Nonlinear theory of parametric excitation of waves

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A consistent nonlinear theory of parametric excitation of waves is constructed on the basis of the exact interaction Hamiltonian of parametric waves (6), in which the interaction of these waves with thermal waves is treated microscopically. A justification is given for the procedure of diagonalizing the Hamiltonian $\mathscr{H}_{\rm int}$ and the phenomenological introduction of damping into the canonical equations of motion for $a_{\mathbf{k}}$, which forms the basis of S-theory [B. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk **114** 609 (1974); Fiz. Tverd. Tela (Leningrad) **11**, 2067 (1969); Zh. Eksp. Teor. Fiz. **59**, 1200 (1970)]. A study is made of the effects that arise when one goes outside the scope of S-theory; in particular, a study is made of the distribution of $n_{\mathbf{k}\omega}$ in ω, $\mathbf{k}$ space, $\langle a_{\mathbf{k}\omega} a^*_{\mathbf{k}'\omega'}\rangle = n_{\mathbf{k}\omega}\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')$, in a wide range of variation of the pumping amplitudes. It is shown in particular that at a certain supercriticality, which is near the threshold value, there is a phase-transition type phenomenon—the precipitation of a "single-frequency condensate", in which all the parametric waves oscillate with the same frequency equal to half the pumping frequency: $\Delta n_{\mathbf{k}\omega} \propto \delta(\omega-\omega_p/2)$. At large supercriticalities, the integrated amplitude of the single-frequency part of the parametric turbulence of the waves appreciably exceeds the integrated amplitude of the many-frequency parametric turbulence. These phenomena must also occur for other methods of excitation of turbulence of waves near the surface of constant frequency $\omega_{\mathbf{k}} = \text{const}$.

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Great interest is now concentrated on phenomena accompanying the paramagnetic excitation of waves in ferromagnets,[1-4] antiferromagnets,[5,6] plasmas,[7] and other nonlinear media. In a number of important cases, the dispersion law $\omega_{\mathbf{k}}$ of the waves is a nondecay law, and the external field, the pumping, can be assumed to be spatially homogeneous and monochromatic:

$$h(\mathbf{r},t)=h(t)=h\exp(-i\omega_p t). \tag{1}$$

In the construction of a nonlinear theory of parametric excitation of waves in this situation, Zakharov, Starobinets, and the author[8,9] proceeded from the classical Hamilton equations of motion for the complex amplitudes $a_{\mathbf{k}}$ of traveling waves:

$$i\frac{\partial a_{\mathbf{k}}}{\partial t}=\frac{\delta\mathscr{H}}{\delta a_{\mathbf{k}}^*}. \tag{2}$$

The Hamiltonian of the problem

$$\mathscr{H}=\int \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* d\mathbf{k}+\mathscr{H}_p+\mathscr{H}_{int} \tag{3}$$

includes interaction of the waves with the pumping:

$$\mathscr{H}_p={}^1/_2\int[h(t)V_{\mathbf{k}}a_{\mathbf{k}}a_{-\mathbf{k}}^*+\text{c.c.}]d\mathbf{k} \tag{4}$$

and their interaction $\mathscr{H}_{\rm int}$ with one another. In the case of a nondecay dispersion law, the pumping gives rise to the excitation of waves in the neighborhood of the "resonance surface"

$$\omega_p=\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}=2\omega_{\mathbf{k}}. \tag{5}$$

It is obvious that the interaction of these "parametric waves" is described by the four-wave Hamiltonian

$$\mathscr{H}_{int}={}^1/_2\int T_{12,34}a_1^*a_2^*a_3a_4\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\mathbf{k}_4. \tag{6}$$

The main approximation made in [8,9] is analogous to the BCS approximation in the theory of superconductivity and consists of a reduction of the interaction Hamiltonian to a form diagonal in the wave pairs $a_{\pm\mathbf{k}}$. It corresponds to the replacement of the exact problem by the problem of the interaction of pairs of waves. The justification for this is the "pairing"—the phase correlation of waves in pairs, which arises as a result of the pumping and leads to the exchange of energy between pairs already in the first order of perturbation theory in $\mathscr{H}_{\rm int}$.

Another important approximation consists of the phenomenological allowance for the interaction between the parametric waves and the "thermal bath" of the remaining waves through the addition of a "dissipative" term to the canonical equations (1). The theory based on these assumptions (which we have called S-theory [10]), enables one to describe in detail the behavior of the waves above the threshold of parametric excitation; it is in good qualitative and quantitative agreement with a number of experiments on parametric excitation of spin waves in ferro- and antiferromagnets (see the review[4], and also [8-19]).

At the same time, there are experimental results (for example, the "hard" excitation of parametric waves[20]) and there could be new accurate experiments (for example, investigation of "noise" emission at frequencies near $\omega_p/2$, ω_p, $2\omega_p$) whose interpretation requires one to go beyond the framework of S-theory. The point is that S-theory describes correctly only the integral characteristics of the system of parametric waves and, for example, does not take into account at all the finite width of the distribution of the parametric waves in $\mathbf{k}$, ω space. In S-theory

$$n_{\mathbf{k}\omega}\propto\delta(k-k_0)\,\delta(\omega-\omega_p/2). \tag{7}$$

However, a priori one could imagine completely different behaviors of parametric waves, resulting in different broadening of the distribution (7) without a change (accurate to small parameters) in the integral (in the modulus $\mathbf{k}$ and ω) magnitude of $n_{\mathbf{k}\omega}$.

The limits of applicability of S-theory also remain obscure. The procedure for diagonalizing the interaction Hamiltonian can clearly provoke serious objections on first examination. For if one adds formally to the equations of S-theory the collision term of the kinetic equation [which is formally of second order in $\mathscr{H}_{\rm int}$ (6)] and substitutes into it the singular distribution (7), it is not small, but infinite. The point is that the kinetic equation is valid if the wave packet has a large frequency width, which guarantees randomization of the phases, whereas the distribution (7) of S-theory is a single-frequency distribution.

In this connection, we encounter fundamental questions that take us outside the scope of the theory of parametric excitation of waves: Can single-frequency turbulence of waves be weak and how can it be described? Can one retain the single-frequency nature of the turbulence in the presence of many-frequency thermal noise and, if yes, why? What are the criteria for randomization of the phases for singular distributions concentrated on a surface or even a line in $\mathbf{k}$ space?

