Three-dimensional quasi-linear relaxation

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The problem of the dynamics of electron-beam relaxation in a plasma is solved. An analytic solution to the three-dimensional quasi-linear equations is found. A numerical simulation of the nonlinear phase of the electron beam-plasma interaction process in the two-dimensional case is carried out. The results of the numerical and analytic solutions coincide. The mechanism of beam-electron capture by the field of the mirror machine is explained.

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1. FORMULATION OF THE PROBLEM

The interaction of an electron beam with a plasma has been investigated by many authors both experimentally[1-4] and theoretically[5-8]. However, thus far only a one-dimensional theory of electron-beam relaxation has been developed, it having been impossible in the three-dimensional case to determine even the steady state[9]. We shall solve the three-dimensional temporal problem of the beam-plasma interaction.

Let on a plasma at the moment of time $t = 0$ be incident an electron beam of velocity u_0 directed along the z axis. The beam density n_b is considerably lower than the plasma density n_0 ($n_b/n_0 \ll 1$), while the beam velocity is considerably higher than the thermal plasma velocity. The magnetic field is sufficiently weak (in the experiments[1], $\omega_{pe} = (3-10)\omega_{He}$), so that it does not affect the permittivity ($\omega_{He} < \omega_{pe}(n_b/n_0)^{1/3}$). The beam excites in the plasma Langmuir oscillations, which grow from a level W_0 close to the thermal level to some level W_∞ considerably exceeding the initial level. An important parameter of the problem is $\ln(W_\infty/W_0)$, which, for the majority of experimental conditions, differs little from the Coulomb logarithm $\Lambda = \ln N_D$, where N_D is the number of particles in the Debye sphere. In a typical laboratory plasma ($n \sim 10^{13}$ cm^{-3}, $T_e \sim 100$ eV), the quantity $\ln(W_\infty/W_0)$ is estimated by the following relation:

$$\ln(W_\infty/W_0) \propto \Lambda \sim 15-20 \gg 1. \tag{1.1}$$

Under the above-described conditions the relaxation of the distribution function is described by the set of quasi-linear equations[10]:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_\alpha} D_{\alpha\beta} \frac{\partial f}{\partial v_\beta}, \tag{1.2}$$

$$\partial W/\partial t = 2\gamma W, \tag{1.3}$$

$$D_{\alpha\beta} = \frac{8\pi^2 e^2}{m^2} \int \frac{k_\alpha k_\beta}{k^2} \frac{E_k^2}{8\pi} \delta(\omega_{pe} - \mathbf{kv})\, d^3k, \tag{1.4}$$

$$\gamma = \frac{\pi}{2} \frac{n_b}{n_0} \frac{\omega_{pe}^3}{k^2} \int \mathbf{k} \frac{\partial f}{\partial \mathbf{v}} \delta(\omega_{pe} - \mathbf{kv})\, d^3k. \tag{1.5}$$

Here f is the beam-electron distribution function, $D_{\alpha\beta}$ is the diffusion tensor,

$$W = \omega \frac{\partial \varepsilon}{\partial \omega} \frac{E_k^2}{8\pi}$$

is the spectral density of the energy of the electrostatic noise, γ is the increment of the oscillations, m and e are the electron mass and charge, and n_b and n_0 are respectively the beam and plasma densities. Although the quasi-linear equations are valid only for sufficiently smeared distribution functions with a thermal spread Δv satisfying the condition for the buildup of the kinetic oscillations,

$$\Delta v > \gamma/k, \tag{1.6}$$

in fact, any beam after a time $t \sim \Lambda/\gamma$ acquires a sufficient spread over velocity to be describable by the Eqs. (1.2)–(1.5).

For a purely one-dimensional noise spectrum ($k = \{0, 0, k_z\}$) we can carry out the integration in the formulas (1.4) and (1.5) in the explicit form. As a result, the set of integro-differential equations (1.2)–(1.5) reduces to the set of one-dimensional quasi-linear equations

$$\frac{\partial f_1}{\partial t} = \frac{4\pi^2 e^2}{m^2} \frac{\partial}{\partial v_z} \left(\frac{W_1}{v_z} \frac{\partial f_1}{\partial v_z} \right), \tag{1.7}$$

$$\frac{\partial W_1}{\partial t} = \frac{\pi \omega_{pe} n_b}{n_0} v_z^2 W_1 \frac{\partial f_1}{\partial v_z}. \tag{1.8}$$

In these equations we have made a change of variables, using the relation $k_z = \omega_{pe}/v_z$, so that W_1 is now a function of v_z.

The problem, described by the system (1.7) and (1.8), of the dynamics of one-dimensional quasi-linear relaxation has been solved by Rudakov and one of the present authors[7]. But certain characteristic parameters of the process can be found without solving the quasi-linear equations. In particular, we can immediately find the steady state of the distribution function, $f_{1\infty}(v_z) = \text{const}$, and the stationary noise level $W_{1\infty}(k_z) = mn_b\omega_{pe}^3/k_z^4 u_0$[5]. Attempts have been made in analogous fashion to determine the steady state in the case of three-dimensional relaxation[11,12], but the steady states thus obtained are too artificial, and are not realized in experiments[9]. It is quite possible that the steady state does not at all exist in the case of three-dimensional relaxation. An answer to this question can be found only by solving the problem of the dynamics of the relaxation of the beam-electron distribution function. Below this process will be considered qualitatively.

In Sec. 3 we find the dependence of the distribution function and the noise level on v_z. Essentially three-dimensional effects—the variation of the spread over the transverse velocities and the spectral width of the noise with respect to $k_\perp$—are considered in Sec. 4. Section 5 is devoted to the numerical experiment.

2. QUALITATIVE ANALYSIS OF RELAXATION

Let us, in accordance with the foregoing, follow the process of relaxation in the dynamics step by step. Let us consider the initial phase of the process, when the noise level is still low and the distribution function can be assumed to be stationary. In this case the oscillations in the plasma are described by the linear equation (1.3) with the time-independent increment (1.5) determined by the specific form of the initial distribution function.

As a model beam-electron distribution function at the initial stage, let us use the Maxwellian function

$$f=\pi^{-3/2}v_T^{-3}\exp\{-(v_z-u_0)^2/v_T^2-v_\perp^2/v_T^2\}, \tag{2.1}$$

where $v_T = (2T/m)^{1/2}$. The increment of the oscillations excited by the beam has the form

$$\gamma=\left(\frac{\pi}{2e}\right)^{1/2}\frac{n_b}{n_0}\frac{u_0^2}{v_{Tz}^2}\omega_{pe}\cos^2\theta, \tag{2.2}$$

where θ is the angle between the wave vector and the z axis. Notice that the increment decreases with increasing $k_\perp$. This means that the skew waves build up more slowly than the longitudinal waves, and will, up to the moment when the process reaches the nonlinear regime, at which time the growth of the oscillations ceases, have a small amplitude.

