Coherent scattering of γ quanta by a hydrogen-like atom

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The differential and total cross sections for the elastic scattering of a photon by a hydrogen-like atom are derived, correct to terms of order $(\alpha Z)^2$ inclusively, in the energy range $\omega \gg m\alpha Z$ (ω is the energy of the γ quantum, m is the electron mass). The formula for the differential cross section is valid for small momentum transfers to the nucleus, $q \sim m\alpha Z$, in which case the cross section is maximal.

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1. INTRODUCTION

The differential cross section for the coherent scattering of photons (CSP) by the electron shell of an atom (Rayleigh scattering) was first calculated by Franz [1] in the nonrelativistic approximation and with the utilization of the free Green's function, i.e., without taking into consideration the binding of the electron to the nucleus in the intermediate state. In this approximation the CSP cross section was expressed in terms of an atomic form factor.

The coherent scattering of γ rays by the K electrons of heavy atoms was investigated by Brenner, Brown, and Woodward. [2] They calculated the differential cross section for the scattering of γ quanta with energy $\omega = 0.32m$ (m is the electron mass) by the K shell of mercury atoms ($\alpha Z = 0.6$). The calculation was numerical with the binding of the electron to the nucleus taken into account in the intermediate state and with screening taken into consideration.

By utilizing the Coulomb Green's function an analytic expression was obtained by Gavrila [3] for the cross section for CSP on the hydrogen atom in the nonrelativistic region, and an expression for the relativistic region is given in [4] to the lowest-order approximation in αZ. The formula derived in [4] is valid for arbitrary scattering angles, i.e., for arbitrary momentum transfer to the nucleus. In the high energy region ($\omega \gg m$) Goldberger and Low [5] derived a formula for the amplitude of the forward ($q = 0$) elastic scattering of photons by the K electron of an atom, correct to terms $\sim(\alpha Z)^5$ inclusively.

The CSP by a hydrogen-like atom is investigated in the present article in the region of relativistic energies ($\omega \gg \eta = m\alpha Z$) and small momentum transfers to the nucleus ($q \sim \eta$). Such values of q give the major contribution to the total cross section. Formulas are obtained for the amplitude, differential and total cross sections of the process with relative accuracy of order $(\alpha Z)^4$, which allows one to utilize them for calculations in the case of scattering by intermediate and heavy ions. In contrast to previous articles, here potential scattering in the Coulomb field of the nucleus (Delbrück scattering) is also taken into consideration, since its contribution to the amplitude is of the same order as the contribution from the Coulomb corrections which take into account the binding of the electron to the nucleus in the initial, final, and intermediate states.

2. AMPLITUDE OF THE PROCESS

Correct to terms $\sim(\alpha Z)^2$ inclusively, the wave function of an electron in the 1s state has the following form in the momentum representation:

$$\langle \mathbf{f}|1s\rangle = N\left\{\left(1+\frac{\tilde{\mathbf{f}}}{2m}\right)\Big|_{\lambda=0}+\sigma\left(\ln\frac{\varepsilon}{2\eta}+\int_\varepsilon^\infty\frac{d\lambda}{\lambda}\right)\right\} \times\left(-\frac{\partial}{\partial\eta}\right)\langle \mathbf{f}|V_{i(\eta+\lambda)}|0\rangle|_{\varepsilon=0}u_0, \tag{1}$$

where $\tilde{\mathbf{f}} = \boldsymbol{\alpha}\cdot\mathbf{f}$ and $\boldsymbol{\alpha}$ denotes the Dirac matrices:

$$N=\left(\frac{1+\gamma}{\pi\Gamma(1+2\gamma)}\right)^{1/2}\frac{\eta^{3/2}}{\Gamma(1+\sigma)}\approx\left(\frac{\eta^3}{\pi}\right)^{1/2}\left(1+\frac{5}{8}\alpha^2Z^2\right); \tag{1a}$$

$$\langle \mathbf{f}|V_{i\mu}|\mathbf{s}\rangle=\frac{4\pi}{(\mathbf{f}-\mathbf{s})^2+\mu^2},\quad \gamma=(1-\alpha^2Z^2)^{1/2},$$
$$\sigma=1-\gamma\approx\alpha^2Z^2/2. \tag{1b}$$

Terms $\sim(\alpha Z)^2\tilde{\mathbf{f}}/2m$ are omitted in the wave function since the region $f \sim \eta$ will give the major contribution to the matrix element.

