Coulomb deceleration of fast protons in a strong magnetic field

G. G. Pavlov and D. G. Yakovlev

A. F. Ioffe Physico-technical Institute, USSR Academy of Sciences
(Submitted August 12, 1975)
Zh. Eksp. Teor. Fiz. 70, 753–767 (March 1976)

The cross section for Coulomb scattering is found and the process of fast-proton deceleration in a plasma with a magnetic field is studied in the case when $\hbar\omega_{Be} \gtrsim kT_e$. The magnetic field affects the deceleration if the Larmor radius ρ_L of the electron of the medium is less than the Debye radius ρ_D after colliding with a proton. For $\rho_L \lesssim \rho_D$ the rate of proton-energy loss is $\propto \Lambda_{\parallel} = \ln(\rho_L/\rho_{\min})$, where $\rho_{\min}$ is the minimum impact parameter. As the field strength increases, the stopping length and time increase, and the regular deviation of the protons from the field lines, which sets in at a rate $\propto \Lambda_{\perp} = \ln(\rho_D/\rho_L)$, begins to play a role. If the field is so strong that a proton moving along it cannot excite an electron to a high Landau level, then it loses energy only after a sharp regular deviation of the trajectory from the direction of the field. The stopping length and time in this case attain maximum values that are roughly $2\ln(M/m)$ times greater than the maximum values attained in zero field. At the cyclotron-resonance points, where $mv^2/2 = n\hbar\omega_{Be}$ ($n = 1, 2, \ldots$), the energy-loss curve may have sharp peaks whose effect on the stopping length and time is negligible.

PACS numbers: 52.20.Hv

1. INTRODUCTION

The strong magnetic fields used in laboratory investigations of plasmas can appreciably influence the processes determined by Coulomb collisions.[1] This influence is even more important in white dwarfs and in neutron stars, the magnetic fields on the surfaces of which can attain strengths of 10^8–10^{13} G. In the present paper we consider the deceleration of a fast ($e^2/\hbar v \ll 1$) nonrelativistic proton in a plasma with a magnetic field, B, so strong that the quantization of the plasma-electron motion is important ($kT_e \lesssim \hbar\omega_{Be}$). This problem is important for the construction of models of x-ray pulsars[2,3] and the study of processes in the atmospheres of magnetic white dwarfs.

In a plasma without a magnetic field,[4] proton deceleration occurs primarily as a result of energy transfer in distant Coulomb collisions with the electrons of the medium. The energy losses and the stopping length L_0 and time τ_0 for decleration along the z axis are then given by the formulas

$$\frac{dE}{dz} = -\frac{4\pi Ne^4}{mv^2}\Lambda_0, \quad L_0 = \frac{mE_0^2}{4\pi Ne^4 M\Lambda_0}, \quad \tau_0 = \frac{mE_0^{3/2}}{3\pi Ne^4(2M)^{1/2}\Lambda_0} \qquad (1)$$

where N is the plasma-electron concentration, v is the velocity of the decelerating proton, E_0 is its initial energy, m and M are the electron and proton masses, $\Lambda_0 = \ln(\rho_D/\rho_{\min})$ is the Coulomb logarithm, ρ_D is the Debye radius, and $\rho_{\min} = \hbar/mv$ is the minimum impact parameter. The average angle of deviation from the initial direction of motion of the proton over the stopping length is equal to zero and the root-mean-square deviation is $\sim (m/M)^{1/2}$.

The proton-deceleration process in a strong magnetic field can be qualitatively different because of the sharp anisotropy of the electronic component of the plasma. With the aid of the cross section obtained in Sec. 2 for Coulomb scattering in a magnetic field, we shall find the energy losses and the stopping length and time for three cases. In Sec. 3 we consider very strong fields $B \gg B_0 = m^2v^2c/2e\hbar = 2.2\times10^{13}(v/c)^2$ G, for which $\hbar\omega_{Be} \gg mv^2/2$ and the transition of a plasma electron to another Landau level is forbidden. In this case for small inclination angles ϑ of the proton trajectory to the magnetic field, distant Coulomb collisions do not lead to energy losses. Therefore, the proton has time to deviate markedly from the lines of force of the field during the deceleration, the most important deviation being, in turns out, the regular ($d\vartheta/dt \neq 0$) angle change, which is connected with an increase in the transverse—to the field—effective electron mass. The stopping length and time increase in comparison with L_0 and τ_0 roughly $2\ln(M/m)$ times. For $B \sim B_0$ (Sec. 4) the energy-loss curve exhibits peaks connected with the cyclotron resonances, $mv^2/2 = n\hbar\omega_{Be}$, usually observed in quantizing magnetic fields. In spite of the increase on the average of the energy losses at small ϑ, the deviation of the proton from the direction of the field is, as before, important. As B_0/B increases, the stopping length slowly decreases.

For $B \ll B_0$ (Sec. 5), an electron after a collision can be treated classically. The magnetic field influences the deceleration of the proton until the Larmor radius of the knocked-on electron $\rho_L < \rho_D$. In this case the deviation of the proton during the deceleration is important if $\Lambda_{\perp} = \ln(\rho_D/\rho_L) \gtrsim \Lambda_{\parallel} = \ln(\rho_L/\rho_{\min})$. For $\Lambda_{\perp} < \Lambda_{\parallel}$, only the Coulomb logarithm changes in (1) ($\Lambda_0 \to \Lambda_{\parallel}$). In Sec. 5 we also derive for the stopping length and time unique formulas which are approximately applicable in an arbitrary magnetic field and which contain the two Coulomb logarithms $\Lambda_{\parallel}$ and $\Lambda_{\perp}$.

2. CROSS SECTION FOR COULOMB SCATTERING IN A MAGNETIC FIELD AND THE GENERAL FORMULAS DESCRIBING PROTON DECELERATION

To solve the problem of interest to us, it is necessary first to find the cross section for Coulomb scattering in a magnetic field. In the Born approximation ($e^2/\hbar v$

Copyright © 1977 American Institute of Physics

$\ll 1$), it is necessary for this purpose to compute the matrix elements of the interaction potential $V(\mathbf{r}_1-\mathbf{r}_2)$ of two particles between the initial and final states. The state of a charged particle in the magnetic field $B=B_z$ will be specified by the quantum numbers n, p_z, and p_x, where n is the number of the Landau level ($n=0, 1, 2, \ldots$), p_z is the component of the particle momentum along the direction of the magnetic field, and p_x determines the y coordinate of the center of the Larmor circle: $y_0=-p_x\rho_0^2\hbar^{-1}$ ($\rho_0=(c\hbar/eB)^{1/2}$ is the Larmor radius in the ground Landau level). The corresponding wave functions and energy levels of a particle of mass m have the form

$$\psi=(L_xL_z)^{-1/2}\exp[i(p_zz+p_xx)/\hbar]\chi_{np_x}(y), \tag{2}$$

$$\chi_{np_x}(y)=(2^nn!\rho_0\sqrt{\pi})^{-1/2}H_n(\eta)\exp(-\eta^2/2),\quad \eta=(y-y_0)/\rho_0; \tag{3}$$

$$E=E_z+E_\perp,\quad E_z=p_z^2/2m,\quad E_\perp=\hbar\omega_B(n+1/2), \tag{4}$$

where L_x and L_z are normalization lengths in the corresponding directions, H_n is a Hermite polynomial, and ω_B is the cyclotron frequency (below ω_{Be} and ω_{Bi} will denote the electron and proton cyclotron frequencies). The formulas (2)–(4) have been written without allowance for spin, since its effect on the deceleration of a nonrelativistic particle is negligible.