To determine the dependence of the noise level on $k_\perp$, let us use the solution to Eq. (1.3)

$$W=W_0e^{2\gamma t} \tag{2.3}$$

with an increment γ determined by the formula (2.2). Substituting into (2.3) the time τ taken to reach the nonlinear regime and estimated with the maximum increment by

$$\tau\sim\frac{1}{2\gamma_{max}}\ln\frac{W_\infty}{W_0},$$

we obtain the $k_\perp$ spectrum of the noise:

$$W(k_\perp)=W(0)\exp\{-k_\perp^2/\Delta k^2\}, \tag{2.4}$$

where $\Delta k = k_z/\Lambda^{1/2}$. It can be seen from this that a beam with an initial Maxwellian distribution function excites in a plasma oscillations in a narrow cone along the direction of propagation of the beam. The angle of taper of the cone is determined by the relation

$$\Delta\theta\sim\Lambda^{-1/2}\ll 1. \tag{2.5}$$

Thus, at the initial stage the spectrum of the noise excited by the beam is nearly one-dimensional. For a one-dimensional spectrum the quasi-linear equations get significantly simplified and go over into the Eqs. (1.7) and (1.8). One may expect that the solutions to the three-dimensional equations with a sufficiently narrow noise spectrum will not significantly differ from the solutions to the one-dimensional equations.

The principal result of the one-dimensional theory of the quasilinear relaxation of an electron beam consists in the fact that there is formed in the distribution function a narrow front that slowly moves toward the region of lower velocities. There occurs at the front a sharp increase in the distribution function from the initial value to the value at the plateau. Behind the front the distribution function of the beam electrons is virtually velocity independent (see Fig. 1). Such a form of the solution can be qualitatively understood if we take into account the fact that in the transition region, where the derivative $\partial f/\partial v$ is large, the electrostatic-noise level, which is very low before the arrival of the front, rapidly increases.

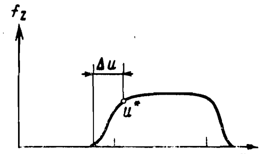

FIG. 1. The distribution function of an electron beam in one-dimensional relaxation.

Behind the front the noise level does not change, since the increment of the oscillations in this region is equal to zero. The high noise level behind the front leads to intense diffusion and the formation of a plateau. The beam particles strive to move into the lower-velocity region, but the too low noise level in the region ahead of the front retards the motion of the front of the distribution function and causes a narrow transition region to be formed.

Thus, in the initial phase we can use the results of the one-dimensional theory. According to the foregoing, a front and a plateau are formed in the one-dimensional distribution function f_z. Let us represent the three-dimensional distribution function in the form

$$f=f_z\frac{1}{\pi v_{T\perp}^2}\exp\left\{-\frac{v_\perp^2}{v_{T\perp}^2}\right\}, \tag{2.6}$$

where f_z is the function shown in Fig. 1. Let us investigate the stability of this distribution (for greater details, see the Appendix).

In the region of the plateau, the increment of the oscillations turns out, as in the one-dimensional case, to be equal to zero. Let us replace f_z in the frontal zone by the model Maxwellian distribution function with $v_{Tz} = \Delta u$ (Δu is the width of the front) and consider the increment of the oscillations excited by a beam with different spreads over the longitudinal and transverse velocities. The angular dependence of the maximum increment in this case is somewhat different from (2.2), and has the form

$$\gamma=\left(\frac{\pi}{2e}\right)^{1/2}\frac{n_b}{n_0}\frac{u_0^2\omega_{pe}}{v_{Tz}(u_0-u)}\left(1+\mathrm{tg}^2\theta\frac{v_{T\perp}^2}{v_{Tz}^2}\right)^{-1}. \tag{2.7}$$

The spectrum of the oscillations excited by such a beam is described, as in the case of equal spreads, by the formula (2.4), but with a different width with respect to $k_\perp$:

$$\Delta k=k_zv_{Tz}/\Lambda^{1/2}v_{T\perp}. \tag{2.8}$$

This leads, as before, to the appearance of noise in a narrow angle range along the direction of propagation of the beam. Thus, it turns out that it is not only in the initial phase that the noise grows only in a narrow cone with an angle of taper

$$\Delta\theta\sim\Delta u/\Lambda^{1/2}v_{T\perp}\ll 1. \tag{2.9}$$

but also during the entire subsequent relaxation process. The narrowness of the noise spectrum at all stages of the electron beam-plasma interaction allows us to fully apply the results of the one-dimensional theory to the three-dimensional relaxation of the beam.

Let us note at once that the estimate for the width of the noise spectrum with respect to the angle $\Delta\theta$, determined by the relation (2.9), is not quite exact. The point is that there exist in any distribution function regions where the angle dependence of the increment is not only weaker than in the formula (2.2), but even inverted, i.e., in these regions the skew waves build up more rapidly than the longitudinal waves. All the sections of the moving front excite waves by turns. As a result, the oscillations develop in a wider cone than is determined by the relation (2.9). However, the main conclusion about the narrowness of the noise spectrum ($\Delta\theta \ll 1$) remains valid.

Let us consider the relaxation at a later stage. On the basis of the analysis of the oscillation increment carried out in the Appendix for functions of the type (2.6), we can conclude that the skew oscillations with $k_z \lesssim \omega_{pe}/u_0$ that appeared at an earlier stage of the relaxation will die down by the time $v_{T\perp}$ increases significantly. In the process, they will give away their energy to the beam particles, somewhat extending the tail of the distribution function. The magnitude of the extension depends essentially on the energy stored in the skew oscillations with $k_z \sim \omega_{pe}/u_0$, i.e., in the initial phase of the relaxation. The skew oscillations are intensely pumped by monoenergetic beams. According to the formula (2.2), beams with large thermal spreads excite at the initial stage of the relaxation oscillations in a narrow cone along the direction of their propagation. Therefore, the elongation of the tail will be appreciable only for monoenergetic beams. In an analytical consideration, it is difficult to take the elongation of the tail of the beam-electron distribution into account, but the numerical computations carried out in Sec. 5 allow this to be done. As can be seen from Figs. 4 and 8 (see below), the qualitative arguments about the substantial elongation of the tail of a monoenergetic beam are correct. In this case the increase in the velocity is 10–15% of u_0.

3. ANALYTIC SOLUTION OF THE THREE-DIMENSIONAL QUASI-LINEAR EQUATIONS

Let us show that the three-dimensional quasi-linear equations (1.2)–(1.5) with a narrow $k_\perp$ spectrum of the noise reduce to the one-dimensional equations (1.7) and (1.8) for the functions $f_z(v_z)$ and $W_z(k_z)$. For this purpose, let us integrate Eq. (1.2) over v_x and v_y and Eq. (1.3) over k_x and k_y. As a result, the quasi-linear equations (1.2)–(1.5) will assume the form

$$\frac{\partial f_z}{\partial t}=\frac{4\pi^2e^2}{m^2}\frac{\partial}{\partial v_z}\int_{-\infty}^{+\infty}k_zI(k_z,v_z)\,dk_z, \qquad (3.1)$$

$$\frac{\partial W_z}{\partial t}=\pi\frac{n_b}{n_0}\omega_{pe}^3\int_{-\infty}^{+\infty}I(k_z,v_z)\,dv_z, \qquad (3.2)$$

$$I(k_z,v_z)=\int\frac{W}{k^2}\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\delta(\omega_{pe}-\mathbf{kv})\,dv_x\,dv_y\,dk_x\,dk_y. \qquad (3.3)$$