In the approximation under consideration the amplitude for scattering by a bound electron is represented by the two Feynman graphs shown in Fig. 1 (plus two graphs with interchanged lines representing the initial and final photons). Let us show that the contribution I_b from the graph of Fig. 1b is, in the region of small $q \sim \eta$, $(\alpha Z)^2$ times smaller than the contribution I_a from the graph of Fig. 1a. It is easy to estimate the major contributions from these graphs by using the explicit form (1) of the wave function $\langle \mathbf{f}|1s\rangle$:

$$I_a\sim\alpha\eta^5\int\frac{d^3\mathbf{f}}{[(\mathbf{f}+\mathbf{q})^2+\eta^2]^2[\mathbf{f}^2+\eta^2]^2}\frac{\hat{f}+\hat{k}_1+m}{(f+k_1)^2-m^2}$$
$$\sim\frac{\alpha\eta^5}{m}\int\frac{d^3\mathbf{f}}{[(\mathbf{f}+\mathbf{q})^2+\eta^2]^2(\mathbf{f}^2+\eta^2)^2}\sim\frac{\alpha\eta^5}{m}\frac{1}{\eta^5}=\frac{\alpha}{m}=r_e$$

(r_e denotes the classical radius of the electron, $\hat{f} = f_0\gamma_0 - \mathbf{f}\cdot\boldsymbol{\gamma}$),

$$I_b\sim\alpha\eta^5\int\frac{d^3\mathbf{s}}{[(\mathbf{s}+\mathbf{q})^2+\eta^2]^2}\frac{\hat{s}+\hat{k}_1+m}{(s+k_1)^2-m^2}\int\frac{d^3\mathbf{f}}{(\mathbf{f}^2+\eta^2)^2}\frac{\alpha Z\gamma_0}{(\mathbf{s}-\mathbf{f})^2}\frac{\hat{f}+\hat{k}_1+m}{(f+k_1)^2-m^2}$$
$$\sim\frac{\alpha\eta^5}{m^2}\alpha Z\int\frac{d^3\mathbf{s}}{[(\mathbf{s}+\mathbf{q})^2+\eta^2]^2}\int\frac{d^3f}{(\mathbf{f}^2+\eta^2)^2(\mathbf{s}-\mathbf{f})^2}$$
$$\sim\frac{\alpha\eta^5}{m^2}\alpha Z\frac{1}{\eta^4}=\frac{\alpha}{m}(\alpha Z)^2=r_e(\alpha Z)^2.$$

In similar fashion one can show that the contribution from the graph containing two Coulomb lines in the intermediate state will be $(\alpha Z)^4$ times smaller than the contribution from the graph shown in Fig. 1a.

Just like the contribution from the graph in Fig. 1b, the graph representing the elastic scattering amplitude in the Coulomb field of the nucleus (Fig. 2) gives a contribution ($\sim r_e\alpha^2Z^2$) to the CSP amplitude. Since the graph shown in Fig. 2 does not contain the parameter η (η denotes the average momentum of the K-electron), the parameters of the expansion with respect to q in the region of small $q \sim \eta$ may be parameters of the type q/m and q/ω. In the approximation under consideration it is necessary to retain only the leading term of the ex-

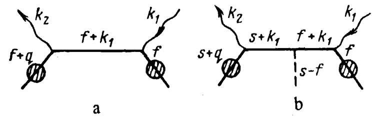

FIG. 1. Feynman graphs for the scattering of a photon by a bound electron (a line containing a circle represents a bound electron).

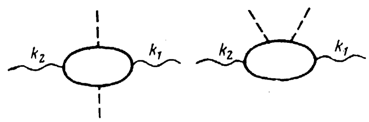

FIG. 2. Feynman graphs for potential scattering.

pansion with respect to q. For this purpose we set q = 0 (forward scattering) in the graph of Fig. 2. This is sufficient to obtain the differential cross section for CSP on an atom with relative accuracy $\sim(\alpha Z)^4$, but it is not sufficient to obtain the total cross section with the same degree of accuracy.

