{1}{\varkappa^2} + \frac{1}{2}\right)\right],$$
$$J_2 = \frac{1}{n}\int_S d\varkappa \frac{d\xi \cos 2\xi}{\varkappa^2(1-\varkappa^2\sin^2\xi)^{1/2}} f'\left[\frac{1}{4} - \frac{1}{\varkappa^2} + \frac{\sin^2\xi}{2}\right], \quad (4.2)$$

where the region of integration S is shown in Fig. 2, and n is the plasma density. For the case where the field is switched on instantaneously at $t = 0$ the coefficients in the first of Eqs. (4.2) equal to following quantities:

$$a_1 = \frac{k^2\tau^2}{2}\bar{u}_0(\tau)|_{\tau=0}, \quad a_2 = \frac{k^2\tau^2}{4}\overline{u_0^2}, \quad (4.3)$$

where $\overline{u_0}$ and $\overline{u_0^2}$ are given by Eqs. (2.27) and (2.28). However, in the case of a field in the form of a packet with a smooth profile in the region $\partial h(z)/\partial z \geq 0$, the coefficients a_1 and a_2 are determined by the following expressions:

$$a_1 = \frac{1}{v_0}\begin{cases} 4E^2(\varkappa)/\pi^2\varkappa^2 + 1/2 - 1/\varkappa^2, & |\varkappa| < 1 \\ \nu^2(\varkappa)(4/\pi^2 - 3/2), & |\varkappa| \geq 1 \end{cases}.$$
$$a_2 = \begin{cases} 4E^2(\varkappa)/\pi^2\varkappa^2, & |\varkappa| < 1 \\ 4\nu^2(\varkappa)/\pi^2, & |\varkappa| \geq 1 \end{cases}. \quad (4.4)$$

After straightforward but cumbersome, calculations one can show that the contribution to J_2 from untrapped particles ($|\varkappa| < 1$) is exactly compensated by the contribution from the trapped particles ($|\varkappa| \geq 1$), so that $J_2 = 0$.

After integration over ξ and $\varkappa$ in Eqs. (4.2) we get the following expressions for $\delta\omega$ for the two different ways to state the problem, which are considered above.

A. When the field is switched on at the initial time instantaneously

$$\delta\omega = -8\frac{\omega_p^2}{k^2c^2}\frac{I}{n} v_g \int_0^\infty \frac{W^3 dW}{\tau(W)}\left(f'' + \frac{\omega_c}{kW^2} f'\right), \quad (4.5)$$

where the quantity I is equal to the integral evaluated in[6]:

$$I = \int_0^1 dx \left\{\frac{x}{K}(2E-K) + \frac{[x^2K + 2(E-K)]^2}{x^6 K}\right\} = 1.63.$$

In contrast to the $\delta\omega$ for Langmuir waves, found in[6], Eq. (4.5) contains not only second, but also first derivatives of the initial distribution function and they appreciably affect the sign of $\delta\omega$.

B. In the case of a smoothly increasing whistler packet we get in the region up to its maximum

$$\delta\omega = -16\frac{\omega_p^2}{k^2c^2} v_g \frac{1}{n}\int_0^\infty \frac{W^3 dW}{\tau(W)}\left[\bar{a}_2 f'' + \frac{\bar{a}_1}{v_0} f'\right], \quad (4.6)$$

where

$$\bar{a}_1 = I_1 + I_2\left(1 - \frac{3\pi^2}{8}\right), \quad \bar{a}_2 = I_1 + I_2 + 0.1,$$

$$I_1 = \int_0^1 \frac{dx}{x^4}[x^2K + 2(E-K)]\left(\frac{4E^2}{\pi^2 x^2} - \frac{1}{x^2} + \frac{1}{2}\right),$$

$$I_2 = \frac{4}{\pi^2}\int_0^1 dx\, x(2E-K)[E-(1-x^2)K]^2.$$

Evaluating $I_{1,2}$ we get $\overline{a_1} = 0.018$, $\overline{a_2} = 0.107$. As a simple, but important example we consider $\delta\omega$ for an anisotropic plasma with a bi-Maxwellian distribution func-