To compute the matrix element, let us expand the potential V in a Fourier integral:

$$V(\mathbf{r}_1-\mathbf{r}_2)=(2\pi\hbar)^{-3}\int d\mathbf{q}\,V_\mathbf{q}\exp[i\mathbf{q}(\mathbf{r}_1-\mathbf{r}_2)/\hbar] \tag{5}$$

and use the equalities

$$\int_{-\infty}^{\infty}\chi^*_{n'p_x'}(y)\exp(iq_yy/\hbar)\chi_{np_x}(y)\,dy=S_{n'n}(p_x'-p_x,q_y)\exp(iq_yy_0/\hbar), \tag{6}$$

$$|S_{n'n}(q_x,q_y)|^2=\frac{n!}{n'!}s^{n'-n}[L_n^{n'-n}(s)]^2e^{-s},\quad s=\frac{(q_x^2+q_y^2)\rho_0^2}{2\hbar^2}=\frac{q_\perp^2\rho_0^2}{2\hbar^2} \tag{7}$$

where $L_n^{n'-n}$ is a Laguerre polynomial. Averaging the square of the modulus of the matrix element of the potential (5) over the initial states and summing over the final states of the particle 2 with allowance for the homogeneity of the distribution of the particles 2 over the coordinate y, we obtain for the probability of transition per unit time of the particle 1 from the state $|n_1p_{1z}p_{1x}\rangle$ into the state $|n_1'p_{1z}'p_{1x}'\rangle$ the expression

$$dw_{1'\leftarrow1}=K\delta(E_1'+E_2'-E_1-E_2)\,dp_{1x}'dp_{1z}', \tag{8}$$

$$K=(4\pi^2\hbar^4L_xL_yL_z)^{-1}\sum_{n_2'n_2p_{2z}}f_{n_2p_{2z}}\int_{-\infty}^{\infty}dq_y|V_\mathbf{q}S_{n_1'n_1}(\mathbf{q}_\perp)S_{n_2'n_2}(\mathbf{q}_\perp)|^2, \tag{9}$$

where $f_{n_2p_{2z}}$ is the distribution function of the particles 2 in the initial state and $q_\alpha=p_{1\alpha}'-p_{1\alpha}=p_{2\alpha}-p_{2\alpha}'$ ($\alpha=x, z$). Let us rewrite (8) in the form

$$dw=dw^{(+)}+dw^{(-)},\quad dw^{(\pm)}=K\delta(q_z-q_\pm)(v_z^2-2\Delta E_\perp\mu^{-1})^{-1/2}dp_{1x}'dp_{1z}', \tag{10}$$

where

$$q_\pm=-\mu v_z\mp(\mu^2v_z^2-2\mu\Delta E_\perp)^{1/2},\quad \Delta E_\perp=E_{1\perp}'+E_{2\perp}'-E_{1\perp}-E_{2\perp},$$

μ is the reduced mass and $v_z=v_{1z}-v_{2z}$ is the relative longitudinal velocity of the particles. The two values of the transferred longitudinal momentum $q_\pm$ that are allowed by the conservation laws correspond to two possible scattering channels. The sign (+) corresponds to scattering, as a result of which the particles in their center-of-mass system reverse their direction of motion along the z axis; in the (−) channel such a change does not occur.

The obtained formulas allow us to find the cross sections for scattering of any nonrelativistic particles in a magnetic field. Let us, for example, consider the scattering of an electron by a stationary proton. Then it is necessary to impose in (9) the condition $\Sigma f|S_{n_2'n_2}|^2=1$. Going over from (8) to the differential scattering cross section, using the Fourier transform of the screened Coulomb potential

$$V_\mathbf{q}=-4\pi e^2/(\mathbf{q}^2\hbar^{-2}+\rho_D^{-2}) \tag{11}$$

and integrating over p_x' and p_z', we obtain for the total cross section in the given channel the expression

$$\sigma^{(\pm)}_{n'\leftarrow n}=(\pi e^4/\hbar\omega_{Be}E_z)W_{n'n}(\xi_\pm)[1-(n'-n)\hbar\omega_{Be}/E_z]^{-1/2},$$

$$\xi_\pm=\rho_0^2(\rho_D^{-2}+\hbar^{-2}q_\pm^2)/2; \tag{12}$$

$$W_{n'n}(\xi)=(n!/n'!)\int_0^\infty ds\,s^{n'-n}[L_n^{n'-n}(s)]^2e^{-s}(s+\xi)^{-2}. \tag{13}$$

This cross section has been found by another method by Ventura.[5] His formula differs from ours only in that the expression for $W_{n'n}(\xi)$ is much more unwieldy than (13). It can, however, be shown that the two expressions are two different representations of the same function.

Let us now consider the scattering of a proton by an electron. For $E_{1\perp}\gg\hbar\omega_{Bi}$ we may neglect the quantization of the proton motion, while for $\omega_{Bi}\rho_D\ll v_1$ we can assume that during one scattering event the proton moves along a rectilinear trajectory and that we can, in deriving (9), replace the proton wave functions by plane waves. Then the cross section for scattering by an electron is given by

$$d\sigma_{\mathbf{p}_1'\leftarrow\mathbf{p}_1}=\frac{4e^4}{v_1(q^2+\hbar^2\rho_D^{-2})^2}\sum_{n_2n_2'p_{2z}}f_{n_2p_{2z}}|S_{n_2'n_2}(\mathbf{q}_\perp)|^2\delta\left(\frac{p_1'^2-p_1^2}{2M}+E_2'-E_2\right)d\mathbf{p}_1',$$

$$\mathbf{q}=\mathbf{p}_1'-\mathbf{p}_1. \tag{14}$$

Of special interest is the case of sufficiently strong magnetic fields, when the electron before the scattering is in the ground Landau state ($kT_e\lesssim\hbar\omega_{Be}$) and the longitudinal velocity of the proton is much higher than the longitudinal velocity of the electron. Then

$$d\sigma_{\mathbf{p}'\leftarrow\mathbf{p}}=\frac{4e^4}{v(q^2+\hbar^2\rho_D^{-2})^2}\sum_{n=0}^{n_{max}}\left[\frac{1}{n!}s^ne^{-s}\right]\delta\left(\frac{p'^2-p^2}{2M}+n\hbar\omega_{Be}+\frac{q_z^2}{2m}\right)d\mathbf{p}'. \tag{15}$$

The maximum number, n_{max}, of the Landau level to which the knocked-on electron can jump is determined by the laws of conservation of energy and momentum. As $mv^2/\hbar\omega_{Be}\to\infty$ we have $n_{max}\gg1$. For large n the expression in the square brackets in (15) is a sharp function, different from zero only near $s=n$, when $n\hbar\omega_{Be}=q_1^2/2m$. Then the delta-function can be taken out from under the summation sign, the summation over n yields unity, and (15) goes over into the usual Rutherford cross section.