Here we have introduced the functions f_z and W_z determined by the relations

$$f_z=\int f\,dv_x\,dv_y, \qquad (3.4)$$

$$W_z=\int W\,dk_x\,dk_y. \qquad (3.5)$$

Let us substitute into Eq. (3.3) the $k_\perp$ spectrum of the noise of the form (2.4) and, furthermore, using the fact that the spectrum is narrow, let us replace k^2 by k_z^2. Carrying out the indicated transformations, and performing the integration in Eqs. (3.1)–(3.3), we obtain the equations for the functions f_z and W_z:

$$\frac{\partial f_z}{\partial t}=\frac{4\pi^2e^2}{m^2}\left[\frac{\partial}{\partial v_z}\left(\frac{W_z}{v_z}\frac{\partial f_z}{\partial v_z}\right)\right], \qquad (3.6)$$

$$\frac{\partial W_z}{\partial t}=\pi\frac{n_b}{n_0}\omega_{pe}v_z^2W_z\frac{\partial f_z}{\partial v_z}. \qquad (3.7)$$

In deriving these equations, we used the expansions of the distribution function f_z at the point $v_z = \omega_{pe}/k_z$ and of the one-dimensional spectrum of the noise, W_z, at the point $k_z = \omega_{pe}/v_z$, and took only the lowest-order terms into account. The smallness parameters with respect to which the expansion is carried out are $(\Delta k/k_z)^2$ and $(\Delta k v_{T\perp}/2k_z\Delta u)^2$, where Δu is the characteristic variation width of the distribution function (the width of the front). In Eqs. (3.6) and (3.7), W_z has been transformed into a function of v_z and, thus, they completely coincide with the one-dimensional quasi-linear equations (1.7) and (1.8).

Let us recall the solution, obtained in[7], to the one-dimensional equations. Let us find the solution in two regions: in the region of the front, where the energy of the noise and the distribution function rise sharply, and at the plateau, where the value of f_z almost does not change. In doing this, we shall assume that the front moves slowly with velocity $u(t)$. We shall obtain the complete solution by matching the solutions found at the point u^* (see Fig. 1).

Let us first consider the region of the front. We shall seek the solution to the Eqs. (3.6) and (3.7) in the form

$$f_z(v_z,t)=f_z(\eta), \qquad (3.8)$$

$$W_z(v_z,t)=W_z(\eta), \qquad (3.9)$$

where $\eta = v_z - u(t)$. Then the derivatives $\partial/\partial t$ and $\partial/\partial v_z$ will respectively go into $-\dot{u}d/d\eta$ and $d/d\eta$, and the equations themselves will assume the form

$$\frac{P}{c_1}\frac{df_z}{d\eta}=\frac{d}{d\eta}\left(G\frac{df_z}{d\eta}\right), \qquad (3.10)$$

$$\frac{P}{c_2}\frac{dG}{d\eta}=G\frac{df}{d\eta}. \qquad (3.11)$$

Here we have introduced the notation:

$$P=-\frac{\dot{u}}{u^2},\quad G=\frac{W_z}{v_z^2},\quad c_1=\frac{\pi\omega_{pe}}{mn_0},\quad c_2=\pi\omega_{pe}\frac{n_b}{n_0}$$

and used the assumption that the front is narrow, i.e., that $\eta \ll u_0 - u$, which justifies the replacement of $P(t)$ by a constant.

Let us integrate both equations over η, and substitute f_z from Eq. (3.11) into (3.10). After some transformations, we obtain

$$\frac{dG}{d\eta}=\frac{P}{c_1}\ln\left(\frac{G}{G_0}\right), \qquad (3.12)$$

$$f_z=\frac{P}{c_2}\ln\left(\frac{G}{G_0}\right). \qquad (3.13)$$

The solution to Eq. (3.12) has the form

$$G_0(\mathrm{li}\,(G/G_0)-1.9)=P\eta/c_1, \qquad (3.14)$$

where li is the integral logarithm. The beginning of the front is reckoned from the moment when the noise increases by a factor of e.

Thus, until the front becomes too steep and the local relation between $\partial f_z/\partial v_z$ and γ gets broken, the shape of the front in the function f_z will be described by the Eqs. (3.13) and (3.14). When the slope of the front exceeds some value, the local connection between $\partial f_z/\partial v_z$ and the increment γ gets broken. Let us discuss this case in greater detail.

For $\Delta u \lesssim \gamma/k_z$, there get excited at the front of the function (2.6), resulting from the electron-beam relaxation process, hydrodynamic oscillations of increment

$$\gamma=\omega_{pe}\left(\frac{n_b}{2n_0}\right)^{1/3}\left(\frac{u}{u_0-u}\right)^{1/3}\cos\theta. \tag{3.15}$$

The noise will then develop not only in the region of the front, but over a distance γ/k_z from it. The growth of the noise will lead to the appearance of a substantial diffusion coefficient ahead of the front and, consequently, to an increase of the width of the front to $\Delta u \approx \gamma/k$.

Thus, because of the appearance of a nonlocal connection between γ and $\partial f_z/\partial v_z$, the minimum width of the front is bounded by the quantity

$$\Delta u\sim u\left(\frac{n_b}{n_0}\right)^{1/3}\left(\frac{u}{u_0-u}\right)^{1/3}. \tag{3.16}$$

This limitation naturally distorts the shape of the front but does not affect its velocity[7]. Qualitatively, this is not difficult to understand from the following simple arguments. The time during which the front traverses a distance of the order of its width (Δu) is equal to the time of growth of the noise from the initial to the final level. Therefore, the velocity $\dot{u}$ of the front can be estimated as follows:

$$\dot{u}\sim-\Delta u/\tau\sim\Delta u\gamma/\Lambda. \tag{3.17}$$

But, as follows from the inequality (3.16), the front can never become too narrow. Its slope is maintained at the buildup boundary of the hydrodynamic and kinetic oscillations ($\Delta u \sim \gamma/k_z$). Therefore, we can always use the kinetic increment (2.7) at the front. Substituting it into the relation (3.17), we obtain an equation determining the velocity of the front:

$$\dot{u}\sim-\frac{n_b}{n_0}\frac{u^2}{u_0-u}\frac{\omega_{pe}}{\Lambda}. \tag{3.18}$$

The width Δu of the front does not enter into this equation; therefore, the velocity $\dot{u}$ turns out to be the same for a fairly wide front ($\Delta u > \gamma/k_z$) as for a front whose width is limited by the transition to the hydrodynamic oscillation-pumping regime (3.17).

Let us now consider the Eqs. (3.6) and (3.7) in the region behind the front. We shall seek the solution in the form of a sum of the value on the plateau of $c(t)$ and a small correction δf_z:

$$f_z=c(t)+\delta f_z. \tag{3.19}$$

The value of the distribution function on the plateau is found from the normalization condition

$$\int_{-\infty}^{+\infty} f_z\,dv_z=1. \tag{3.20}$$

Using the fact that the front is narrow, let us evaluate the integration in this expression over only the region of the plateau. As a result, we obtain the value of the distribution function behind the front:

$$c(t)=1/(u_0-u). \tag{3.21}$$

Let us find the correction δf_z by substituting the solution in the form (3.19) into Eq. (3.6):

$$\delta f_z=-\int_{u^*}^{u_0}\frac{c(t)(u_0-v_z)}{c_1G(v_z)v_z^2}dv_z. \tag{3.22}$$

Let us estimate the value of the integral. Let us substitute into it $c(t)$ in the form (3.21), and let us replace the product $G(v_z)v_z^2$ by its minimum value, which is attainable at the matching point u^*:

$$\left|\int_{u^*}^{u_0}\frac{c(t)(u_0-v_z)}{c_1G(v_z)v_z^2}dv_z\right|\ll c(t)\frac{\dot{u}(u_0-u)}{c_1G(u^*)u^{*2}}. \tag{3.23}$$