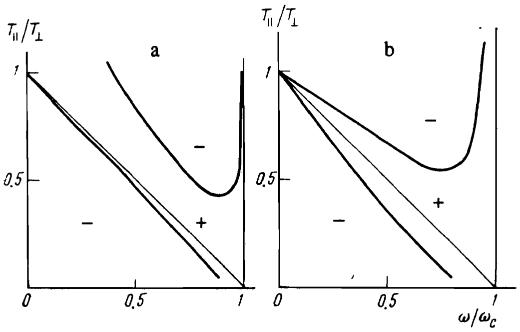

FIG. 3. The regions where $\delta\omega$ is positive (+) and negative (–) for $g=\omega_p^2\overline{W}^2/\omega_c^2c^2=0.2$; a—in the case where the field is switched on instantaneously, b—for a packet with a smoothly increasing amplitude.

tion (i.e., with different temperatures along and at right angles to the magnetic field: $T_{\parallel}$, $T_{\perp}$). Formulae (4.5) and (4.6) then give

$$\delta\omega=-\frac{\gamma_L b(\overline{W})}{\Gamma}P(\Gamma), \tag{4.7}$$

where γ_L is the growth rate of a whistler wave in the linear approximation (see, e.g., [16]), $b(\overline{W})$ is the parameter (2.18) where we substitute for W the thermal transverse velocity $\overline{W}=(2T_{\perp}/m)^{1/2}$, $\Gamma=1-T_{\parallel}/T_{\perp}-\omega/\omega_c$, while $P(\Gamma)$ has the form

$$P(\Gamma)=2.4\left[\frac{5T_{\perp}}{2T_{\parallel}}\Gamma^2+\Gamma+\frac{(T_{\parallel}-T_{\perp})}{4T_{\perp}}\frac{\omega g}{(\omega_c-\omega)}\right], \tag{4.8}$$

$$g=\omega_p{}^2\overline{W}^2/\omega_c{}^2c^2,$$

for the case of an instantaneously switched on field while

$$P(\Gamma)=0.63\left[\frac{5T_{\perp}}{2T_{\parallel}}\Gamma^2+0.21\frac{\omega\omega_c{}^2g}{(2\omega+\omega_c)(\omega_c-\omega)^2}\Gamma+\frac{5(T_{\parallel}-T_{\perp})\omega g}{4T_{\perp}(\omega_c-\omega)}\right] \tag{4.9}$$

for a smooth increase in the field.

It is well known that γ_L and Γ always have the same sign, i.e., when $\Gamma>0$ the plasma is unstable, and when $\Gamma<0$ it is stable. Therefore, we have always $\gamma_L/\Gamma>0$ and the sign of $\delta\omega$ is the opposite of that of $P(\Gamma)$. By way of illustration, Fig. 3 depicts the regions in the $(T_{\parallel}/T_{\perp}, \omega)$ plane where $\delta\omega$ is positive or negative for $T_{\parallel}/T_{\perp}<1$ (as often occurs in a magnetospheric plasma).

When applying these results to a magnetospheric plasma we must, however, bear in mind that they are only applicable to that part of the equatorial region of the magnetosphere where the plasma is sufficiently uniform. The effects of the nonuniformity may lead to an appreciable modification of the nonlinear resonance interaction between the wave and the particles and, in particular, to a different way in which the frequency shift depends on the amplitude. (For details about this see[17]).

We finally consider the condition for the validity of neglecting the term $\varphi(z, t)$ in Eq. (2.8) when evaluating the distribution function. One checks easily that if we do not neglect this term there occur in the right-hand side of (2.22) extra terms of the order of $\partial\omega/k\partial t \propto \delta\omega/k$. Neglecting the phase in (2.8) is thus valid when $b/\tau \gg \delta\omega$. This is always satisfied in our case, as we everywhere assume that $\gamma_L\tau \ll 1$.

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[1)]In the region where $\partial h(z)/\partial z<0$, there is the inverse change in the distribution function, caused by the particles leaving the capture region. The influence of this effect on the distribution function (neglecting terms $\propto b$) was studied in detail in[12]. One can easily take terms $\propto b$ into account, following the method given in[12].

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