The proton deceleration process is determined by the rate of change of its energy and of the angle of inclination, ϑ, of the trajectory to the magnetic field. The small changes in the energy ΔE, the angle $\Delta\vartheta$, and the square of the angle $\Delta\vartheta^2$ in one scattering event are equal to

$$\Delta E=-\frac{q_z^2}{2m}-n\hbar\omega_{Be},\quad \Delta\vartheta=\frac{q_x\cos\vartheta-q_z\sin\vartheta}{Mv},\quad \Delta\vartheta^2=\frac{q_\perp^2}{M^2v^2}+2\vartheta\Delta\vartheta, \tag{16}$$

the latter formula being valid for $\vartheta\ll 1$ and the system of coordinates having been chosen such that $p_y=0$. Let us average these quantities with the cross section (15). For the integration over $\mathbf{p}'$, let us represent the cross section in the form of a sum over the (+) and (−) channels (see (10)) and introduce the dimensionless quantities.

$$\varepsilon=\frac{B_0}{B}=\frac{mv^2}{2\hbar\omega_{Be}},\quad \mathbf{u}=\frac{\mathbf{q}_\perp}{mv},\quad \delta=\frac{\hbar}{mv\rho_D},$$
$$u_\pm=-\cos\vartheta\mp\left(\cos^2\vartheta-2u_x\sin\vartheta-\frac{n}{\varepsilon}-\frac{m}{M}u^2\right)^{1/2}. \tag{17}$$

As a result, for the rates of change of the energy, the angle, and the square of the angle, we obtain

$$\frac{dE}{dz}=-\frac{2\pi Ne^4}{mv^2\cos\vartheta}F,\quad \frac{d\vartheta}{dz}=\frac{4\pi Ne^4}{mMv^4\cos\vartheta}Q, \tag{18}$$

$$\frac{d\vartheta^2}{dz}=\frac{4\pi Ne^4}{mMv^4}\left[2\vartheta Q+\frac{m}{M}(G+2\Lambda_p)\right]. \tag{19}$$

The factor $\cos\vartheta$ in the denominators of (18) arose because of the transition from the variation of the quantities (16) along the trajectory to variation in the direction of the magnetic field. In (19) we have included a term taking into account the variation of ϑ^2 as a result of the proton-proton collisions (Λ_p is the corresponding Coulomb logarithm). The function F has the form

$$F=\sum_{n=0}^{n_{\max}}F_n,\quad F_n=F_n^{(+)}+F_n^{(-)},\quad F_n^{(\pm)}=\int d\mathbf{u}\,(u_\pm^2+n/\varepsilon)R_n^{(\pm)}, \tag{20}$$

$$R_n^{(\pm)}=(\pi n!)^{-1}(\cos^2\vartheta-2u_x\sin\vartheta-n/\varepsilon-u^2m/M)^{-1/2}$$
$$\times(u^2+u_\pm^2+\delta^2)^{-2}(\varepsilon u^2)^n\exp(-\varepsilon u^2). \tag{21}$$

Let us split up the functions Q and G in similar fashion:

$$Q_n^{(\pm)}=\int d\mathbf{u}\,(u_x\cos\vartheta-u_\pm\sin\vartheta)R_n^{(\pm)},\quad G_n^{(\pm)}=\int d\mathbf{u}\,u^2R_n^{(\pm)}. \tag{22}$$

As will be shown, in a strong magnetic field in which $\vartheta\neq 0,\ \pi/2$, the anisotropy of the dispersive medium leads to a preferred direction of variation for the angle ϑ, so that the mean rate of variation of the angle—a rate which is proportional to the function Q—can be different from zero. As $\vartheta\to 0$, obviously, $Q\to 0$, and only the mean rate of change of ϑ^2 determined in (19) by the terms containing m/M is different from zero. In zero field[4] $F=G=2\Lambda_0$ and $Q=0$ (see (1)).

The integration domain in (20) and (22) is determined by the positiveness of the radicand in (21), the contribution of the values of $\varepsilon u^2>n+\frac{1}{2}$ being exponentially small. If $\varepsilon\sin^2\vartheta>(n+\frac{1}{2})(m/M)^2$, then

$$u_x<u_0=\frac{\cos^2\vartheta-n/\varepsilon}{2\sin\vartheta},\quad n\leqslant\varepsilon(1+\sin\vartheta)^2. \tag{23}$$

According to the first inequality in (23), the integration domain is a half-plane; the second inequality determines the set of n values that make a substantial contribution to the sums (20) and (22). When the inequalities in (23) are fulfilled, the quasi-classicality condition for the proton, $\varepsilon\sin^2\vartheta\gg(m/M)^2$, is always valid, and we can neglect in (17) and (20)–(22) the terms u^2m/M under the radical sign. We shall, in the main, restrict ourselves to this case. The case when these approximations break down is interesting for very strong fields $B\gtrsim B_0M/m$, and is considered at the end of Sec. 3.

3. PROTON DECELERATION AT $B\gg B_0$

According to (17) and (23), in a sufficiently strong magnetic field $B\gg B_0$ ($\varepsilon\ll 1$), the energies of colliding particles are not high enough for the excitation of an electron to another Landau level to be possible. Therefore, in the cross section (15) and the functions F, Q, and G only the terms with $n=0$ play a role. Such a cross section differs from the Rutherford cross section only by the presence of the exponential function and in the argument of the delta-function. For $\varepsilon<\vartheta^2$ we can replace the exponential function in (21) by unity, after which the functions F, Q, and G are computed by performing the integration first over u_x and then over u_y. For $\varepsilon>\vartheta^2$ the exponential function should be retained, and the integration should be performed in cylindrical coordinates after expanding the integrand in a series in ϑ^2. Matching the results for the two cases, we obtain

$$F=1+\Lambda_B\sin^2\vartheta,\quad Q=\Lambda_B\sin\vartheta\cos\vartheta,\quad G=2\Lambda_B, \tag{24}$$

where

$$\Lambda_B=\ln\,[\delta(\sqrt{\varepsilon}+\sin\vartheta)^2]^{-1}=\ln\,[\rho_D/(\rho_0+\rho_{\min}\sin\vartheta)]$$
$$+\ln\,[\rho_{\min}/(\rho_0+\rho_{\min}\sin\vartheta)] \tag{25}$$

is the Coulomb logarithm for $B\gg B_0$, which is assumed to be sufficiently large in comparison with unity. The dependence of G on Λ_B is the same as on Λ_0 for $B=0$. The angular dependence of Q has a simple meaning: In a sufficiently strong field the effective "transverse mass" of the electron increases, so that $q_z\gg q_x$. Then it follows from the laws of conservation of energy and longitudinal momentum that $q_z=-2mv\cos\vartheta$, and, according to (16), $\Delta\vartheta=(m/M)\sin 2\vartheta$, which corresponds with (24). As $\vartheta\to 0$, the function $F\to 1$, which corresponds to losses due to "head-on" collisions, which, for $\vartheta\to 0$, are determined by the (+) channel; distant collisions (the (−) channel) do not in this case lead to energy losses, because of the conservation laws. As ϑ increases, the quantity $\Lambda_B\sin^2\vartheta$ begins to play a role

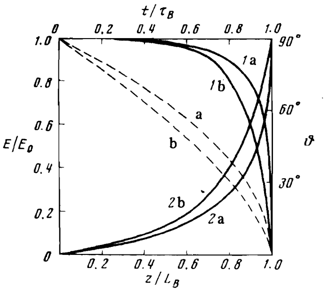

FIG. 1. Decrease of the energy E and increase of the angle ϑ during the deceleration of a proton. The continuous curves depict the dependence of E/E_0 and ϑ on the relative stopping length z/L_B (the curves 1a and 2a) and relative stopping time t/τ_B (1b and 2b) for $B \gg B_0$, $\Lambda_p = \Lambda_B$, and $\vartheta_0^2 \ll 2m/M$. The dashed curves give the dependence of E/E_0 on the relative range (a) and time (b) for $B = 0$.

in F. The first term in Λ_B is due to the (−) channel; the second, to the (+) channel. The role of the minimum impact parameter is played in both cases by $\rho_0 + \rho_{\min} \sin\vartheta$, since the minimum dimension of the interaction region is limited in the direction of the magnetic field by the de Broglie wavelength $\rho_{\min}$ in the transverse direction by the value $\rho_0 < \rho_{\min}$. For $\vartheta^2 > \Lambda_B^{-1}$ the role of the distant collisions becomes decisive. Notice that when $\varepsilon \ll 1$ the functions F, Q, and G depend on the quantity B only logarithmically.