These and other questions are considered in the present paper. In it, for the case of parametric excitation, a consistent nonlinear theory is constructed for single-frequency and many-frequency weak turbulence of waves on the basis of the exact Hamiltonian of the problem (2), (4), (6), in which the interaction of parametric waves with thermal waves is treated microscopically.[1)] We use the canonical diagram technique developed by Zakharov and the author,[22] which generalizes the technique which Wyld[23] developed for problems of hydrodynamic turbulence. The weakness of the interaction of parametric waves relative to their dispersion $TN(k\partial\omega/\partial k)^{-1}$ enables us to use only a selective summation of diagrams that do not renormalize vertices in the Dyson equations, and obtain for the binary correlation functions $n_{\mathbf{k}\omega}$ and $\sigma_{\mathbf{k}\omega}$:

$$\begin{aligned}\langle a_{\mathbf{k}\omega}a^*_{\mathbf{k}'\omega'}\rangle &= n_{\mathbf{k}\omega}\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega'),\\ \langle a_{\mathbf{k}\omega}a_{\mathbf{k}'\omega'}\rangle &= \sigma_{\mathbf{k}\omega}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega'-\omega_p),\end{aligned} \tag{8}$$

closed integral equations that generalize the equations of S-theory (Sec. 1).

Let us describe qualitatively the picture of the phenomena found by the analysis of these equations. In the absence of pumping, the waves relax to thermodynamic equilibrium: $n_{\mathbf{k}\omega} \to n_{\mathbf{k}\omega}^0$, $\sigma_{\mathbf{k}\omega} \to 0$. Under the influence of pumping (even below the threshold) the waves are heated in the neighborhood of the resonance surface; with increasing h there is narrowing of the packet $(n_{\mathbf{k}\omega} - n_{\mathbf{k}\omega}^0)$ with respect to $\omega_{\mathbf{k}}$ and ω around $\omega_p/2$. At a supercriticality h^* somewhat higher than the threshold value h_1 there is a phase-transition type phenomenon—"precipitation of a condensate" in ω space: On the background of the many-frequency turbulence described above there arises a new wave packet that oscillates with a well defined frequency $\omega = \omega_p/2$:

$$\begin{aligned}n_p(\mathbf{k},\omega) &\equiv n_{\mathbf{k}\omega}-n_{\mathbf{k}\omega}^0 = n_S(\mathbf{k})\delta(\omega-\omega_p/2)+n_t(\mathbf{k},\omega),\\ \sigma_p(\mathbf{k},\omega) &\equiv \sigma_{\mathbf{k}\omega} = \sigma_S(\mathbf{k})\delta(\omega-\omega_p/2)+\sigma_t(\mathbf{k},\omega).\end{aligned} \tag{9}$$

Here, the subscript S is appended to the single-frequency, and the subscript t to the many-frequency part of the parametric turbulence of the waves. The theory of many-frequency turbulence for $h < h^*$ is constructed in Sec. 2; parametric turbulence of waves for $h > h^*$ is considered in Sec. 3. In particular, it is shown that with increasing supercriticality the integrated magnitude of the single-frequency part of the turbulence, N_S, increases (for $h \lesssim h_1[(\mathbf{k}\partial\omega_{\mathbf{k}}/\partial\mathbf{k})\gamma_{\mathbf{k}}^{-1}]^{1/2}$ in accordance with S-theory), while the many-frequency part N_t decreases, so that $N_t \ll N_S$ already when $h \approx 2h_1$. Note that the distribution

$$n_t(\mathbf{k}) = \int n_t(\mathbf{k}\omega)\,d\omega$$

with respect to the modulus of $\mathbf{k}$ is a Lorentz function, and the distribution $n_S(\mathbf{k})$ is the square of a Lorentz function with a common maximum on the resonance surface, which is renormalized by the interaction, and the same width ν, which increases with increasing supercriticality approximately as $h^{3/2}$ and at the limit of applicability of S-theory, i.e., for

$$h = h_S \approx h_1[(k\partial\omega_{\mathbf{k}}/\partial k)\gamma_{\mathbf{k}}^{-1}]^{1/2},$$

reaches the value $\Delta\mathbf{k}\,\partial\omega_{\mathbf{k}}/\partial\mathbf{k} \approx \gamma_{\mathbf{k}}$.

It must be emphasized that in this paper we have restricted ourselves to studying spatially homogeneous and stationary turbulence, for which $\langle a_{\mathbf{k}\omega}a^*_{\mathbf{k}'\omega'}\rangle \propto \delta \times (\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')$. Of course, in a spatially homogeneous system when the external conditions are stationary such solutions always exist, although they may be unstable. Such an instability does indeed arise in some cases, as is shown in [24], and its development leads to auto-oscillations, which can be observed experimentally.[18,19]

We shall assume that the coefficients of the Hamiltonian are such that the instability against breakdown of spatial inhomogeneity does not arise. It is then natural to assume that the parametric turbulence after the pumping has been switched off relaxes to the homogeneous stationary state (9), which is the state investigated in this paper.

1. BASIC EQUATIONS

We obtain averaged equations that describe parametric turbulence of waves by means of the canonical diagram technique.[22,23,25] For this, into the equations of motion (2) and (3) for $a_q = a_{\mathbf{k}\omega}$ it is necessary to introduce a small damping $\gamma_{\mathbf{k}}^0$ of the waves and a Gaussian random force with Langevin correlation function

$$\langle f_q f_{q'}^*\rangle = F_q\delta(q-q'),\quad F_q = \gamma_{\mathbf{k}}^0 T/\pi\omega_{\mathbf{k}} \tag{10}$$

(T is the temperature of the medium) and represent the solution of (2)–(3) in the form of a formal series in powers of $f_{q'}$. To take into account the pairing ($\sigma_q \neq 0$) it is necessary, as in [25], to introduce, besides the normal Green's function G_q, the anomalous Green's function L_q:

$$\rightarrow = G_q = \left\langle\frac{\delta a_q}{\delta f_q}\right\rangle,\quad \rightarrow\leftarrow = L_q = \left\langle\frac{\delta a_q}{\delta f_q^+}\right\rangle,\qquad \leftarrow = G_q^*,\quad \leftrightarrow = L_q^*, \tag{11}$$

where + denotes the transition to the adjoint and the substitution $q = \bar{q} \equiv -k,\ \omega_p - \omega$.