Substituting into this expression the value of $G(u^*)$ determined from Eq. (3.14), which describes the region of the front, $G(u^*)\approx P\Lambda\eta^*/c_1$, we obtain

$$\left|\int_{u^*}^{u_0}\frac{c(t)(u_0-v_z)}{c_1G(v_z)v_z^2}dv_z\right|\ll c(t)\frac{u_0-u}{\Lambda\eta^*}. \tag{3.24}$$

It can be seen from the estimates above that at distances from the front greater than $(u_0 - u)/\Lambda$, the correction δf_z becomes considerably smaller than the value of the function on the plateau. Thus, there exists a point u^*, located at a distance of $(u_0 - u)/\Lambda$ from the beginning of the front, at which we can use the solution (3.19), which is valid on the plateau, as well as the solution (3.13)–(3.14) describing the front, the use of the latter solution being admissible on account of the smallness of η ($\eta \le u - u^* = (u_0 - u)/\Lambda \ll u_0 - u$). The width of the front is determined by the equality

$$\Delta u=(u_0-u)/\Lambda. \tag{3.25}$$

Let us match the solutions at the point u^*. Let us, to the left of u^*, use the formula (3.13) and, to the right, take the value of the distribution function on the plateau. Equating these quantities, we find the equation of motion of the front

$$\frac{\dot{u}}{u^2}\frac{1}{c^2}\ln\left(\frac{G(u^*)}{G_0}\right)=\frac{1}{u_0-u}. \tag{3.26}$$

We can replace with logarithmic accuracy $\ln(G(u^*)/G_0)$ by Λ. As a result, we obtain

$$-\dot{u}\frac{u_0-u}{u^2}=\frac{\pi\omega_{pe}}{\Lambda}\frac{n_b}{n_0}. \tag{3.27}$$

This equation coincides up to a factor of π with the Eq. (3.18), which was derived from qualitative arguments. The solution to Eq. (3.27) has the form

$$\frac{u_0}{u}+\ln\frac{u}{u_0}=1+\frac{\pi}{\Lambda}\frac{n_b}{n_0}\omega_{pe}t. \tag{3.28}$$

Thus, the exact solution to Eqs. (3.1)–(3.3) has been found, and the functions f_z and W_z have been shown to behave like the distribution function and the spectrum of the noise in one-dimensional relaxation.

4. THE TRANSVERSE CHARACTERISTICS OF THE RELAXATION

In solving the three-dimensional quasi-linear equations, we assumed that: (1) the $k_\perp$ spectrum of the noise was narrow ($\Delta k \ll k_z$) and (2) the condition (3.8) for the admissibility of the f_z and W_z expansions was valid. Let us show that both conditions remain valid during the entire relaxation process.

For this purpose, let us use Eqs. (1.3) and (1.5) with a distribution function of the form (2.6) and the dependence $f_z(v_z)$ as determined by the above-given solution. Carrying out the integration over v_x and v_y in the formula (1.5), and substituting the result into Eq. (1.3), we obtain

$$\frac{\partial W}{\partial t}=\pi^{1/2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k^2k_\perp}\frac{W}{v_{T\perp}}\int_{-\infty}^{+\infty}\left(k_z\frac{\partial f_z}{\partial v_z}+\frac{2(k_zv_z-\omega_{pe})}{v_{T\perp}^2}f_z\right)\times\exp\left\{-\frac{(\omega_{pe}-k_zv_z)^2}{k_\perp^2v_{T\perp}^2}\right\}dv_z. \tag{4.1}$$

Let us, in integrating over v_z, use the fact that $k_\perp v_{T\perp}$ is small and retain in the expansion terms of the order of $v_{T\perp}^2 k_\perp^2$.

After the integration Eq. (4.1) assumes the form

$$\frac{\partial W}{\partial t}=\pi\frac{n_b}{n_0}\omega_{pe}{}^3W\left(\frac{1}{k_z{}^2}\frac{\partial f_z}{\partial v_z}+\frac{k_\perp{}^2v_{T\perp}{}^2}{4k_z{}^4}\frac{\partial^3 f_z}{\partial v_z{}^3}\right). \quad (4.2)$$

The steady-state level of the noise is determined by integrating Eq. (4.2) over the region of the front. Using the narrowness condition for the front ($\Delta u \ll u_0 - u$), let us, as in Eqs. (3.10) and (3.11), go over to the variable $\eta = v_z - u(\tau)$. Furthermore, let us replace f_z by G with the aid of Eq. (3.13). Carrying out the indicated transformations, we reduce Eq. (4.2) to the form

$$\frac{d\ln W}{d\eta}=\frac{d\ln G}{d\eta}+\frac{k_\perp{}^2v_{T\perp}{}^2}{4k_z{}^2}\frac{c_1}{P}\frac{d^4G}{d\eta^4}. \quad (4.3)$$

Here we have used the notation introduced in Eq. (3.11). Solving Eq. (4.3), we obtain

$$W(k_\perp)=W(0)\exp\left(\frac{k_\perp{}^2v_{T\perp}{}^2}{4k_z{}^2}\frac{c_1}{P}\frac{d^3G}{d\eta^3}\right). \quad (4.4)$$

Let us find the steady-state spectrum of the noise by substituting into this formula the value of the derivative $d^3G/d\eta^3$ at the matching point u^*. As has been assumed, the $k_\perp$ spectrum of the noise has the form (2.4), but the exact value of the width Δk turns out to be larger than the value that was obtained from the qualitative estimates:

$$\Delta k_{st}\approx 2k_z\eta^*/v_{T\perp}=2k_z(u_0-u)/v_{T\perp}\Lambda. \quad (4.5)$$

It is important to note that the spectrum of the noise in the region of the front is considerably narrower than the steady-state spectrum. Indeed, in the case when $\ln(G/G_0) \gtrsim 1$ the formula (4.5) can be used to estimate the spectral width after replacing in it η^* by the running value of η:

$$\Delta k\approx 2k_z\eta/v_{T\perp}<\Delta k_{st}. \quad (4.6)$$

Thus, the expansion parameters used in the derivation of Eqs. (3.6) and (3.7) are indeed small. In the region of the front this follows from the estimate $\Delta k v_{T\perp} \approx 2k_z\eta \lesssim k_z\Delta u$. On the plateau, the fulfilment of a significantly weaker inequality, $\Delta k v_{T\perp} \lesssim k_z(u_0 - u)$, which can be derived directly from the formula (4.5), is required. From the formula (4.5) also follows the narrowness of the $k_\perp$ spectrum of the noise, but to show this it is necessary to know the particle spread over the transverse velocities at each moment of time. The determination of the dependence $v_{T\perp}(t)$ is of interest in itself, since it is precisely the growth of the spread over the transverse velocities that essentially distinguishes the three-dimensional relaxation from the one-dimensional process.