Let us substitute (24) into (18) and (19) and, neglecting the dependence of the Coulomb logarithm on E and ϑ, integrate the resulting equations in the two cases. For $\vartheta \ll 1$ we shall seek the change in the angle of inclination of the proton trajectory with the aid of $d\vartheta^2/dz$. For $\vartheta^2 \gg \beta$ we shall use the formula for $d\vartheta/dz$. Here $\beta = (1 + \Lambda_p/\Lambda_B)m/M$. For reasonable values of the Coulomb logarithm $\beta \ll 1$. In both cases it is easy to find the dependence $E(\vartheta)$ and, since the regions of their applicability overlap, it is not difficult to represent the result in the form of a single formula, valid for any ϑ:

$$E = E_0 A^{\gamma/2} \cos\vartheta / \cos\vartheta_0, \qquad A = (\beta + \mathrm{tg}^2\,\vartheta_0)/(\beta + \mathrm{tg}^2\,\vartheta). \tag{26}$$

Here E_0 and ϑ_0 are the initial energy and angle and $\gamma = \Lambda_B^{-1}$. Substituting (26) into (18) and (19) and taking (1) into account, we can similarly obtain the ϑ dependence of the z-coordinate of the proton trajectory:

$$z = \frac{4L_0}{b\cos^2\vartheta_0} \int_{\vartheta_0} \frac{A^{\gamma} \sin\vartheta\, d\vartheta}{\beta + \mathrm{tg}^2\,\vartheta} = \frac{4L_0 \cos\vartheta_0}{b(3+2\gamma)} \left[\Phi(\vartheta_0) - \left(\frac{\cos\vartheta}{\cos\vartheta_0}\right)^3 A^{\gamma} \Phi(\vartheta) \right], \tag{27}$$

where $\Phi(\vartheta) = F(1, \frac{3}{2}; \gamma + \frac{5}{2}; \cos^2\vartheta - \beta)$ is a hypergeometric function and $b = \Lambda_B/\Lambda_0$. Taking into account the fact $dz/dt = v\cos\vartheta$, we can similarly obtain the time dependence of ϑ:

$$t = \frac{2\tau_0}{b(1+\gamma)} \left[\Phi_1(\vartheta_0) - \left(\frac{\cos\vartheta}{\cos\vartheta_0}\right)^{3/2} A^{3\gamma/4} \Phi_1(\vartheta) \right],$$
$$\Phi_1(\vartheta) = F(1, 3/4; 7/4 + 3\gamma/4; \cos^2\vartheta - \beta). \tag{28}$$

It is assumed in (27) and (28) that the proton coordinate $z_0 = 0$ at the initial moment of time $t_0 = 0$. It follows from (27) and (26) that the proton deceleration is accompanied by the increase of ϑ and that the proton comes finally to rest ($E = 0$) when $\vartheta = \pi/2$. The distance L_B along the magnetic field over which the deceleration occurs and the stopping time τ_B are therefore given by the formulas (27) and (28) if we discard in these formulas the second terms in the square brackets. Using the transformation formulas for the hypergeometric functions, we obtain for the values of $\vartheta_0^2 + \beta \ll 1$ from (27) and (28) the expressions

$$\frac{L_B}{L_0} = \frac{2}{b\gamma} \{1 - (\vartheta_0^2 + \beta)^{\gamma} [1 + 2\gamma(1 - \ln 2)]\},$$
$$\frac{\tau_B}{\tau_0} = \frac{2}{b\gamma} \left\{1 - (\vartheta_0^2 + \beta)^{3\gamma/4} \left[1 + \frac{3\gamma}{4}\left(\frac{\pi}{2} - \ln 8\right)\right]\right\}. \tag{29}$$

For sufficiently large values of Λ_B, when $\gamma \ln(\beta + \vartheta_0^2)^{-1} \ll 1$, the formulas (26)–(29) get significantly simplified:

$$\frac{E}{E_0} = \frac{\cos\vartheta}{\cos\vartheta_0}, \qquad \frac{z}{L_0} = \frac{4}{b\cos^2\vartheta_0} \left\{ \cos\vartheta - \cos\vartheta_0 + \frac{1}{2} \ln \frac{\beta + 4\,\mathrm{tg}^2(\vartheta/2)}{\beta + 4\,\mathrm{tg}^2(\vartheta_0/2)} \right\}, \tag{30}$$

$$\frac{t}{\tau_0} = \frac{3}{b(\cos\vartheta_0)^{3/2}} \left\{ \mathrm{arctg}\left(\frac{\sqrt{\cos\vartheta} - \sqrt{\cos\vartheta_0}}{1 + \sqrt{\cos\vartheta\cos\vartheta_0}} \right) + \frac{1}{2} \ln \frac{\beta + 4\sin^2(\vartheta/2)}{\beta + 4\sin^2(\vartheta_0/2)} + \ln \frac{1 + \sqrt{\cos\vartheta_0}}{1 + \sqrt{\cos\vartheta}} \right\}. \tag{31}$$

The stopping length and time obtained from (20) and (31) for $\vartheta_0^2 + \beta \ll 1$ (see also (29)) are equal to

$$L_B/L_0 = 2\{\ln[4/(\beta + \vartheta_0^2)] - 2\} b^{-1}, \qquad \tau_B/\tau_0 = {}^3/_2\{\ln[8/(\beta + \vartheta_0^2)] - \pi/2\} b^{-1}. \tag{32}$$

If $\Lambda_p = \Lambda_B$ and $\vartheta_0^2 \ll 2m/M$, then $L_B/L_0 = 12.4 b^{-1}$ and $\tau_B/\tau_0 = 11 b^{-1}$. If the quantity Λ_B is not very large, then a more exact value for L_B is obtainable from (29). For example, for $\Lambda_B = \Lambda_p = 10$ the ratio $L_B/L_0 = 10.5 b^{-1}$. The decrease in the coefficient b^{-1} in comparison with the case of very large Λ_B is connected with the increase in the role of energy losses at small values of the angle ϑ. The results obtained above are illustrated in Figs. 1 and 2.

Thus, for $B \gg B_0$ the stopping length and time for a proton that initially moves along the field increase in comparison with L_0 and τ_0. This happens because of the drastic decrease in the energy losses, which, for $\vartheta^2 < \Lambda_B^{-1}$, are determined not by distant, but by head-on, collisions. If the proton moves along the direction of the field, then $L_B/L_0 = \tau_B/\tau_0 = 2\Lambda_0$. In fact, these ratios are smaller, since the variation of ϑ plays an important role. As can be seen from (19) and (24), so long as $\vartheta^2 < \beta$, the deviation occurs in the usual diffusive manner; then the effective mechanism of regular devia-

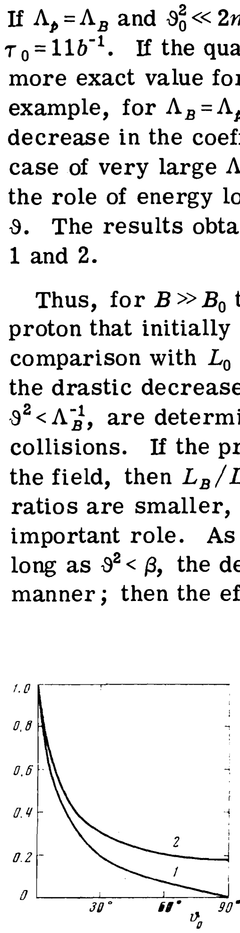

FIG. 2. Decrease of the stopping length L_B and time τ_B as the initial angle ϑ_0 is increased. The curves 1 and 2 depict the dependences $L_B(\vartheta_0)/L_B(\vartheta_0 = 0)$ and $\tau_B(\vartheta_0)/\tau_B(\vartheta_0 = 0)$ for $B \gg B_0$ and $\Lambda_p = \Lambda_B$. When $\vartheta_0 = \pi/2$, the quantity $\tau_B = 2\tau_0 \Lambda_0/\Lambda_B$.