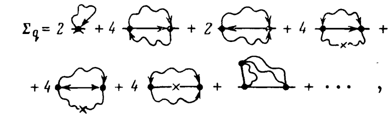
Summing, as usual, the reducible graphs, we arrive at the system of Dyson equations for G_q, L_q and n_q, σ_q:

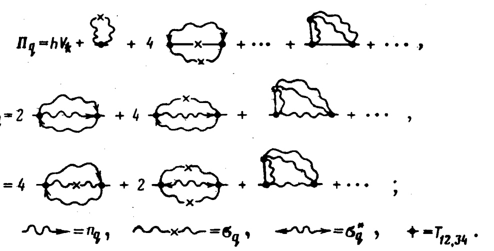
$$\begin{aligned}&G_q[\omega-\omega_{\mathbf{k}}-\Sigma_q+i\gamma_{\mathbf{k}}^0]-L_q^+\Pi_q=1,\\ &-G_q\Pi_q^+ + L_q^+[\omega_p-\omega-\omega_{\mathbf{k}}-\Sigma_q^+-i\gamma_{\mathbf{k}}^0]=0;\end{aligned} \tag{12}$$

$$\begin{aligned}n_q &= G_q[(F_q+\Phi_q)G_q^* + \Psi_q L_q^+] + L_q[\Psi_{\bar{q}}G_q^* + (F_q^+ + \Phi_q^+)L_q^+],\\ \sigma_q^+ &= L_q^+[(F_q+\Phi_q)G_q^* + \Psi_q L_q^+] + G_q^+[\Psi_{\bar{q}}G_q^* + (F_q^+ + \Phi_q^+)L_q^+].\end{aligned} \tag{13}$$

Here, Σ_q, Π_q, Φ_q, and Ψ_q denote the sums of irreducible diagrams; Σ_q and Φ_q are normal diagrams, i.e., they preserve the direction of the arrows; Π_q and Ψ_q are anomalous irreducible diagrams.

The diagrams of Φ_q and Ψ_q, in contrast to the diagrams for Σ_q and Π_q, can be cut into two parts only with respect to the dashed lines f_q. We expand the irreducible diagrams in powers of the matrix element $T_{12,34}$ of the interaction Hamiltonian:





(14)

Subsequently we shall show that for not too large supercriticality the diagrams of (14) containing three or more vertices T can be ignored. This means that we obtain closed nonlinear integral equations for G_q, L_q and n_q, σ_q. We write down analytically the relations (14):

$$\Sigma_q=2\int T_{kk'}n_{q'}dq'+2\int T_{k1,23}\{T_{k1,23}[G_1^*n_2n_3+n_1(G_2n_3+n_2G_3)]+2T_{k\bar{2},\bar{1}3}[\sigma_1^*(\sigma_2G_3+L_2n_3)+L_1^*\sigma_2n_3]\}\delta(q+q_1-q_2-q_3)\,dq_1\,dq_2\,dq_3,$$

$$\Pi_q=P_k+2\int T_{k1,23}\{T_{k1,23}[\sigma_1^*(\sigma_2L_3+\sigma_3L_2)+L_1^*\sigma_2\sigma_3]+2T_{k\bar{2},\bar{1}3}[n_1(G_2\sigma_3+G_3\sigma_2)+G_1^*n_2\sigma_3]\}\delta(q+q_1-q_2-q_3)\,dq_1\,dq_2\,dq_3, \qquad (15)$$

$$P_k=hV_k+\int S_{kk'}\sigma_{q'}\,dq',$$

where

$$T_{kk'}=T_{kk',kk'},\quad S_{kk'}=T_{k\bar{k},\bar{k}'k'}.$$

Further,

$$\Phi_q=2\int[|T_{k1,23}|^2n_1n_2n_3+T_{k1,23}T^*_{k\bar{2},\bar{1}3}\times\sigma_1^*(n_2\sigma_3+\sigma_2n_3)]\delta(q+q_1-q_2-q_3)\,dq_1\,dq_2\,dq_3, \qquad (16)$$

$$\Psi_q=2\int[T^2_{k1,23}\sigma_1^*\sigma_2\sigma_3+T_{k1,23}T_{k\bar{2},\bar{1}3}n_1(\sigma_2n_3+\sigma_3n_2)]\delta(q+q_1-q_2-q_3)\,dq_1\,dq_2\,dq_3.$$

In the absence of pumping, when $\Pi_q = 0$, $L_q = 0$, and $\sigma_q = 0$, the system of equations obtained here goes over into the kinetic equation for waves of [21,22]. This means that the system of interacting waves evolves to thermodynamic equilibrium:

$$n_k\equiv\int n_{k\omega}\,d\omega\to n_k^0=T/\tilde{\omega}_k,\quad \tilde{\omega}_k\equiv\omega_k+\mathrm{Re}\,\Sigma_{k,\tilde{\omega}_k}, \qquad (17)$$

i.e., to the Rayleigh-Jeans distribution. At the same time, as is shown in [22],

$$n_{k\omega}^0=\frac{1}{\pi}\frac{\gamma_kn_k^0}{(\omega-\tilde{\omega}_k)^2+\gamma_k^2}; \qquad (18)$$

where $\gamma_k = \gamma_k^0 - \mathrm{Im}\,\Sigma_{k,\tilde{\omega}_k}$ is the decay rate of the waves due to their interaction with one another. In equilibrium,

$$\Phi_{k\omega}\approx\Phi_{k\omega}^0=\gamma_kn_k^0/\pi \qquad (19)$$

and this function is virtually independent of ω.

Under the influence of the pumping, the waves are heated in the neighborhood of the resonance surface:

$$n_p(k\omega)\equiv n_{k\omega}-n_{k\omega}^0>0.$$

if $|\omega-\omega_p/2|$ and $|\tilde{\omega}_k-\omega_p/2|$ are less or of the order of hV. The correlation functions $n_p(k\omega)$ and $\sigma_p(k\omega)\equiv\sigma_{k\omega}$ characterize the parametric waves. Generally speaking, the pumping also changes the amplitude of the "thermal" waves whose frequencies ω and ω_k are not near $\omega_p/2$.[26] However, in a number of cases this does not occur and in this paper we shall for simplicity assume that the thermal waves are in thermodynamic equilibrium.