The growth of $v_{T\perp}$ occurs in the region of the plateau, where the noise level does not change. Therefore, the problem of the diffusion of the distribution function behind the front can be solved. The diffusion tensor is determined by the relation (1.4) with a time-independent noise level. Computing the diffusion coefficients and substituting them into Eq. (1.2), we obtain

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial v_z}\left(D\frac{\partial f}{\partial v_z}\right)+\frac{\partial}{\partial v_z}\left(\frac{\alpha}{2}D_1\left(\frac{v_x}{v_z}\frac{\partial f}{\partial v_x}+\frac{v_y}{v_z}\frac{\partial f}{\partial v_y}\right)\right)$$

$$+\left(\frac{\partial}{\partial v_y}\frac{v_y}{v_z}+\frac{\partial}{\partial v_x}\frac{v_x}{v_z}\right)\frac{\alpha}{2}D_1\frac{\partial f}{\partial v_z}+\frac{\alpha}{2}D\left(\frac{\partial^2 f}{\partial v_x{}^2}+\frac{\partial^2 f}{\partial v_y{}^2}\right), \quad (4.7)$$

where D and D_1 are the diffusion coefficients, determined by the equalities

$$D=\frac{4\pi^2e^2}{m^2}\frac{W_z}{v_z},\quad D_1=\frac{4\pi^2e^2}{m^2}\left(\frac{W_z}{v_z}+\frac{\partial W_z}{\partial v_z}\right),$$

while $\alpha(v_z) = \Delta k^2(v_z)v_z^2/\omega_{pe}^2$ is a small parameter. In this equation Δk is regarded as a function of v_z, since this is virtually the spectral width of the noise produced upon the passage of the front through the point v_z.

Since the diffusion coefficients D and D_1 are of the same order of magnitude, while into the last three terms on the right-hand side of Eq. (4.7) enters the small parameter α, the dominant term is the first term, which describes the longitudinal diffusion. This leads to a significant difference in the gradients of the distribution function along the longitudinal and transverse directions:

$$\frac{\partial f}{\partial v_z}\ll\frac{\partial f}{\partial v_x},\quad \frac{\partial f}{\partial v_z}\ll\frac{\partial f}{\partial v_y}. \quad (4.8)$$

These inequalities, together with the smallness of the ratio $v_\perp/v_z$, allow us to neglect the second and third terms on the right-hand side of Eq. (4.7) in comparison with the last.

Let us rewrite the equation, retaining only the dominant terms:

$$\frac{\partial f}{\partial t}=\frac{\alpha}{2}D\left(\frac{\partial^2 f}{\partial v_x{}^2}+\frac{\partial^2 f}{\partial v_y{}^2}\right)+\frac{\partial}{\partial v_z}\left(D\frac{\partial f}{\partial v_z}\right). \quad (4.9)$$

This equation describes the region $u(\tau) < v_z < u_0$ with the moving boundary $u(\tau)$ determinable from Eq. (3.28). The coefficient D in Eq. (4.9) is a known function of v_z, while the parameter α depends, as can be seen from the formula (4.5), on the spread over the transverse velocities, which is determinable only from the solution to the equation. The moving boundary and the $v_{T\perp}$-dependent diffusion coefficient $\alpha D/2$ make the search for the exact analytic solution considerably difficult. But the presence of the small parameter $\alpha(v_z)$ allows us to make important estimates.

As has already been noted, in the region of the plateau the intense longitudinal diffusion is the decisive effect. It leads to the smoothing out of the difference in the magnitudes of the spread over the transverse velocities, a difference which is caused by the dependence of the transverse-diffusion coefficient on v_z. Assuming in the zeroth approximation that the distribution function does not depend on v_z, let us integrate Eq. (4.9) over v_z from $u(\tau)$ to u_0. We obtain as a result an equation describing the transverse diffusion:

$$\frac{\partial f}{\partial t}=D_\perp(t)\left(\frac{\partial^2 f}{\partial v_x{}^2}+\frac{\partial^2 f}{\partial v_y{}^2}\right). \quad (4.10)$$

In this equation we have introduced the mean diffusion coefficient

$$D_\perp(t)=\frac{1}{2(u_0-u(t))}\int_{u(t)}^{u_0}\alpha(v_z)D(v_z,u)\,dv_z. \quad (4.11)$$

The solution to Eq. (4.10) with some initial distribution function $f_0(v_x, v_y)/(u_0 - u(t))$ has the form

$$f=\frac{1}{4\pi\chi(t)(u_0-u(t))}\int f_0(u_x,u_y)\exp\left\{-\frac{(v_x-u_x)^2+(v_y-u_y)^2}{4\chi^2(t)}\right\}du_x\,du_y,$$

$$\chi^2(t)=\int_0^t D_\perp(\tau)\,d\tau. \quad (4.12)$$

At large times such that $\chi(t) \gtrsim v_{T0}$, where v_{T0} is the initial velocity spread, the distribution function determined by the formula (4.12), approaches a Maxwellian distribution with a transverse temperature $v_{T\perp} \sim 2\chi(t)$.

The solution (4.12) cannot be considered to be final, since the dependence $\chi(t)$ has not yet been determined. It can be determined exactly if the initial distribution was Maxwellian. In this case the Maxwellian transverse-velocity distribution is preserved during the entire relaxation process, while $v_{T\perp}^2$ is determined according to the formula

$$v_{T\perp}^2 = v_{T0}^2 + 4\chi^2(t). \tag{4.13}$$

Substituting the obtained solution into the formula (4.5), determining the width of the steady-state spectrum of the noise, we obtain an integral equation for α:

$$v_{T0}^2 + 2\int_0^t \frac{d\tau}{u_0 - u(\tau)} \int_{u(\tau)}^{u_0} \alpha(v_z) D(v_z, u)\, dv_z = \frac{4(u_0 - u)^2}{\Lambda^2 \alpha(u)}. \tag{4.14}$$

This equation can be reduced to a third-order differential equation:

$$-\frac{\Lambda^3}{2} u^3 \alpha(u) = \frac{d}{du}\left\{u^2 \frac{d}{du}\left[u(u_0 - u)\frac{d}{du}\frac{(u_0 - u)^2}{\alpha(u)}\right]\right\}, \tag{4.15}$$

which can be solved by a numerical method.

A family of integral curves of Eq. (4.15) for different v_{T0} are shown in Fig. 2. It can be seen that the parameter α remains small during the entire relaxation process, i.e., the noise is always excited in a narrow cone around the direction of propagation of the beam ($\Delta k \ll k_z$), in complete agreement with the qualitative conclusions.

Of considerable interest are the transverse-velocity spread versus front location curves shown in Fig. 3. From the figure can be seen the most important result of the analysis of the process of the transverse diffusion of the distribution function, a result which could not be obtained with the aid of the one-dimensional theory. This is the appearance, in the final phase of the relaxation, of an electron beam with a transverse velocity comparable to the initial longitudinal velocity.

Notice that the axial symmetry of the problem was implicitly used in the above-expounded arguments. This manifested itself, in particular, in the fact that both the initial beam-electron distribution function (2.1) and the function (2.6), with the aid of which the $k_\perp$ spectrum of the noise that develops at the front was computed, were assumed to be axially symmetric.

In the general case the initial distribution function

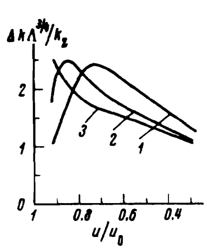

FIG. 2

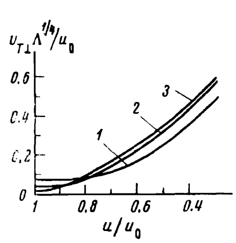

FIG. 3

FIG. 2. The dependence of the spectral width Δk of the noise on the location of the front in the distribution function. 1) $v_{T0} = 0.08u_0$, 2) $v_{T0} = 0.04u_0$, 3) $v_{T0} = 0.02u_0$.