tion is switched on. The rate of the deceleration exceeds that of the deviation only at large angles. Therefore, along the main part of the path the increase of the angle occurs almost without a change in the energy (Fig. 1), the domain part of the energy being dumped at the very end of the path, when the trajectory of the proton becomes circular. This is why the main results of the present section for $\vartheta_0 \ll 1$ can be obtained by solving Eqs. (18), (19), and (24) only for $\vartheta \ll 1$. Then, for example, for L_B/L_0 and τ_B/τ_0 we obtain the formulas (32) if we replace in them the expressions in the curly brackets by $\ln(\beta+\vartheta_0^2)^{-1}$, which gives an error of less than 10%.

The increase in the path length when $B \gg B_0$ was first pointed out by Basko and Syunyaev,[3] but the value $L_B/L_0 = 54b^{-1}$ obtained by them is greater than our value by almost a factor of five. The discrepancy is explained by the fact that the effect of regular variation of the angle in a strong magnetic field was not noticed in[3], and the losses due to head-on collisions at $\vartheta \ll 1$ were not taken into account.

The above-obtained results are, strictly speaking, inapplicable at $B > B_0 M/m$ and at $\varepsilon\vartheta^2 < (m/M)^2$, when the quantization of the proton motion in the magnetic field should be taken into account. When $\varepsilon\vartheta^2 < (M/M)^2$, the number of the Landau proton level before scattering is not large. However, if in this case $B \ll B_0 M/m$, then $mv^2 \gg 2\hbar\omega_{Bi}$, and transitions of the proton to high Landau levels are energetically allowed. However, after the first collision, which is unimportant for the stopping length or time, the proton reaches the already considered region. Therefore, the obtained formulas for L_B and τ_B remain valid.

For $B > B_0 M/m$ the energy involved in a proton-electron collision is not high enough for the excitation of the proton to another Landau level to be possible. The cross section for scattering by an electron in a lower energy state can be obtained from (8)–(10), it being determined by the (+) channel. Knowing the cross section, we can easily compute the energy losses and the proton range. In particular, if the proton is in the Landau ground level, then

$$d\sigma_{p_z'\leftarrow p_z} = \frac{\pi e^4}{m^2 v_z^4}\delta(q_z+2mv_z)\,dp_z', \qquad \frac{dE}{dz} = -\frac{2\pi N e^4}{m v_z^2}, \qquad \frac{L_B}{L_0} = \frac{\tau_B}{\tau_0} = 2\Lambda_0, \tag{33}$$

while $E_\perp$ does not change during collisions with electrons. Notice that the same results are obtained if, not quantizing the motion of the proton, we consider its deceleration at a constant small ϑ. If $B \gg B_0(M/m)^2$, then a slight increase in $E_\perp$ occurs as a result of collisions with the protons of the medium. For $B \gg B_0(M/m)^2$, when $Mv^2 \ll \hbar\omega_{Bi}$, transitions to other Landau levels are forbidden even in collisions with the protons of the medium, and there is no transfer of transverse energy.

4. PROTON DECELERATION AT $B \sim B_0$

If the energy of the impinging proton exceeds the threshold value (see (17) and (23)), at which $\varepsilon \approx \varepsilon_n = n(1+\sin\vartheta)^{-2}$, then a new scattering channel corresponding to an electron transition to the n-th Landau level is switched on.

In the most interesting case of $\vartheta_0 \ll 1$, to estimate the stopping length and time, it is sufficient to assume (as in the case when $B \gg B_0$) that $\vartheta \ll 1$. If in this case $(\varepsilon - n)^2 \gg \vartheta^2\varepsilon(n+\frac{1}{2})$ (the proton energy is not too close to the threshold value), then the integrands in (20) and (22) can be expanded in a series in powers of $u_x\vartheta/(1-n/\varepsilon)$ and integrated over the entire plane (u_x, u_y). Then

$$F_n^{(\pm)} = \frac{\varkappa_n^{(\pm)}+n}{\sqrt{1-n/\varepsilon}} W_n(\xi_n^{(\pm)}), \qquad F_0^{(-)} = \frac{\vartheta^2}{2} W_1(\xi_0^{(-)}), \tag{34}$$

$$G_n^{(\pm)} = \frac{(n+1)\,W_{n+1}(\xi_n^{(\pm)})}{\sqrt{1-n/\varepsilon}}$$

$$W_n(\xi) = \frac{1}{n!}\int_0^\infty \frac{s^n e^{-s}\,ds}{(s+\xi)^2}$$

$$= -\frac{(-\xi)^{n-1}}{n!}\left[(n+\xi)e^\xi \,\mathrm{Ei}(-\xi)+1-\sum_{k=0}^{n-2}\frac{k!\,(n-k-1)}{(-\xi)^{k+1}}\right], \tag{35}$$

where $\varkappa_n^{(\pm)} = \varepsilon(1\pm\sqrt{1-n/\varepsilon})^2$, $\xi_n^{(\pm)} = \varkappa_n^{(\pm)} + \varepsilon\delta^2$.

$$\text{For } \xi \ll n+{}^1/_2 \quad W_0=\xi^{-1},\ W_1=-\ln\xi - C - 1,\ W_{n>1}=1/n(n-1). \tag{36}$$

$$\text{For } \xi \gg n+{}^1/_2 \quad W_n=\xi^{-2}.\ \text{For } n\gg 1 \quad W_n=(n+\xi)^{-2}. \tag{37}$$

Here $\mathrm{Ei}(\xi)$ is an exponential integral function and C is the Euler constant. The functions Q_n are given by the expressions

$$Q_n^{(\pm)} = \frac{\vartheta}{2\sqrt{1-n/\varepsilon}}\left\{\frac{(n+1)\,W_{n+1}(\xi_n^{(\pm)})}{1-n/\varepsilon}\right.$$

$$\left. +(n+\varkappa_n^{(\pm)})\left[W_n(\xi_n^{(\pm)}) \pm 2(n+1)\,U_{n+1}(\xi_n^{(\pm)})\right]\right\}, \tag{38}$$

$$U_n(\xi) = \frac{1}{n!}\int_0^\infty \frac{s^n e^{-s}\,ds}{(s+\xi)^3}$$

$$= -\frac{(-\xi)^{n-2}}{2n!}\left\{\left[(\xi+n)^2-n\right]e^\xi\,\mathrm{Ei}(-\xi)+2n-1+\xi\right.$$

$$\left. -\sum_{k=0}^{n-3}\frac{k!\,(n-k-2)\,(n-k-1)}{(-\xi)^{k+1}}\right\}. \tag{39}$$

For

$$\xi \ll n \quad U_1=\xi^{-1},\ U_2=-(2\ln\xi+2C+3)/4.\ U_{n>2}=1/n(n-1)(n-2). \tag{40}$$

For

$$\xi \gg n \quad U_n=\xi^{-3}.\ \text{For } n\gg 1 \quad U_n=(n+\xi)^{-3}. \tag{41}$$