2. MANY-FREQUENCY PARAMETRIC TURBULENCE OF WAVES

In this section, we shall study the parametric turbulence of waves at small supercriticality. For this, in the expressions for Σ_q, Φ_q, Π_q, and Ψ_q it is necessary to substitute $n_{k\omega} = n_{k\omega} + n_p(k\omega)$, $\sigma_{k\omega} = \sigma_p(k\omega)$ and investigate the resulting expressions. The result obtained in the zeroth approximation in n_p, σ_p is known—it is Eqs. (17)–(19). The equation

$$\tilde{\omega}_k^0=\omega_k+2\int T_{kk'}n_{q'}^0\,dq'$$

describes the frequency of the parametric waves as a function of the temperature of the medium. We shall assume that this dependence is already included in the definition of ω_k, so that

$$\tilde{\omega}_k=\omega_k+2\int T_{kk'}n_p(k'\omega')\,dk'\,d\omega'. \qquad (20)$$

The expressions (15) and (16) for γ_k, Φ_k, and Ψ_k are quadratic in the coefficients of the interaction Hamiltonian $T_{12,34}$. This enables us in the first stage of the investigation to restrict their calculation to the zeroth approximation in the amplitude of the parametric turbulence. The limit of applicability of this approximation in the amplitudes n_p and σ_p and the effects that occur at large amplitudes will be considered below.

Substituting γ_k and Φ_k^0 from (19) and $\Psi_k = 0$ into the Dyson equations (12) and (13), we obtain

$$n_{k\omega}=\frac{\gamma_kn_k^0}{\pi|\Delta_{k\omega}|^2}[(\tilde{\omega}_k+\omega-\omega_p)^2+\gamma_k^2+|P_k|^2],$$

$$\sigma_{k\omega}=\frac{2P_k\gamma_kn_k^0}{\pi|\Delta_{k\omega}|^2}\left(i\gamma_k+\tilde{\omega}_k-\frac{\omega_p}{2}\right), \qquad (21)$$

$$\Delta_{k\omega}=[(\omega-\tilde{\omega}_k+i\gamma_k)(\omega_p-\omega-\tilde{\omega}_k-i\gamma_k)]-|P_k|^2.$$

Note that these equations are integral equations in four-dimensional $(\mathbf{k}, \omega)$ space. However, using the narrowness of the packets $n_p(\mathbf{k},\omega)$ and $\sigma_p(\mathbf{k}, \omega)$ with respect to ω and ω_k, we can reduce them to two-dimensional integral equations for quantities that are integral with respect to ω and ω_k:

$$N_p(\Omega)=k_\Omega^2\int n_p(k\omega)\,d\omega\,d\varkappa,$$

$$\Sigma_p(\Omega)=k_\Omega^2\int\sigma_p(k\omega)\,d\omega\,d\varkappa. \qquad (22)$$

Here, k_Ω is the radius of the resonance surface $2\tilde{\omega}_k = \omega_p$ at the point with angular coordinate Ω, $\varkappa$ is the deviation of k from k_Ω along the normal to the surface. For this it is necessary, ignoring the dependence of γ_k and p_k on $\varkappa$, to integrate (21) with respect to $\varkappa$ and ω:

$$N_p(\Omega)=\frac{\pi k_\Omega^2n_0|P_\Omega|^2}{v_\Omega\nu_\Omega},\quad \Sigma_p(\Omega)=\frac{i\pi k_\Omega^2n_0\gamma_\Omega P_\Omega}{v_\Omega\nu_\Omega}$$

$$\mathbf{v}_\Omega=\left.\frac{\partial\omega_k}{\partial\mathbf{k}}\right|_{\varkappa=0},\quad \nu_\Omega^2=\gamma_\Omega^2-|P_\Omega|^2. \qquad (23)$$

Substituting here $\Sigma_p(\Omega)$ into (15), we obtain a nonlinear integral equation for the selfconsistent pumping P_Ω on the resonance surface:

$$P_\Omega=hV_\Omega+i\pi n_0\int S_{\Omega\Omega'}\frac{k_{\Omega'}^2\gamma_{\Omega'}P_{\Omega'}\,d\Omega'}{v_{\Omega'}(\gamma_{\Omega'}^2-|P_{\Omega'}|^2)^{1/2}}, \qquad (24)$$

which differs from the erroneous self-consistency conditions obtained in [27].

Solving this equation, we can determine from Eq. (23) the integral characteristics $N_p(\Omega)$ and $\Sigma_p(\Omega)$ of the parametric waves on the resonance surface and, using Eqs. (21), investigate the structure of the distributions $n_{k\omega}$

and $\sigma_{k\omega}$ near it. In particular, integrating (21) with respect to ω, we have

$$n_p(\mathbf{k}) = n_k^0 \frac{|P_\Omega|^2}{(\tilde\omega_k - \omega_p/2)^2 + \nu_\Omega^2}, \quad \sigma_p(\mathbf{k}) = n_k^0 \frac{P_\Omega(\tilde\omega_k - \omega_p/2 + i\gamma_\Omega)}{(\tilde\omega_k - \omega_p/2)^2 + \nu_\Omega^2}. \tag{25}$$

Thus, the distribution of $n_p(\mathbf{k})$ and $\sigma_p(\mathbf{k})$ with respect to the modulus of $\mathbf{k}$ has the form of a Lorentz function with maximum on the resonance surface and half-width $\nu_\mathbf{k}$.

Equations (25) can be transformed identically to the form

$$\gamma_k n_k + \mathrm{Im}\,(P_k^* \sigma_k) = \gamma_k n_k^0, \quad [\gamma_k + i(\tilde\omega_k - \omega_p/2)]\sigma_k + iP_k n_k = 0. \tag{26}$$

These equations were written down for the first time earlier in [10] on the basis of intuitive arguments and investigated there in detail. Equations (26) differ from the basic equations of S-theory[9] only by the inhomogeneous term $\gamma_\mathbf{k} n_\mathbf{k}^0$, which describes the influence of the thermal fluctuations on the system of parametric waves.