FIG. 3. The dependence of the transverse-velocity spread $v_{T\perp}$ on the location of the front. The meanings of the curves are the same as in Fig. 2.

cannot be assumed to be axially symmetric. As a result, the $k_\perp$ spectrum of the noise will be different from the expression (2.5):

$$W(k_x, k_y) = W(0, 0) \exp\{-k_x^2/\Delta k_x^2 - k_y^2/\Delta k_y^2\}. \tag{4.16}$$

Such a change will almost not be reflected in Eqs. (3.6) and (3.7), which determine the one-dimensional functions W_z and f_z. In these equations we need only to replace Δk^2 by $\frac{1}{2}(\Delta k_x^2 + \Delta k_y^2)$. But Eq. (4.9), which describes the diffusion of the distribution function in the region of the plateau will change slightly. It will lose its axial symmetry, and will have the form

$$\frac{\partial f}{\partial t} = \frac{\alpha_x}{2} D \frac{\partial^2 f}{\partial v_x^2} + \frac{\alpha_y}{2} D \frac{\partial^2 f}{\partial v_y^2} + \frac{\partial}{\partial v_z}\left(D \frac{\partial f}{\partial v_z}\right), \tag{4.17}$$

where $\alpha_x = \Delta k_x^2 / k_z^2$ and $\alpha_y = \Delta k_y^2 / k_z^2$. The solution to this equation is similar to the solution to Eq. (4.9), with the only difference that now the integral curves $v_{Tx}(u)$ and $v_{Ty}(u)$ are different, since the initial spreads over the velocities in the directions of the x and y axes are not the same. However, this difference turns out to be insignificant, since after a short time the quantity $v_{T\perp}$ "forgets" about the initial distribution (see Fig. 3).

Thus far, we have considered beams with a sufficiently large initial transverse-velocity spread $v_{T0} \gtrsim \gamma/k$. Let us qualitatively consider beams with distribution functions not satisfying this condition ($\Delta v \lesssim (n_b/n_0)^{1/3} u_0$). A monoenergetic beam excites in the plasma a hydrodynamic instability with increment

$$\gamma \sim (n_b/n_0)^{1/3} \omega_{pe}. \tag{4.18}$$

In this case the noise that arises over a time $\tau \sim \Lambda/\gamma$ increases the particle-velocity spread to values $\Delta v \gtrsim (n_b/n_0)^{1/3} u_0$. Subsequently, the relaxation of the distribution function is described by Eqs. (1.2)–(1.5), but with somewhat different initial conditions: intense noise exists in the region

$$|k_z - \omega_{pe}/u_0| \leqslant (n_b/n_0)^{1/3} \omega_{pe}/u_0.$$

This noise does not lead to any new effects in the solution of the one-dimensional problem of the relaxation of the distribution function[6], but they exert considerable influence in three-dimensional relaxation. The point is that the spectrum of the initial hydrodynamic noise is far from one-dimensional. This can easily be verified by analyzing the increment (4.18). Since the magnitude of the increment does not depend on the angle, there gets established in a wide range of angles almost the same noise level. If as a model distribution of the noise energy over $k_\perp$ we use the expression (2.4), then the spectral width Δk should be taken to be larger than k_z.

Let us consider the consequences to which the appearance of noise with a wide spectrum will lead. The solution to Eqs. (3.6) and (3.7) in the region of the front does not differ in any way from the earlier-obtained solution, since in this region only the spectral width Δk of the noise that develops at the front enters into the equations. Thus, as in the relaxation of beams with a large initial velocity spread, a front is formed in the distribution function during the relaxation of monoenergetic beams. However, the process will change greatly in the region of the plateau. Because of the fact that the hydrodynamic noise arises in a wide angle region, the coefficients of longitudinal and transverse diffusion get to be of the same order of magnitude. This leads to the rapid growth of the transverse-velocity spread comparable to $u_0 - u(\tau)$. Such is the qualitative difference

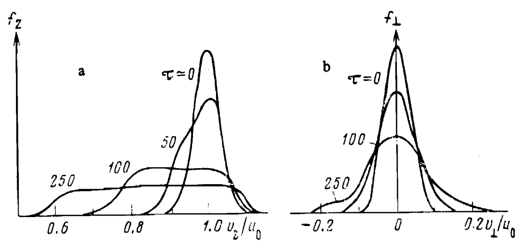

FIG. 4. Relaxation of an electron beam ($v_{T0} = 0.05u_0$; $n_b = 0.001\, n_0$). (a) The longitudinal- and (b) transverse-velocity distribution functions ($\tau = \omega_{pe}t$).

between the relaxation of monoenergetic beams and the earlier-described relaxation of beams with a large thermal spread.

5. NUMERICAL SIMULATION

Lately, the methods of numerical simulation have come to be widely used in plasma physics. With their aid have, in particular, been solved a number of problems related to the interaction of an electron beam with a plasma. In these applications the method of total and partial simulation were both used[13-16]. However, thus far, only largely one-dimensional problems have been solved. In the present paper we develop a method of two-dimensional partial numerical simulation which allows us to confirm the main results of the theory of three-dimensional electron-beam relaxation, as well as to investigate the relaxation of an initially monoenergetic electron beam. An important distinctive feature of the method used is that the transition to the two-dimensional and three-dimensional problems does not require an increase in the number of particles, an increase which was necessary in total simulation by, for example, the particles-in-a-cell method. The number of assigned waves increases, as in the case of total simulation.

The method of partial numerical simulation allows any a priori given part of the distribution function to be described by the individual particles. In the present problem, in particular, the electron beam is simulated by the particles:

$$n_b=\sum_p \delta(\mathbf{r}-\mathbf{r}_p(t)), \tag{5.1}$$

where $\mathbf{r}_p(t)$ is the coordinate of the p-th particle. The rest of the plasma is considered in the linear approximation to be a continuous medium. The electric-field potential is given in the form of a sum of Fourier components:

$$\varphi=\sum_k \varphi_k(t)\exp\{-i\omega_k t+i\mathbf{kr}+i\alpha_k(t)\}, \tag{5.2}$$

where $\varphi_k(t)$ and $\alpha_k(t)$ are the slowly varying ($|\dot{\varphi}_k| \ll \omega_k\varphi_k$, $|\dot{\alpha}_k| \ll \omega_k$) amplitude and phase of the k-th harmonic of the oscillations.

The method of deriving the equations for the numerical simulation is similar to the method used in[15]. Let us write out the final equations:

$$\frac{d\Phi_k}{d\tau}=-\frac{1}{2A_k^2}\frac{n_b}{n_0}\frac{1}{N}\sum_p \sin(\nu_k+\mathbf{A}_k\boldsymbol{\rho}_p),$$
$$\frac{d\nu_k}{d\tau}=A_{kz}-1-\frac{1}{2A_k^2}\frac{n_b}{n_0}\frac{1}{N\Phi_k}\sum_p \cos(\nu_k+\mathbf{A}_k\boldsymbol{\rho}_p),$$
$$\frac{d\boldsymbol{\rho}_p}{d\tau}=\mathbf{W}_p,\qquad \frac{d\mathbf{W}_p}{d\tau}=\sum_k \mathbf{A}_k\Phi_k\sin(\nu_k+\mathbf{A}_k\boldsymbol{\rho}_p), \tag{5.3}$$

where we have introduced the following notation:

$$\Phi_k=e\varphi_k/mu_0^2,\quad \tau=\omega_{pe}t,\quad \mathbf{A}_k=u_0\mathbf{k}/\omega_{pe},$$
$$\nu_k=(\mathbf{ku}_0-\omega_{pe})t+\alpha_k,\quad \boldsymbol{\rho}_p=(\mathbf{r}_p-\mathbf{u}_0t)\,\omega_{pe}/u_0,$$
$$\mathbf{W}_p=(\mathbf{v}_p-\mathbf{u}_0)/u_0,\quad \mathbf{k}=\{n2\pi/L_x,\ m2\pi/L_y\}.$$

The initial conditions for the equations are the values of all the variables at $\tau = 0$. The initial wave amplitude was prescribed to be from 10^{-3} to 10^{-6} of the maximum value, the phases are arbitrary, the particle velocities in the coordinate system moving with velocity u_0 are equal to zero, and the particle coordinates are distributed over a rectangle with sides L_x and L_y. To get rid of the intense noise connected with the fact that the number of particles simulating the beam is small, the beam-particle coordinates are assigned consistently with the initial wave amplitudes. The equations are solved by the Runge-Kutta method. The numerical experiments were carried out with a particle number of up to 500 and a wave number of up to 200.