If $\varepsilon \gg 1$, then

$$F_0=\vartheta Q_0=\vartheta^2\Lambda^{(0)}, \qquad G_0=2\Lambda^{(0)}, \qquad Q_1=-\vartheta F_1/2=-\vartheta\Lambda^{(1)}. \tag{42}$$

The formulas (42) are valid for large—in comparison with unity—values of the logarithms

$$\Lambda^{(0)}=\ln(1+\rho_D/\rho_0), \qquad \Lambda^{(1)}=\ln\frac{1+\rho_D/\rho_0}{1+\rho_D/\rho_L}. \tag{43}$$

For $n_{\max} \sim \varepsilon \gtrsim 1$ the quantity $\rho_L \approx \rho_0\sqrt{2n_{\max}+1}$ coincides with the Larmor radius of the electron after scattering. If the proton energy considerably exceeds the energy of the n-th threshold, then, without allowance for the Debye screening,

$$F_{n>1}=1/(n-1), \qquad G_{n>0}=1/n, \qquad Q_{n>1}=-\vartheta/2n(n-1). \tag{44}$$

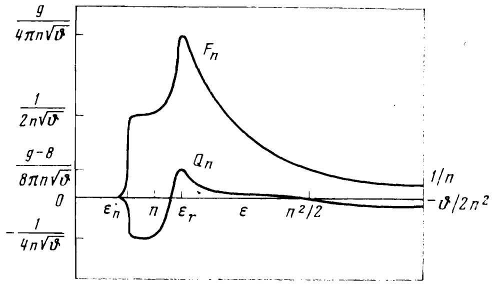

FIG. 3. Schematic representation of the dependences $F_n(\varepsilon)$ and $Q_n(\varepsilon)$ for $n \gg 1$ and $\vartheta \ll 1$. The maximum values of the functions, which are determined by the quantity $g = 17\ln 2 + C + \ln n = 12.4 + \ln n$, are attained at $\varepsilon = \varepsilon_r \approx n(1+2\vartheta)$; $\varepsilon_n \approx n(1-2\vartheta)$ is the threshold value of the dimensionless energy ε.

The expressions for F_n and G_n are valid for $\varepsilon \gg n$, while the expression for Q_n is valid only when $\varepsilon \gg n^2$, since for $n < \varepsilon < n^2$ we must take into account in the asymptotic expressions for W_n and U_n the next corrections in powers of ξ/n. Notice that the functions (42) and (44) are determined primarily by the (–) channel. For $\rho_D < \rho_0$, the Debye screening leads to a considerable decrease in these functions for $n < (\rho_0/\rho_D)^2$, not changing them for $n > (\rho_0/\rho_D)^2$.

In $2n\vartheta \ll \varepsilon - n \ll n$ (not very far from the threshold), then the two channels make the same contribution, and the Debye screening is not important, since $\delta = (\rho_{\min}/\rho_D) \ll 1$. Then

$$F_n = \frac{4nW_n(n)}{\sqrt{1-n/\varepsilon}}, \quad G_n = \frac{2(n+1)W_{n+1}(n)}{\sqrt{1-n/\varepsilon}}, \quad Q_n = \frac{\vartheta(n+1)W_{n+1}(n)}{(1-n/\varepsilon)^{3/2}}. \tag{45}$$

For $n \gg 1$ the formulas (34) and (38) assume the form

$$F_n = \frac{1}{n\sqrt{1-n/\varepsilon}}, \quad G_n = \frac{2-n/\varepsilon}{2}F_n, \quad Q_n = -\frac{\vartheta}{2n^2} + \frac{\vartheta}{4\varepsilon(1-n/\varepsilon)^{3/2}}. \tag{46}$$

Thus, when, decreasing, the proton energy approaches the n-th threshold, the functions F_n, G_n, and Q_n increase, the sharpest increase occurring in Q_n.

To compute F_n, Q_n, and G_n for the "near-threshold energy" $|\varepsilon - n| \lesssim 2n\vartheta$, we can set $\varepsilon \approx n$ in the slowly varying parts of the integrands in (20) and (22) and integrate in cylindrical coordinates over the azimuthal angle. The result is expressible in terms of the elliptic integrals $E(k)$ and $K(k)$. The subsequent integration over $|u|$ for $n \sim 1$ can be carried out only numerically. For $n \gg 1$ in the function (21)

$$(s^n/n!)\exp(-s) \approx (2\pi n)^{-1/2}\exp[-(s-n)^2/2n] \to \delta(s-n), \quad s = \varepsilon u^2. \tag{47}$$

Then for $n(1-2\vartheta) \leqslant \varepsilon < n(1+2\vartheta)$, we obtain

$$2G_n = F_n = \mathrm{K}(k)/(\pi n\sqrt{\vartheta}), \quad Q_n = [\mathrm{K}(k) - 2\mathrm{E}(k)]/(2\pi n\sqrt{\vartheta}), \quad k = [(1 + u_0\sqrt{\varepsilon/n})/2]^{1/2}, \tag{48}$$

where u_0 is given by the formula (23). For $\varepsilon_r - \varepsilon \ll 2n\vartheta (\varepsilon_r \equiv n(1-\sin\vartheta)^{-2})$,

$$F_n = \zeta/(2\pi n\sqrt{\vartheta}), \quad Q_n = (\zeta - 4)/(4\pi n\sqrt{\vartheta}), \quad \zeta = \ln(64n\vartheta/|\varepsilon - \varepsilon_r|). \tag{49}$$

By writing out the expressions similar to (48) for $\varepsilon > \varepsilon_r$, we can easily verify that they go over into (46) when $\varepsilon - \varepsilon_r \gg 2n\vartheta$. The functions (49) increase logarithmically as $\varepsilon \to \varepsilon_r$. Their values at $\varepsilon = \varepsilon_r$ (see Fig. 3) can be obtained if in integrating in the region $u_x - \sqrt{n/\varepsilon_r} \lesssim \vartheta$ we take the finite width of the function (47) into account. This same width determines the finite magnitude of the interval $\Delta\varepsilon \sim \vartheta\sqrt{n}$ in which the sharp change in the functions (20) and (22) occur in going through the threshold ε_n.

When $n \sim 1$, the function (47) ceases to be so sharp. Therefore, although the behavior of F_n, Q_n, and G_n remains qualitatively the same, their dependence on ε inside the near-threshold region becomes smoother (cf. Figs. 3 and 4). The functions F_n, G_n, and Q_0 are positive everywhere, while Q_n for $n \geqslant 1$ is negative when $\varepsilon \lesssim \varepsilon_r$ and $\varepsilon \gtrsim n^2$. The dependence $F(\varepsilon)$ is shown in Fig. 4. The presence of the peaks is due to the switching of the scattering channels to new n. For $\vartheta \ll 1$ and $n \sim 1$, the width of the peaks $\sim \varepsilon_r - \varepsilon_n \sim 4n\vartheta$, and for $4n\vartheta < 1$ it is less than the threshold spacing. The peak height $\sim 1/n\sqrt{\vartheta}$ (while $\vartheta \gtrsim m/M$) can significantly exceed the value $2\Lambda_0$, which the function F assumes for $B = 0$. As n increases, the peak height falls off in proportion to n^{-1}, and for $\varepsilon \gg 1$ the function $F \to 2\Lambda_0$. As ϑ increases at fixed n, the thresholds become smeared and the maxima of the peaks move towards the region of higher energies. For $n \gg 1$ the structure of the near-threshold region becomes more complex—the threshold-peak separation ε_n at $\varepsilon \simeq \varepsilon_r$ increases (Fig. 3); for $n \gtrsim \vartheta^{-1}$ the near-threshold regions for neighboring n overlap.