It follows from Eqs. (26) in particular[10] that in the case of spherical symmetry and for $h - h_1 > \xi^2 h_1$ (i.e., above the formal "threshold" $h_1 V = \gamma$)

$$SN_p = \gamma\left[\left(\frac{h}{h_1}\right)^2 - 1\right]^{1/2}\left(1 + \xi^2 \frac{h_1^4}{2(h^2 - h_1^2)^2}\right), \quad \nu = \xi\gamma h_1 (h^2 - h_1^2)^{-1/2}. \tag{27}$$

Here, ξ is a small parameter of the problem, which characterizes the influence of thermal fluctuations:

$$\xi = (2\pi)^2 k^2 S n_0 v^{-1}, \quad S = (4\pi)^{-1} \int S_{\Omega\Omega'}\, d\Omega'.$$

Thus, the integrated amplitude N_p differs little from the quantity predicted by S-theory.[10] The fundamental difference from S-theory is the occurrence of a finite width of the wave packet $n_\mathbf{k}$ with respect to the modulus k, this being due to the influence of the thermal fluctuations of the medium. However, if $h - h_1 \gg \xi^2 h_1$, the width, as follows from (27), is small: $\nu \ll \gamma$.

In S-theory, the turbulence of parametric waves is described in terms of $n_\mathbf{k}$ and $\sigma_\mathbf{k}$ and the question of the distribution of $n_{\mathbf{k}\omega}$ and $\sigma_{\mathbf{k}\omega}$ with respect to ω simply does not arise. Nevertheless, clarification of the structure of this distribution is of considerable interest and it has a direct bearing on the question of the noise of parametric amplifiers. In addition, it can be directly measured; for it follows from the expression (4) for the Hamiltonian of the interaction of parametric waves with the external field that the power P_ω of the electromagnetic emission from a sample at frequencies ω near the pumping frequency ω_p (with polarization parallel to $\mathbf{H}_0$) is determined by the expression

$$P_\omega = A\left|\int V_k \sigma_{k,\omega/2}\, d\mathbf{k}\right|^2 = A\left|\int V_\Omega \sigma_\Omega\left(\frac{\omega}{2}\right) d\Omega\right|^2, \quad \sigma_\Omega(\omega) = k_\Omega^2 \int \sigma_{k\omega}\, d\varkappa,$$

where A is a constant. Integrating in (21) with respect to κ, we obtain

$$\sigma_\Omega(\omega) = \frac{2^{1/2}\gamma_\Omega \nu_\Omega \Sigma_p(\Omega)}{\{[(x^2 - \nu_\Omega^2)^2 + 4\gamma_\Omega^2 x^2]^{1/2} + (\nu_\Omega^2 - x^2)\,[(x^2 - \nu_\Omega^2)^2 + 4\gamma_\Omega^2 x^2]^{1/2}\}^{1/2}}, \tag{28}$$

where $x = \omega - \omega_p/2$. We can see from this that for $\nu_\Omega \ll \gamma_\Omega$ the half width of the distribution $\sigma_\Omega(\omega)$ with respect to ω is $\nu_\Omega^2/2\gamma_\Omega$ and, as one would expect, the maximum corresponds to $\omega = \omega_p/2$.

Let us consider why the parametric turbulence of waves has been found to be of a many-frequency kind. It can be seen from the Dyson equations (13) that the distribution $n_{\mathbf{k}\omega}$ to within factors nonsingular in ω—the Green's functions—repeats the distribution with respect to ω of $\Phi_{\mathbf{k}\omega}$ and $\Psi_{\mathbf{k}\omega}$, which are the correlation functions of the random force. To calculate them, we used the lowest approximation in $n_p(\mathbf{k}\omega)$ and $\sigma_p(\mathbf{k}\omega)$ and found that $\Psi_{\mathbf{k}\omega} = 0$, while $\Phi_{\mathbf{k}\omega} = \Phi_\mathbf{k}^0$ is virtually independent of ω. It is easy to see that the corrections $\Phi_{\mathbf{k}\omega}^{(1)}$ and $\Phi_{\mathbf{k}\omega}^{(2)}$ to $\Phi_{\mathbf{k}\omega}{}^0$ calculated in the linear and quadratic approximations in n_p and σ_p are also independent of ω. However, the term $\Phi_{\mathbf{k}\omega}^{(3)}$ cubic in $n_p(\mathbf{k}\omega)$ has a sharp peak at $\omega = \omega_p/2$. This can be seen by substituting $n_p(\mathbf{k}\omega) \propto \delta(\omega - \omega_p/2)$ into (16), which gives $\Phi_{\mathbf{k}\omega}^{(3)} \propto \delta(\omega - \omega_p/2)$. This means that the width of the distribution $\Phi_{\mathbf{k}\omega}^{(3)}$ with respect to ω is of the same order as the width of the distribution $n_p(\mathbf{k}\omega)$ with respect to ω. Therefore, the results obtained above are true for amplitudes $N_p \ll N^*$ for which $\Phi_{\mathbf{k}\omega}^{(3)} \ll \Phi_\mathbf{k}^0$. For $h > h^*$, when $N_p > N^*$, the spectrum of the random force $\Phi_{\mathbf{k}\omega}$ is far from the value $\Phi_\mathbf{k}{}^0$ of thermodynamic equilibrium, and is almost completely determined by $\Phi_{\mathbf{k}\omega}^{(3)}$.

We show that in this case the system of parametric waves is unstable against collapse in ω space. The point is that a random contraction of the packet $n_p(\mathbf{k}\omega)$ with respect to ω gives rise to a contraction of the spectrum of the random force $\Phi_{\mathbf{k}\omega}$, and this, in its turn, enhances the contraction of the packet $n_p(\mathbf{k}\omega)$, and as a result the function $\Phi_{\mathbf{k}\omega}$ becomes even narrower in ω space, etc. This process continues until the widths of the distributions $n_p(\mathbf{k}\omega)$ and $\Phi_{\mathbf{k}\omega}$ with respect to ω is zero. This process cannot broaden the packets n_p and Φ with respect to ω. This is prevented by the Green's functions, which have a small width in ω: $\Delta\omega \propto \nu^2/2\gamma$. It is important that this width $\Delta\omega$ does not depend on the structure of the distributions $\sigma_p(\mathbf{k}\omega)$ and $n_p(\mathbf{k}\omega)$ with respect to ω and is determined solely by the integrals of them with respect to ω.