The wave spacing in k space that does not destroy the continuity of the spectrum is determined by the width of the wave-particle interaction region, $\Delta v \sim \gamma/k_z$, from which it follows that

$$\Delta k_z/k_z\sim\gamma/\omega_{pe}. \tag{5.4}$$

To determine the wave spacing in the transverse direction, let us turn to Eq. (1.4). Besides the width of the interaction zone, which is determined by the width of the δ-function (of the order of γ), into this equation enters another factor: $E_k^2\cos^2\xi$, where ξ is the angle between $\partial f/\partial \mathbf{v}$ and $\mathbf{k}$.

A good model for the continuous spectrum can be obtained only when with each velocity-space point of interest in the present problem will interact several waves with different $\mathbf{k}$ directions and with a spacing in angle space so small as to make the resolution of the structure of $E_k^2\cos^2\xi$ possible. For the structure of

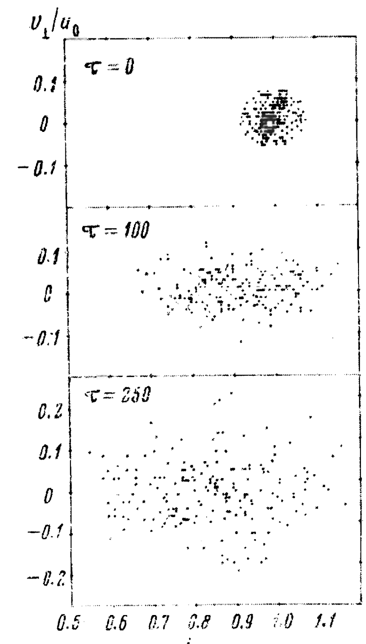

FIG. 5. Relaxation of an electron beam ($v_{T0} = 0.05u_0$; $n_b = 0.001n_0$). Velocity space ($\tau = \omega_{pe}t$).

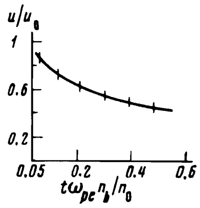

FIG. 6. Motion of the front. The curve corresponds to the motion of the front in one-dimensional relaxation, as described by (3.28). The short lines indicate the front's motion obtained from the numerical simulation.

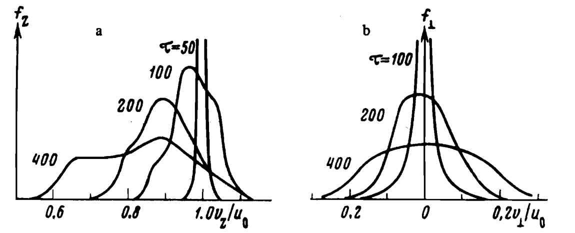

FIG. 7. Relaxation of a monoenergetic electron beam ($n_b = 0.001 n_0$). (a) The longitudinal- and (b) transverse-velocity distribution functions ($\tau = \omega_{pe} t$).

cos ξ to be resolvable, it is sufficient for the waves to be spaced 20—30° apart. The angle necessary for the resolution of E_k^2 can be ascertained only in the course of the solution. In practice, the angle interval for velocities $v \sim u_0$ was set to be $\sim 15°$, while angle intervals of up to 5° were used for lower v.

The extent of the k_z-defined spectrum depends on the relaxation region in velocity space of interest to us, and can be estimated from the duration τ of the computational run with the aid of the equation

$$\frac{k_z}{k_{z0}} + \ln\frac{k_{z0}}{k_z} = 1 + \frac{\pi}{\Lambda}\frac{n_b}{n_0}\tau. \qquad (5.5)$$

The electron beam-plasma interaction was simulated in the case when at the initial moment of time the beam had a sufficient velocity spread (a kinetic beam), as well as when the beam was initially monoenergetic.

The results of the numerical simulation of the interaction of a kinetic beam with a plasma completely corroborate the result of the three-dimensional quasi-linear theory. The one-dimensional distribution function f_z is shown at different moments of time in Fig. 4a and the transverse-velocity distribution function $f_\perp$ at the same moments of time is shown in Fig. 4b. The total distribution function, shown in Fig. 5, graphically illustrates the two-dimensional relaxation of an electron beam. It can be seen from Fig. 4a that the longitudinal-velocity distribution function has a steep front. The front moves with a velocity that coincides with the velocity obtained from the analytic solution (3.28) (see Fig. 6).

The relaxation of an initially monoenergetic beam differs somewhat from the relaxation of a kinetic beam, since the buildup of the noise in the initial phase occurs in a wide range of angles. The results of the numerical simulation corroborate the qualitative conclusions arrived at in Sec. 4. Figure 7a shows the longitudinal-velocity distribution function, while Fig. 7b shows the transverse-velocity distribution function at different moments of time. It is not difficult to see that in this case also the function f_z acquires a steep front, but that because of some difference in the initial phase of the relaxation, the broadening of the transverse-velocity distribution function occurs somewhat more rapidly than in the case of a kinetic beam. The complete form of the distribution function is shown in Fig. 8.

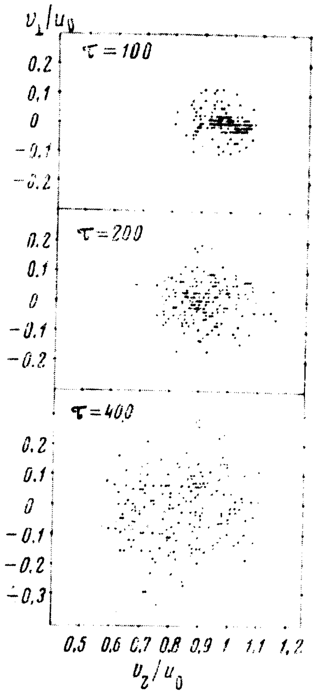

FIG. 8. Relaxation of a monoenergetic electron beam ($n_b = 0.001 n_0$). Velocity space ($\tau = \omega_{pe} t$).