The appearance of peaks is connected with the root singularities in the number density of the final states and, by the same token, in the cross section (15) for those collisions after which the electron has, relative to the proton, zero longitudinal energy. The very fact that such singularities appear is due to the quantizability of the electron motion across the magnetic field (see, for example, [6]). The peaks were missed in[3], because the ordinary Rutherford cross section was used there in place of (15) to compute F. Notice that the region $\delta\varepsilon \sim E_b/\hbar\omega_{Be}$ near the peak centers is determined by those collisions after which the longitudinal particle energy is so small that the formation of quasi-bound particle states of binding energy E_b is possible.[7] In this region the Born approxmation, used by us, can break down, but for $\hbar\omega_{Be} \gg me^4/\hbar^2$ ($B \gg 2\times10^9$ G) the quantity $\delta\varepsilon \ll 1$, and the obtained results are not quite

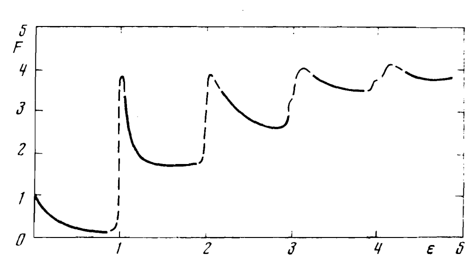

FIG. 4. Dependence of the energy loss (i.e., of the function F) on ε for $\vartheta \approx (m/M)^{1/2}$. The dashed sections of the curve in the region of the peaks, where the accuracy of the computation is not high, are illustrative in character.

exact only at the very centers of the peaks.

Let us proceed to the analysis of the proton range. Let us first consider the $B > B_0$ case, when electron transitions to levels with $n \geq 1$ are not yet allowed. Then from (34)–(39) we obtain

$$F^{(-)}=\vartheta^2\Lambda_B,\quad F^{(+)}=1+\xi e^{\xi}\,\mathrm{Ei}\,(-\xi),\quad \vartheta G=2Q=2\vartheta\Lambda_B,\quad \Lambda_B\approx\ln\,(\rho_D/\rho_0), \tag{50}$$

where $\xi = \xi_0^{(+)} = 4\varepsilon$. Here we have allowed for the fact that $Q_0^{(-)} \gg Q_0^{(+)}$. For $B \gg B_0$ these expressions go over into (24), while for $B \gtrsim B_0$ they differ from (24) only by the quantity $F^{(+)}$. For an initial proton energy E_0 close to the first threshold, $F^{(+)}$ increases in the course of the deceleration from 0.16 to 1 (the role of the head-on collisions increases as B_0/B decreases; see Fig. 4). Since $F^{(+)}$ determines the proton deceleration only along that part of the trajectory where $\vartheta^2\Lambda_B \lesssim 1$ and $E \sim E_0$, let us, for the purposes of making estimates, set $F^{(+)}(E) = F^{(+)}(E_0)$ in (50). Then Eqs. (18) and (19) for $\vartheta \ll 1$ can easily be integrated, and yield

$$E=E_0A^{\gamma/2},\quad z=(2L_0/b)\,(1-A^{\gamma})\,\gamma^{-1}\rightarrow(2L_0/b)\,\ln A^{-1}, \tag{51}$$

where the latter relation is valid if $\gamma = F^{(+)}(E_0)/\Lambda_B \ll 1$. The closer E_0 is to the first threshold, the more exactly it is fulfilled. To estimate the stopping length in (51), let us set $\vartheta \sim 1$. Then L_B is given by the formula (51) is $A \rightarrow \beta + \vartheta_0^2$ in it, which coincides with (29) and (32) to within $\sim 1/\ln(M/m)$, an accuracy which is inevitable in the approximation $\vartheta \ll 1$. Thus, the general picture of the deceleration remains as before, while L_B for $B > B_0$ only slightly exceeds L_B for $B \gg B_0$, more exactly coinciding with the asymptotic value (32). In both cases there occurs at the initial stage of the deceleration a deviation of the proton from the direction of the field at almost constant energy.

Let us consider for $B < B_0$ the proton deceleration in the region of the most powerful threshold $n = 1$, where the rates of change of the energy and the angle attain their extremum values and undergo abrupt changes. Let the proton reach the near-threshold region from the higher-energy side $\varepsilon \sim 1 + 2\vartheta$. If $\vartheta^{3/2}\Lambda_B \lesssim 1$, then $F \approx F_1 \sim \vartheta^{-1/2}$ and $Q \approx Q_1$. If the proton were decelerated at constant ϑ, then, according to (18), the distance which it would traverse, remaining in the near-threshold region, is $\delta L_E \sim L_0 b^{-1}\vartheta^{3/2}$. If the energy of the proton were fixed, then the angle ϑ would tend to a value ϑ_c at which (Fig. 3) $Q(\vartheta_c) = 0$, irrespective of whether the initial angle was greater or smaller than ϑ_c. Since for $\vartheta \sim \vartheta_c$ the function $Q(\vartheta) \sim -(\vartheta - \vartheta_c)\vartheta^{-3/2}$, the characteristic length over which $\vartheta \rightarrow \vartheta_c$ is, according to (19), equal to $\delta L_\vartheta \sim \delta L_E$. Consequently, E and ϑ vary at roughly the same rate, and take the proton over the indicated distance out of the near-threshold region. The ratio of δL to the subsequent stopping length (32) $\sim \vartheta^{3/2}\Lambda_B/\ln(M/m) \lesssim 1/\ln(M/m)$. Therefore, the crossing of the near-threshold region $n = 1$ by the proton does not make a substantial contribution to the stopping length (and time). A still smaller contribution is obviously made by the crossing of the thresholds with $n > 1$. Thus, it is sufficient to find the stopping length and time, using the functions F, Q, and G smoothed over the near-threshold peaks, which is what we shall do in the following section.

5. PROTON DECELERATION AT $B \ll B_0$ AND EXTRAPOLATION TO THE CASE OF ARBITRARY FIELDS

For $B \ll B_0$ ($\varepsilon \gg 1$), as follows from (17) and (23), electron transitions to high Landau levels $n \lesssim n_{\max} = \varepsilon(1 + \sin\vartheta)^2$ are allowed. Let us compute the functions F_n, Q_n, and G_n for all possible n and an arbitrary ϑ.

If $\varepsilon\cos^2\vartheta - n \gg \sqrt{\varepsilon(n+1)}\sin\vartheta$, then the dominant contribution to the scattering is made by the (–) channel, it being possible then to extend the integration in (20) and (22) over the entire (u_x, u_y) plane. For $\vartheta \rightarrow \pi/2$, the (±) channels make the same contribution, and it is convenient to perform the integration over the half-plane $u_x \leq 0$ in cylindrical coordinates. The same expressions, which can be considered to be valid for any ϑ, are obtained in the two cases. For $n = 0$ and $n = 1$,

$$F_0=\Lambda^{(0)}\sin^2\vartheta,\quad F_1=\Lambda^{(1)}(1+\cos^2\vartheta);\quad Q_0=\Lambda^{(0)}\sin\vartheta\cos\vartheta,\quad Q_1=-\Lambda^{(1)}\sin\vartheta\cos\vartheta. \tag{52}$$