Thus, for some supercriticality $h = h^*$, when $\Phi_{\mathbf{k}\omega_p/2}^{(3)} \approx \Phi_\mathbf{k}{}^0$, single-frequency turbulence arises, i.e., the correlation functions $n_p(\mathbf{k}\omega)$ and $\sigma_p(\mathbf{k}\omega)$ take the form (8). Substituting $n_p(\mathbf{k}\omega)$ from (21) into Eq. (16), we obtain for $\Phi_{\mathbf{k}\omega}$ the estimate

$$\Phi_{\mathbf{k}\omega_p/2}^{(3)} \approx T^2 N_p^3 \gamma / k^3 \nu^2.$$

Equating $\Phi_{\mathbf{k}\omega_p/2}^{(3)}$ and $\Phi_\mathbf{k}{}^0 = \gamma n_\mathbf{k}{}^0/\pi$ and substituting successively $n_{\mathbf{k}\omega}$ from (21) and ν and N_p from (23), we obtain

$$(h^* - h_1)/h_1 \approx \left(k\frac{\partial\omega}{\partial k}\Big/\gamma\right)^{1/4} \xi^{2/3}. \tag{29}$$

Assuming for ferromagnets $\xi \approx 10^{-3} - 10^{-2}$ and $\gamma(k\partial\omega_\mathbf{k}/\partial\mathbf{k})^{-1} \approx 10^{-3} - 10^{-4}$, we obtain $h^* - h_1 \approx (0.1 - 0.01)h_1$, i.e., a "phase transition" occurs when the "formal" threshold $h_1 = \gamma/V$ has been slightly exceeded.

3. THEORY OF SINGLE-FREQUENCY TURBULENCE OF PARAMETRICALLY EXCITED WAVES [21]

1. Basic Equations

We shall show below that at small supercriticalities, when $h - h_1 \gg h - h^*$, the amplitudes of single-frequency turbulence $n_S(\mathbf{k})$ are appreciably greater than those of many-frequency turbulence:

$$n_t(\mathbf{k})=\int n_t(\mathbf{k}\omega)\,d\omega.$$

Therefore, at these supercriticalities it is natural to calculate $n_S(\mathbf{k})$ and $\sigma_S(\mathbf{k})$ in the zeroth approximation in $n_t(\mathbf{k})$, and then calculate $n_t(\mathbf{k})$, assuming that $n_S(\mathbf{k})$ and $\sigma_S(\mathbf{k})$ are given.

In the case of single-frequency turbulence, the Dyson equations (12) and (13) simplify to the form

$$\begin{gathered}n_S(\mathbf{k})=(|G_\mathbf{k}|^2+|L_\mathbf{k}|^2)\Phi_\mathbf{k}+2\operatorname{Re}G_\mathbf{k}L_\mathbf{k}^*\Psi_\mathbf{k},\\ \sigma_S(\mathbf{k})=2L_\mathbf{k}G_\mathbf{k}\Phi_\mathbf{k}+L_\mathbf{k}^2\Psi_\mathbf{k}^*+G_\mathbf{k}^2\Psi_\mathbf{k}.\end{gathered}\tag{30}$$

Here, $G_\mathbf{k}$ and $L_\mathbf{k}$ are the values of the Green's functions $G_\mathbf{q}$ and $L_\mathbf{q}$ for $\omega=\omega_p/2$:

$$\begin{gathered}G_\mathbf{k}=-(i\Gamma_\mathbf{k}-\tilde{\omega}_\mathbf{k}+\omega_p/2)/\Delta_\mathbf{k},\ L_\mathbf{k}=\Pi_\mathbf{k}/\Delta_\mathbf{k},\ \Gamma_\mathbf{k}=\operatorname{Im}\Sigma_{\mathbf{k}\omega_p/2},\\ \Delta_\mathbf{k}=(\tilde{\omega}_\mathbf{k}-\omega_p/2)^2+\nu_\mathbf{k}^2,\ \nu_\mathbf{k}^2=\Gamma_\mathbf{k}^2-|\Pi_\mathbf{k}|^2,\end{gathered}\tag{31}$$

and $\Phi_\mathbf{k}$ and $\Psi_\mathbf{k}$ are determined by the equations

$$\Phi_{\mathbf{k}\omega}=\Phi_\mathbf{k}\delta(\omega-\omega_p/2),\ \Psi_{\mathbf{k}\omega}=\Psi_\mathbf{k}\delta(\omega-\omega_p/2).$$

It follows that the distributions $n_S(\mathbf{k})$ and $\sigma_S(\mathbf{k})$ have the form of the square of a Lorentz function with peak at $2\tilde{\omega}_\mathbf{k}=\omega_p$, in contrast to the corresponding distributions $n_t(\mathbf{k})$ and $\sigma_t(\mathbf{k})$, which have the form of Lorentz functions.

Using the narrowness of the packets, we can, as before, integrate the Dyson equations (30) along the normal to the resonance surface and obtain equations for the integral quantities $N_S(\Omega)$ and $\Sigma_S(\Omega)$ [see (23)]:

$$\begin{gathered}N_S(\Omega)=\frac{\pi k_\Omega^2\Gamma_\Omega}{v_\Omega\nu_\Omega^3}(\Gamma_\Omega\Phi_\Omega+\operatorname{Im}\Pi_\Omega^*\Psi_\Omega),\\ \Gamma_\Omega\Sigma_S(\Omega)+i\Pi_\Omega N_S(\Omega)=0.\end{gathered}\tag{32}$$

Using the notation

$$\tilde{N}(\Omega)=\pi k_\Omega^2\Phi_\Omega/v_\Omega\nu_\Omega,\ \tilde{\Sigma}(\Omega)=\pi k_\Omega^2\Psi_\Omega/v_\Omega\nu_\Omega\tag{33}$$

we can transform these equations identically to the form

$$\begin{gathered}\Gamma_\Omega(N_S(\Omega)-\tilde{N}(\Omega))+\operatorname{Im}[\Pi_\Omega^*(\Sigma_S(\Omega)-\tilde{\Sigma}(\Omega))]=0\\ \Gamma_\Omega\Sigma_S(\Omega)+i\Pi_\Omega N_S(\Omega)=0,\end{gathered}\tag{34}$$

which differs from the stationary equations of S-theory[9]

$$\begin{gathered}\gamma_\Omega N_S(\Omega)+\operatorname{Im}P_\Omega^*\Sigma_S(\Omega)=0,\\ \gamma_\Omega\Sigma_S(\Omega)+iP_\Omega N_S(\Omega)=0\end{gathered}\tag{35}$$

by the renormalization of the damping of the parametric waves and the pumping: $\Gamma_\Omega=\gamma_\Omega+\ldots$, $\Pi_\Omega=P_\Omega+\ldots$, and also by the presence of the additional "noise" terms $\tilde{N}$ and $\tilde{\Sigma}$.