6. CONCLUSION

Thus, it has been shown that the three-dimensional quasi-linear equations have an analytic solution. The problem of the interaction of an electron beam with a plasma has also been solved by the method of two-dimensional partial numerical simulation. The results of the solutions coincide. The results obtained are also corroborated by experiments on the interaction of an electron beam with a plasma located in a magnetic field of corkscrew configuration[1,4]. In these experiments the beam parameters were: U = 30 kV and I = 20 A. The magnetic field was weak, so that $\omega_{pe} \gg \omega_{He}$. It was shown[1,4] that the relaxation of a beam proceeds in accordance with the one-dimensional quasi-linear theory[7], although a manifestly three-dimensional problem was considered ($\omega_{pe} \gg \omega_{He}$). This result confirms the correctness of the solutions obtained by us. On the other hand, we can explain, on the basis of the obtained solution to the three-dimensional quasi-linear equations, the appearance of trapped particles with a high transverse energy ($T_\perp \sim 200$ keV) observed in the above-discussed experiments. It can be seen from Figs. 5 and 8 that over times $\sim 200\omega_{pe}^{-1}$ the beam particles acquire a considerable transverse-velocity spread. The portion of the particles having a higher transverse velocity is trapped by the field of the mirror device, after which they are accelerated to energies ~200 keV. A possible mechanism of acceleration of the trapped particles is described in[4].

The obtained results may also be useful in the investigation of the interaction of high-power electron beams with a plasma for the purposes of, for example, beam heating or plasma chemistry.

APPENDIX

The increment of the oscillations excited by an electron beam with a distribution function of the form (2.6) is determined by the formula (1.5). After the integration over the transverse velocities this formula assumes the form

$$\gamma=\frac{\sqrt{\pi}}{2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k^2k_\perp v_{T\perp}}\int_{-\infty}^{+\infty}\left(k_z\frac{\partial f_z}{\partial v_z}+\frac{2(k_zv_z-\omega_{pe})}{v_{T\perp}^2}f_z\right)\times\exp\left\{-\frac{(\omega_{pe}-k_zv_z)^2}{k_\perp^2v_{T\perp}^2}\right\}dv_z.$$

The integration in this expression can be performed by parts:

$$\gamma=\frac{\sqrt{\pi}}{2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k_zk_\perp v_{T\perp}}\int_{-\infty}^{+\infty}\frac{\partial f_z}{\partial v_z}\exp\left\{-\frac{(\omega_{pe}-k_zv_z)^2}{k_\perp^2v_{T\perp}^2}\right\}dv_z. \qquad (A.1)$$

For oscillations with sufficiently small $k_\perp$:

$$\frac{\partial f_z}{\partial v_z}\Big/\frac{\partial^3 f_z}{\partial v_z^3}\gg\frac{k_\perp^2v_{T\perp}^2}{k_z^2}, \qquad (A.2)$$

and the increment gets transformed to the "one-dimensional" form

$$\gamma=\frac{\pi}{2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k_z^2}\frac{\partial f_z}{\partial v_z}.$$

To obtain the $k_\perp$-dependence of the increment, let us take higher-order terms of the expansion into account:

$$\gamma=\frac{\pi}{2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k_z^2}\left(\frac{\partial f_z}{\partial v_z}+k_\perp^2\frac{v_{T\perp}^2}{4k_z^2}\frac{\partial^3 f_z}{\partial v_z^3}\right).$$

It can be seen from this formula that the $k_\perp$-dependence of the increment is determined by the sign of $\partial^3 f_Z/\partial v_Z^3$. Since the sign of the third derivative changes, there exist in any distribution function regions where the increment of the transverse oscillations is higher than the increment of the longitudinal oscillations. In particular, for a Maxwellian distribution function of the form (2.1), the sign of $\partial^3 f_Z/\partial v_Z^3$ is negative in the regions

$$v_z<u_0-v_T(3/2)^{1/2}\,(\mathrm{I}) \text{ and } u_0<v_z<u_0+v_T(3/2)^{1/2}\,(\mathrm{II})$$

and positive in the regions

$$u_0-v_T(3/2)^{1/2}<v_z<u_0\,(\mathrm{III}) \text{ and } u_0+v_T(3/2)^{1/2}<v_z\,(\mathrm{IV}).$$

Therefore, in the region I the transverse oscillations will build up more slowly, and in the region II more rapidly, then the longitudinal oscillations. In the region III the transverse oscillations will attenuate more rapidly, and in the region IV more slowly, than the longitudinal.

Let us consider the increment for distribution functions with sufficiently large $v_{T\perp}$, for which the condition (A.2) is violated. Since in this case the approximation of a local connection between γ and $\partial f_Z/\partial v_Z$ is not valid, the form of the slopes of the function (see Fig. 1) does not play an important role. Therefore, we shall assume that

$$f_z=\begin{cases}c\exp\{-(v_z-u)^2/v_{T1}^2\} & \text{for } v_z<u\\ c & \text{for } u<v_z<u_0\\ c\exp\{-(v_z-u_0)^2/v_{T2}^2\} & \text{for } u_0<v_z.\end{cases}$$

After this, the integration in the formula (A.1) can easily be performed, and the increment assumes the form

$$\gamma=\frac{\sqrt{\pi}}{2}\frac{n_b}{n_0}\frac{\omega_{pe}^3}{k_z^3}k_\perp v_{T\perp}c\left[\frac{1}{v_{T1}^2\mu_1}e^{-\nu_1^2}\xi\left(\frac{\nu_1}{\mu_1}\right)-\frac{1}{v_{T2}^2\mu_2}e^{-\nu_2^2}\xi\left(\frac{\nu_2}{\mu_2}\right)\right], \qquad (A.3)$$

$$\mu_1=(1+v_{T\perp}^2k_\perp^2/v_{T1}^2k_z^2)^{1/2},\quad \mu_2=(1+v_{T\perp}^2k_\perp^2/v_{T2}^2k_z^2)^{1/2},$$
$$\nu_1=(\omega_{pe}-k_zu)/k_\perp v_{T\perp},$$
$$\nu_2=(k_zu_0-\omega_{pe})/k_\perp v_{T\perp},\quad \xi(x)=1-\pi^{1/2}xe^{x^2}[1-\Phi(x)],$$

and $\Phi(x)$ is the probability integral. For $x \ll 1$, the function $\xi(x)$ can be represented in the form of a series:

$$\xi(x)=1-\sqrt{\pi}\,x+2x^2-\ldots,$$

while for $x \gg 1$ it has the asymptotic form

$$\xi(x)=1/2x^2-3/4x^4+\ldots$$

It can be seen from the formula (A.3) that the increment has its maximum value in the region of phase velocities

$$v_{\varphi z}=\omega_{pe}/k_z\approx u-v_{T1}$$

and its minimum value at $v_{\varphi Z}\approx u_0+v_{T2}$. It does not, however, vanish in the region of phase velocities coinciding with the plateau in the distribution function, i.e., in the region $u<v_{\varphi Z}<u_0$. But it is considerably reduced in this region. The main factors decreasing the absolute value of the increment outside the regions $k_Z\sim\omega_{pe}/u$ and $k_Z\sim\omega_{pe}/u_0$ are $\exp(-\nu_1^2)$ and $\exp-\nu_2^2)$. Therefore, for oscillations with phase velocities falling in the region of the plateau at points whose distances $\Delta v_{\varphi Z}$ from the edges of the plateau satisfy the condition

$$\Delta v_{\varphi z}>v_{T\perp}k_\perp/k_z,$$

the increment is exponentially small. It is important to note that the range of phase velocities $v_{\varphi Z}$ at which the increment is substantial broadens as $k_\perp$ increases. Therefore, the transverse waves with $k_\perp\sim k_Z$ can build up even with phase velocities $v_{\varphi Z}$ significantly different from the velocity of the front and die down at $v_{\varphi Z}$ considerably lower than u_0. When $k_\perp v_{T\perp}\gg k_Z(u_0-u)$, the increment will, in general, vanish.

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