Using Eq. (1) we obtain the exact expression for the graphs of Fig. 1 (taking the crossed diagrams into account):

$$A_a = I_{a(1,2)} + I_{a(2,1)} = r_e(\mathbf{e}_2\mathbf{e}_1)\chi_2^* \left(\frac{\mu^2}{\mathbf{q}^2+\mu^2}\right)^2 \Big\{1 + (\alpha Z)^2\left(1+\frac{1}{2}\ln\frac{\mathbf{q}^2+\mu^2}{\mu^2}+\frac{q^2-\mu^2}{2q\mu}\operatorname{arctg}\frac{q}{\mu}\right)+\frac{i\sigma[\mathbf{n}\times\mathbf{q}]}{4m}\Big\}\chi_1, \quad (2)$$

$$A_b = I_{b(1,2)} + I_{b(2,1)} = -r_e(\mathbf{e}_2\mathbf{e}_1)(\alpha Z)^2\frac{\mu^2}{\mathbf{q}^2+\mu^2}. \quad (3)$$

Here $\mathbf{e}_1$ and $\mathbf{e}_2$ are the polarization vectors of the incident and scattered photons, $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$ is the momentum transfer to the nucleus, $\mu = 2\eta$, $\mathbf{n} = \mathbf{k}_1/\omega = \mathbf{n}_1 + \mathbf{n}_2$, σ denotes the Pauli matrices, and χ_1 and χ_2^* are the spin functions of the electron in the initial and final states, normalized by the condition $\chi_i^*\chi_i = 1$. Terms of order $\mathbf{q}^2/m\omega$ are omitted in the imaginary part of the amplitude A_a since their contribution to the differential cross section turns out to be a quantity of order $(\mathbf{q}^2/m\omega)^2 \sim (\alpha Z)^4$.

The amplitude A_e for CSP by a bound electron is the sum of expressions (2) and (3):

$$A_e = A_a + A_b = r_e(\mathbf{e}_2\mathbf{e}_1)\chi_2^*\left(\frac{\mu^2}{\mathbf{q}^2+\mu^2}\right)^2 \times\Big\{1-\frac{\mathbf{q}^2}{4m^2}+\frac{1}{2}(\alpha Z)^2\left(\ln\frac{\mathbf{q}^2+\mu^2}{\mu^2}+\frac{\mathbf{q}^2-\mu^2}{|\mathbf{q}|\mu}\operatorname{arctg}\frac{|\mathbf{q}|}{\mu}\right)+\frac{i\sigma[\mathbf{n}\times\mathbf{q}]}{4m}\Big\}\chi_1. \quad (4)$$

If one sets q = 0 in expression (4), $A_e = r_e(\mathbf{e}_2\cdot\mathbf{e}_1)\times[1-(1/2)(\alpha Z)^2]$, which coincides with the formula derived by Goldberger and Low, [4] if only the terms $\sim(\alpha Z)^2$ are retained there.

The amplitude $A_p(\omega, \mathbf{q})$ for potential scattering contains real and imaginary parts (the latter differs from zero only for $\omega > 2m$). The zero angle scattering amplitude, $A_p(\omega, 0)$, was derived by Bethe and Rohrlich [6] and has the form

$$A_p = A_p(\omega, 0) = r_e(\mathbf{e}_2\mathbf{e}_1)(\alpha Z)^2[a_1(\omega, 0)+ia_2(\omega, 0)], \quad (5)$$

where $a_1(\omega, 0)$ and $a_2(\omega, 0)$ are real. Expressions for these quantities are also given in [7].

3. DIFFERENTIAL AND TOTAL CROSS SECTIONS OF THE PROCESS

The differential cross section for CSP may be represented as the sum of three terms: the terms corresponding to Rayleigh scattering and potential scattering, and the interference term. Averaging over the initial states and summing over the final polarizations of the photons and of the electron, we obtain

$$d\sigma = \frac{1}{4}\sum|A_e+A_p|^2\,d\Omega = d\sigma_e + d\sigma_{int} + d\sigma_p, \quad (6)$$

$$d\sigma_e = r_e^2\left(\frac{\mu^2}{\mathbf{q}^2+\mu^2}\right)^4\Big\{1-\frac{q^2}{2\omega^2}-\frac{q^2}{4m^2} + (\alpha Z)^2\left(\ln\frac{q^2+\mu^2}{\mu^2}+\frac{q^2-\mu^2}{q\mu}\operatorname{arctg}\frac{q}{\mu}\right)\Big\}d\Omega, \quad (6a)$$

$$d\sigma_{int} = r_e^2(\alpha Z)^2\left(\frac{\mu^2}{\mathbf{q}^2+\mu^2}\right)^2\cdot 2a_1(\omega, 0)\,d\Omega, \quad (6b)$$

$$d\sigma_p = r_e^2(\alpha Z)^4[a_1^2(\omega, \mathbf{q})+a_2^2(\omega, \mathbf{q})]\,d\Omega, \quad (6c)$$

where $\mathbf{q}^2 = 2\omega^2(1-\cos\theta)$, θ is the photon scattering angle, $d\Omega = 2\pi\sin\theta\,d\theta = 2\pi q\,dq/\omega^2$, and $q = |\mathbf{q}|$.