Expressions for these quantities can be obtained by substituting into (15) the correlation functions $n_{\mathbf{k}\omega}$ and $\sigma_{\mathbf{k}\omega}$ in the form

$$n_{\mathbf{k}\omega}=n_{\mathbf{k}\omega}^0+n_S\delta(\omega-\omega_p/2),\quad \sigma_{\mathbf{k}\omega}=\sigma_S(\mathbf{k})\delta(\omega-\omega_p/2).$$

As before, it is easy to show that the basic quantities are the terms of the zeroth and the third order in n_S and σ_S. The terms of zeroth order are important because the integral amplitude of the thermal waves is much greater than that of the parametric waves, while the terms of the third order are important because they have a "resonance" nature: The number of δ functions in the integrals is greater than the number of integrations. As a result

$$\begin{gathered}\Gamma_\Omega=\gamma_\Omega+\frac{2\pi}{k_\Omega v_\Omega}\int\left[|T_{\Omega1,23}|^2\frac{N_1\Gamma_2N_3}{\nu_2}+\ldots\right]\delta(n+n_1-n_2-n_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3,\\ \Pi_\Omega=P_\Omega+\frac{2\pi}{k_\Omega v_\Omega}\int\left[T^2_{\Omega1,23}\frac{\Sigma_1^*\Pi_2\Sigma_3}{\nu_2}+\ldots\right]\\ \times\delta(n+n_1-n_2-n_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3,\\ \tilde{N}_\Omega=\frac{2\pi}{k_\Omega v_\Omega}\int[|T_{\Omega1,23}|^2N_1N_2N_3+\ldots]\delta(n+n_1-n_2-n_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3,\\ \tilde{\Sigma}_\Omega=\frac{2\pi}{k_\Omega v_\Omega}\int[T^2_{\Omega1,23}\Sigma_1^*\Sigma_2\Sigma_3+\ldots]\delta(n+n_1-n_2-n_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3.\end{gathered}\tag{36}$$

Here we have omitted terms of the same order; their structure can be readily understood by comparing (28) and (15). We have $n_j\equiv\mathbf{k}_j/|\mathbf{k}_\Omega|$ and the indices 1, 2, 3 replace $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, the angular coordinates of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ on the resonance surface, $N_1=N_S(\Omega_1)$, etc. Thus, we obtain the closed system of integral equations (34) and (36) for $N_S(\Omega)$, $\Sigma_S(\Omega)$, $\Gamma(\Omega)$, and $\Pi(\Omega)$.

Let us turn to an investigation of its solutions.

2. S-Theory

We show first that Eqs. (34) and (36), which describe single-frequency turbulence of parametric waves, admit, like Eq. (26) for many-frequency turbulence, a passage to the limit of the equations of S-theory. Indeed, omitting in (36) the terms that contain N_S and Σ_S to powers higher than the first, we obtain $\Gamma_\Omega=\gamma_\Omega$, $\Pi_\Omega=P_\Omega$, $\Phi_\Omega=\Psi_\Omega=\tilde{N}(\Omega)=\tilde{\Sigma}(\Omega)=0$. The following results are then obtained:

a) the damping of the parametric waves is not renormalized and can be calculated from the ordinary kinetic equation for the waves in the absence of pumping;[2]

b) for self-consistent pumping; the ordinary expression (19) of S-theory is obtained;

c) $N_S(\Omega)$ and $\Sigma_S(\Omega)$ are nonzero only for the directions Ω for which $\nu_\Omega=0$ [see Eq. (32); this means that the distribution of the surface waves in $\mathbf{k}$ space is singular, $n_S(\mathbf{k})\neq 0$, only on a resonance surface satisfying the condition of external stability in S-theory: $2\omega_\mathbf{k}=\omega_p$, and then only at its points where $\gamma_\Omega=|P|$];

d) $N_S(\Omega)$ and $\Sigma_S(\Omega)$ at these points are determined by Eqs. (34), which go over in this approximation into Eq. (35) of S-theory.

3. Fine Structure of the Distribution $n_S(\mathbf{k})$ and Limit of Applicability of S-Theory

At first glance it might appear that our approximation "linear in $T_{12,34}$" in Eqs. (36), which leads to S-theory and, in particular, to singular Green's functions in which $\nu=0$, is incorrect. Indeed, the contribution of the following $\sim T^2$ diagrams calculated by means of it differs, as can be seen from (36), by the factor $\Gamma TN/\nu kv$, which is not small but in fact diverges. We shall show that nevertheless S-theory does correctly describe the integral quantities N_S and Σ_S and the structure of the distributions $N_S(\Omega)$ and $\Sigma_S(\Omega)$ if the supercriticality is not too large: $h<h_S$, where

$$h_S=h_1(kv/\gamma)^{1/2}.\tag{37}$$

Indeed, from (36) we obtain the estimate

$$\tilde{N}/N_S\approx|\tilde{\Sigma}|/|\Sigma_S|\approx(TN_S)^2/\nu kv.\tag{38}$$

Hence and from (32) and (33) we obtain an estimate for

ν which, in accordance with (30), characterizes the width of the distributions $n_S(\mathbf{k})$ and $\sigma_S(\mathbf{k})$ with respect to k: $v\Delta k = \nu$, where

$$\left(\frac{\nu}{\gamma}\right)^3 \approx \frac{(TN_s)^2}{\gamma kv} \approx \frac{\gamma}{kv}\frac{h^2-h_1^2}{h_1^2}. \qquad (39)$$