These formulas are valid if $\Lambda^{(0)}$ and $\Lambda^{(1)} \gg 1$ (see (43)). For $n \geq 2$, without allowance for the Debye screening, $F_n \approx n^{-1} \gg Q_n\vartheta^{-1} \sim n^{-2} + \varepsilon^{-1}$. These formulas are valid for $n < \varepsilon(1 - \sin\vartheta)^2$, but when $\vartheta \sim 1$, they are also valid when $\varepsilon(1 - \sin\vartheta)^2 \ll n < n_{\max}$. Owing to the influence of the Debye screening, a substantial contribution to the sum over n is made by the terms with $n \gtrsim n_{\min} \sim (\rho_0/\rho_D)^2$. Therefore (with allowance for the expression given in (44) for G_n), we obtain

$$\sum_{n=2}^{n_{\max}} F_n \approx \sum_{n=2}^{n_{\max}} G_n \approx \ln\frac{n_{\max}}{n_{\min}} \approx 2\ln\frac{\rho_D/\rho_{\min}}{1+\rho_D/\rho_0} \gg \vartheta^{-1}\sum_{n=2}^{n_{\max}} Q_n. \tag{53}$$

For electron transitions to levels with $n \rightarrow n_{\max}$, the considered fixed value of the proton energy approaches the n-th threshold. Therefore, for $\vartheta \ll 1$, the functions F_n increase, deviating from the law $1/n$. It is easy to show, however, that the contribution to the sum over n due to this additional growth is small compared to the large logarithm (53), which is determined by the large number of terms with $n < n_{\max}$. The values of Q_n increase in magnitude in precisely the same way, as $n \rightarrow n_{\max}$, and may change sign at $n \approx n_{\max}$. However, if we average the dependence $Q(E)$ over the near-threshold discontinuities, then the negative values of Q_n will cancel out the positive values, and the sum will not contain the large logarithm.

Summing (42) and (52) with (53), we obtain

$$F=2\Lambda_{\parallel}+\Lambda_{\perp}\sin^2\vartheta,\quad Q=\Lambda_{\perp}\sin\vartheta\cos\vartheta,\quad G=2(\Lambda_{\parallel}+\Lambda_{\perp}), \tag{54}$$

where

$$\Lambda_{\perp}=\ln(1+\rho_D/\rho_L),\quad \Lambda_{\parallel}=\ln\frac{\rho_D/\rho_{\min}}{1+\rho_D/\rho_L}. \tag{55}$$

Since $\Lambda_{\parallel} + \Lambda_{\perp} = \Lambda_0$, the expression for G is the same as for $B=0$. If $\rho_L \gg \rho_D$, then $\Lambda_\perp \to 0$ and $\Lambda_\parallel = \Lambda_0$, i.e., the magnetic field has no effect on the deceleration process. If $\rho_L \ll \rho_D$, then $\Lambda_\perp = \ln(\rho_D/\rho_L)$ and $\Lambda_\parallel = \ln(\rho_L/\rho_{\min})$. The last logarithm arises in the problem of temperature relaxation in classical magnetized plasmas.[1] Notice that in (54) we have used only the asymptotic forms of F_n, Q_n, and G_n far from the corresponding thresholds, where the Born approximation is applicable also in a weak magnetic field.

If, instead of (55), we take

$$\Lambda_\perp = \ln\left[1+\frac{\rho_D\rho_{\min}}{(\rho_0+\rho_{\min}\sin\vartheta)^2}\right], \quad \Lambda_\parallel = \frac{1}{2}+\ln\frac{(1+\rho_{\min}/\rho_L)(\rho_D/\rho_{\min})}{1+\rho_D/\rho_L}, \tag{56}$$

then for $B \ll B_0$ (56) goes over into (55), while for $B \gg B_0$, $\Lambda_\perp = \Lambda_B$, $\Lambda_\parallel \to \frac{1}{2}$, and (54) goes over into (24). Then we can assume that the formulas (54) are approximately valid for any relation between B and B_0 if we neglect the discontinuities in the functions F, Q, and G near the thresholds, which, as shown in the preceding section, have little effect on the stopping length and time. The value of $\Lambda_\parallel$ virtually determines the smoothed-out energy-loss curve at small angles. Thus, the process of proton deceleration and deflection is, generally speaking, determined by the two Coulomb logarithms $\Lambda_\parallel$ and $\Lambda_\perp$, and only in the limiting cases $B \gg B_0$ and $B \ll B_0$ is it determined by one of them. The quantity $\Lambda_\perp$ influences to a great extent the deflection of the proton, and is due to transitions to the levels with $n=0$ and $n=1$. Such a privileged position of the two levels is connected with the original formulation of the problem—the electron before scattering is in the level with $n=0$. The quantity $\Lambda_\parallel$, on the other hand, affects primarily the deceleration of the proton, and is due to transitions to levels with $n \geq 1$.

Let us substitute (54) into (18) and (19) and solve the resulting equations. It is easy to see that the solution is given by the formulas (26)–(32) if we set in them

$$b=\Lambda_\perp/\Lambda_0, \quad \gamma=2\Lambda_\parallel/\Lambda_\perp, \quad \beta=[1+(\Lambda_\parallel+\Lambda_p)/\Lambda_\perp](m/M). \tag{57}$$

These three parameters determine the ratios L_B/L_0 and τ_B/τ_0 for all the considered cases.

Notice that considerable increase in the stopping length and time, as compared to L_0 and τ_0, is also possible in the "classical" case of $B \ll B_0$ if $\Lambda_\perp \gtrsim \Lambda_\parallel$. For example, for protons with $E_0 = 50$ MeV ($B_0 = 2.3 \times 10^{12}$ G) and $\vartheta_0^2 \ll m/M$, decelerating in a plasma with $N = 10^{21}$ cm^{-3}, $T = 10^7$ K, and $B = 10^{11}$ G (such conditions are possible on the surface of a neutron star), we obtain: $\varepsilon \approx n_{\max} = 23$; $\Lambda_0 = 8.3$; $\Lambda_\perp = 4.4$; and $\Lambda_\parallel = 3.9$. Then $b = 0.54$, $\gamma = 1.7$, and, since $\beta \sim m/M$ and $\gamma \ln \beta^{-1} \gg 1$, the second term in the curly brackets in (29) is unimportant. Therefore, the quantities $L_B = 2.2L_0 = 14$ m and $\tau_B = 2.2\tau_0 \approx 1.9\times10^{-7}$ sec differ from (1) only by the substitution $\Lambda_0 \to \Lambda_\parallel$. Under the same conditions, but in a field $B = 10^{13}$ G, $\varepsilon \approx 0.23$; $n_{\max} = 0$; $\Lambda_\perp = \Lambda_B \approx 8.7$; $\gamma \approx 0.11$; $b \approx 1.05$, and from (32) $L_B = 10.4L_0 = 66$ m; $\tau_B = 9.4\tau_0 = 8.2\times10^{-7}$ sec.

The authors are grateful to Yu. N. Gnedin and R. A. Syunyaev for useful discussions.

1. V. P. Silin, Vvedenie v kineticheskuyu teoriyu gazov (Introduction to the Kinetic Theory of Gases), Nauka, 1971, Ch. 10.
2. M. M. Basko and R. A. Syunyaev, Preprint IKI, Pr-208, 1974.
3. M. M. Basko and R. A. Syunyaev, Zh. Eksp. Teor. Fiz. **68**, 105 (1975) [Sov. Phys. JETP **41**, 52 (1975)].
4. D. V. Sivukhin, in: Voprosy teorii plazmy (Problems of Plasma Theory), No. 4, Atomizdat, 1964, p. 81.
5. J. Ventura, Phys. Rev. **A8**, 3021 (1973).
6. V. Canuto and H. Y. Chiu, Space Sci. Rev. **12**, 3 (1971).
7. V. Canuto and D. C. Kelly, Astrophys. Space Sci. **17**, 277 (1972).

Translated by A. K. Agyei.