Expression (6a) goes over into the formula derived in [4] if the terms $\sim\alpha^2Z^2$ are neglected. As is clear from Eqs. (6a) and (6b), the interference between potential scattering and Rayleigh scattering gives the same relative contribution ($\sim\alpha^2Z^2$) to the cross section as the Coulomb corrections to the wave function and to the electron Green's function. The contribution $d\sigma_{int}$ of the interference term increases with increasing photon energy ω ($a_1(\omega, 0) \sim \omega/m$ for $\omega \gg m$) whereas the electron part of the scattering, $d\sigma_e$, remains constant (for q = 0).

One should discard the purely potential part of the scattering, $d\sigma_p$, in the region of small $q \sim \eta$ and medium energies $\omega \sim m$, but its contribution to the cross section increases with increasing photon energy (a_2 will be $\sim(\omega/m)\ln(2\omega/m)$ for $\omega \gg 2m$ [6, 7]). Furthermore, Rayleigh scattering and the interference part of the scattering fall rapidly with increasing values of q (for $q \sim \omega \sim m$ we have $d\sigma_e \sim r_e^2(\alpha Z)^8$, $d\sigma_{int} \sim r_e^2(\alpha Z)^6$) whereas in the range $\omega \sim m$ the potential scattering remains a quantity of order $r_e^2(\alpha Z)^4$ for arbitrary values of q.

Formulas (6a) and (6b) are only valid in the region of small $q \sim \eta$; however, they may be used to obtain the total cross section since it is precisely this range of variation of q which gives the major contribution to the integral over the solid angle (or with respect to q). The major contribution to the total cross section from (6a) turns out to be a quantity of order $\sigma_0(\alpha Z)^2$, but the correction terms in (6a) and the interference term (6b) are quantities of order $\sigma_0(\alpha Z)^4$. The total cross section for potential scattering is of the same order, but for its computation one must know $a_1(\omega, \mathbf{q})$ and $a_2(\omega, \mathbf{q})$ over the entire range of variation of q. Thus, the total cross section for CSP by a hydrogen-like atom has the form

$$\sigma = \sigma_e + \sigma_{int} + \sigma_p, \quad (7)$$

$$\sigma_e = \sigma_0(\alpha Z)^2\frac{m^2}{2\omega^2}\Big\{1+(\alpha Z)^2\left(\frac{7}{6}-\frac{3\pi^2}{16}-\frac{m^2}{\omega^2}\right)\Big\}, \quad (7a)$$

$$\sigma_{int} = \sigma_0(\alpha Z)^4\frac{3m^2}{\omega^2}a_1(\omega, 0), \quad (7b)$$

$$\sigma_p = \sigma_0(\alpha Z)^4\frac{3}{4}\int_0^{2\omega}\frac{q\,dq}{\omega^2}[a_1^2(\omega, \mathbf{q})+a_2^2(\omega, \mathbf{q})], \quad (7c)$$

where $\sigma_0 = (8/3)\pi r_e^2$ is the Thomson cross section.

Formula (7) may be utilized in experiments for extraction of the purely potential part of the scattering, i.e., for measurement of the Delbrück scattering cross section.

For the coherent scattering of photons on a neutral atom, the contribution of the Rayleigh scattering grows with increasing Z; however, the relative contribution of the potential scattering decreases.

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[1] W. Franz, Z. Phys. **98**, 314 (1935).
[2] S. Brenner, G. E. Brown, and J. B. Woodward, Proc. R. Soc. A**227**, 59 (1954).
[3] M. Gavrila, Phys. Rev. **163**, 147 (1967).
[4] V. G. Gorshkov, A. I. Mikhailov, V. S. Polikanov, and S. G. Sherman, Phys. Lett. **30A**, 455 (1969).
[5] M. L. Goldberger and F. E. Low, Phys. Rev. **176**, 1778 (1968).
[6] H. A. Bethe and F. Rohrlich, Phys. Rev. **86**, 10 (1952).
[7] A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya elektrodinamika (Quantum Electrodynamics), Gostekhizdat, 1959 (English Transl., Interscience Publishers, 1965).

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