At the same time, we have assumed that $\Gamma \approx \gamma$ and $|\Sigma_S| \approx N_S$. Let us show that these relations are satisfied for $h \ll h_S$ when $\nu \ll \gamma$. Indeed, it follows from (36) that

$$1-\frac{\gamma}{\Gamma} \approx \frac{(TN_s)^2}{\nu kv} \approx \left(\frac{\nu}{\gamma}\right)^2 \ll 1. \qquad (40)$$

Further, it can be seen from (32) that $\Gamma|\Sigma_S| = |\Pi|N_S$. Bearing in mind that $\nu^2 = \Gamma^2 - |\Pi|^2$, we obtain

$$1-\frac{|\Sigma_s|}{N_s} \approx \frac{\nu^2}{2\gamma^2} \ll 1. \qquad (41)$$

Further, from (38) and (39),

$$\frac{N}{\tilde{N}} \approx \left|\frac{\tilde{\Sigma}}{\Sigma}\right| \approx \left(\frac{\nu}{\gamma}\right)^2 \approx \left(\frac{\gamma}{kv}\frac{h^2-h_1^2}{h_1^2}\right)^{2/3}. \qquad (42)$$

From (36) and (39),

$$\left|\frac{\Pi-P}{\Pi}\right| \approx \left(\frac{\nu}{\gamma}\right)^2 \approx \left(\frac{\gamma}{kv}\frac{h^2-h_1^2}{h_1^2}\right)^{2/3}. \qquad (43)$$

The relations (40), (42), and (43) mean that the relative deviation of the coefficients of Eqs. (34) from Eqs. (35) of S-theory is small in the parameter $(\nu/\gamma)^2$ and, therefore, the results of S-theory differ from the exact results by a quantity which is small in the same parameter. In particular, it follows from (39) and (41) that for $h < h_S$ the phase correlations in pairs are preserved almost completely.

In [21] it is shown that the estimates (38)–(43) are valid right up to $h = h_S$, when $\nu \approx 0.5\gamma$.

4. Role of Higher Diagrams.

Hitherto we have restricted ourselves to considering irreducible diagrams proportional to T and T^2 in the series (14) for Σ_q, Π_q, Φ_q, and Ψ_q. Estimating the diagrams $\propto T^3$, T^4, etc, in the interaction, we can readily show that they can be arranged in a series in accordance with the parameter $\lambda = \Gamma TN/\nu\Delta kv$. Here, Δk is the maximal size of the distribution of the waves in k space. For $h \lesssim h_S$, substituting here (39) and (40), we obtain

$$\frac{N_t}{N_s} \approx \xi\left(\frac{\gamma^4 kv}{S^5 N_s^5}\right)^{1/3} \approx \xi\left(\frac{kv}{\gamma}\right)^{1/3}\left(\frac{h_1^2}{h^2-h_1^2}\right)^{5/6}. \qquad (44)$$

At large supercriticalities ($h \gg h_S$) it follows from (32), (33), (38), and (40) that

$$\Gamma \approx \nu \approx (TN_s)^2/\Delta kv, \qquad (45)$$

i.e., $\lambda = TN_S/\Delta kv$. Therefore, as long as the nonlinear shift of the frequency TN_S is much less than the dispersion Δkv of the waves, all the necessary diagrams have been retained in Eqs. (34)–(36). For a wave packet concentrated near a point in k space, the diagram method developed here does not apply.

5. Fluctuations of Single-Frequency Turbulence

To conclude this section, let us consider many-frequency turbulence of parametric waves far from the "phase transition" for $h - h_1 \gg h^* - h_1$. We shall show below that then $N_t \ll N^*$, where N^* is an amplitude of N_S for which $\Phi^{(3)}_{\mathbf{k}\omega_p/2} \approx \Phi^{(0)}_{\mathbf{k}}$. In the case $N_t \ll N_S$ one can calculate $n_t(\mathbf{k}\omega)$ as the nonlinear reaction of the system of parametric waves to a many-frequency random force with Langevin correlation function $\Phi^0_{\mathbf{k}} = \gamma n^0_{\mathbf{k}}/\pi$. Then for $n_t(\mathbf{k}\omega)$ and $\sigma_t(\mathbf{k}\omega)$ the results obtained for small supercriticality are valid: Eqs. (21), (23), (25) and (28), in which ν_Ω is now determined by the single-frequency turbulence and is given in order of magnitude by Eq. (39). In particular, for the ratio N_t/N_S we obtain the estimate

$$\lambda = \frac{(\gamma TN)^{1/3}}{(\Delta kv)^{2/3}} \lessdot \left(\frac{\gamma}{\Delta kv}\right)^{1/2} \ll 1. \qquad (46)$$

Hence and from (29) it can be seen that $N_t \approx N_S \approx N^*$ for $h = h^*$, and our treatment is invalid. The investigation of the nature of the "phase transition" at $h = h^*$ is a complicated problem. The possibility cannot be excluded that there is a "phase transition of the first kind which is nearly one of the second kind", i.e., at $h = h^*$ the single-frequency turbulence arises abruptly. The discontinuity is small and ΔN_S cannot exceed N^*, where

$$SN^* \approx \xi^{3/5}(kv)^{1/5}\gamma^{4/5}.$$

For $h - h_1 \gg h - h_1^*$, the amplitude of the single-frequency part N_S of the parametric-wave turbulence increases in accordance with S-theory, and the many-frequency part N_t decreases in accordance with Eq. (46).

When the supercriticality is not small, i.e., $h - h_1 \gtrsim h_1$, the many-frequency part of the parametrically excited turbulence is a negligible part of the single-frequency part and can be regarded as fluctuations on the background of the latter.

In addition, the amplitudes and phases of the single-frequency part of the turbulence fluctuate with frequencies of order $(h^2V^2 - \gamma^2)^{1/2}$. These fluctuations are the thermally excited collective degrees of freedom of the system of parametric waves considered in [17–19,24]. These collective oscillations may be unstable, and they are then excited spontaneously. The resulting auto-oscillations have a noise nature in a number of cases. They can be regarded as giant fluctuations that break up the single-frequency turbulence and lead to strong many-frequency turbulence with characteristic frequency $\sim(h^2V^2 - \gamma^2)^{1/2}$ of the motion, which agrees with the width $\Delta\omega_{\mathbf{k}}$ of the excited region in $\mathbf{k}$ space.

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1)A preliminary communication with some results of this paper was published in [21].
2)This result disagrees with the opinion of Tsukernik and Yankelevich [27].

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