Stationary distribution of magnons following parametric excitation in ferromagnetic substances

V. M. Tsukernik and R. P. Yankelevich

Physico-technical Institute, Ukrainian Academy of Sciences and Physico-technical Institute of Low Temperatures, Ukrainian Academy of Sciences
(Submitted March 3, 1974; resubmitted October 1, 1974)
Zh. Eksp. Teor. Fiz. 68, 2116–2124 (June 1975)

The stationary distribution of magnons induced by parametric excitation by a uniform high frequency magnetic field parallel to the anisotropy axis in a ferromagnetic crystal is investigated. If the magnon system attains a stationary state, then this state can be defined in a self-consistent manner. On the other hand if the self-consistency conditions are not satisfied, this signifies the absence of a steady state. In this case a periodic regime sets in, with a period and amplitude that depend on the high-frequency field amplitude and on the magnitude of the interaction between the magnons.

PACS numbers: 75.30.Fv

Spin waves are excited in a ferrodielectric placed in a high-frequency homogeneous magnetic field parallel to the easy axis. At field amplitudes exceeding a certain threshold value, the number of spin waves increases exponentially in time (parametric excitation).[1-3]. Interactions in a system of magnons (including their interaction with other subsystems) limit this growth. In addition, dissipative processes can lead to establishment of a stationary magnon distribution. If a stationary distribution is established in a magnon system, then the system goes over into a saturation state and ceases to absorb energy.

We have obtained in our study the conditions that must be satisfied in order for such a stationary distribution to exist. If these conditions are not satisfied, then a nonstationary regime is established in the system. By way of example, we consider a simplified situation in which this regime is periodic.

The Hamiltonian of the system considered by us is of the following form[4]:

$$\mathscr{H}=\sum_k\left[\varepsilon_k a_k^+a_k+\frac{1}{2}(V_k a_k a_{-k}e^{i\omega t}+V_k^* a_k^+a_{-k}^+e^{-i\omega t})\right]+\mathscr{H}_{int},$$

$$\varepsilon_k=(A_k^2-|B_k|^2)^{1/2},\quad V_k=\mu h_0 B_k/2\varepsilon_k, \tag{1}$$

where a_k^+ and a_k are the Bose creation and annihilation operators; in the long-wave approximation we have

$$A_k=\theta_c a^2k^2+\mu(H+\beta M_0)+|B_k|,$$
$$B_k=2\pi\mu M_0\sin^2\vartheta_k\exp(2i\varphi_k),$$

M_0 is the saturation magnetization, β is the anisotropy constant, μ is the Bohr magneton, a is the lattice constant, θ_c is the exchange constant, ϑ_k and φ_k are the polar angles in wave-vector space, h_0 is the amplitude of the alternating field, and H is the constant field applied along the selected axis. We have retained in the Hamiltonian (1) only the resonant terms that describe the decay of the photon into two magnons and the inverse process. The remaining terms, which contain the alternating field, and for which the energy conservation law $2\epsilon_k = \hbar\omega$ is not satisfied, can be taken into account by perturbation theory, since it is assumed that $\mu h_0/\epsilon_k \ll 1$.

The time evolution of the system in question is described by the equation for the density matrix ρ:

$$i\hbar\frac{\partial\rho}{\partial t}=[\mathscr{H},\rho]. \tag{2}$$

With the aid of the unitary transformation

$$\tilde\rho=U^{-1}\rho U,\quad U=\exp\left[-\frac{i\omega t}{2}\sum_k a_k^+a_k\right] \tag{3}$$

the explicit dependence on the time is eliminated from (2), as a result of which we get

$$i\hbar\frac{\partial\tilde\rho}{\partial t}=[\tilde{\mathscr{H}},\tilde\rho], \tag{4}$$

$$\tilde{\mathscr{H}}=U^{-1}\mathscr{H}U-\frac{\hbar\omega}{2}\sum_k a_k^+a_k=\tilde{\mathscr{H}}_0+\mathscr{H}_{int}, \tag{5}$$

$$\tilde{\mathscr{H}}_0=\sum_k\left[\left(\varepsilon_k-\frac{\hbar\omega}{2}\right)a_k^+a_k+\frac{1}{2}(V_k a_k a_{-k}+V_k^* a_k^+a_{-k}^+)\right]. \tag{6}$$

Under the transformation (3), the interaction Hamiltonian H_{int} remains unchanged, since we confine ourselves to Bose-operator fourth order terms that are of exchange origin and therefore contain equal numbers of creation and annihilation operators:

$$\mathscr{H}_{int}=\sum_{1234}\Phi_{1,2;3,4}a_1^+a_2^+a_3a_4. \tag{7}$$

On going from the operators connected with the spin deviation to operators a_k of "true" magnons (diagonalization with account taken of the dipole interaction), there appear in the interaction Hamiltonians also terms with unequal numbers of creation and annihilation operators, which are proportional to ΦB_k and which we have neglected, assuming that $|B_k| \ll A_k$.

In the Hamiltonian $\tilde{\mathscr{H}}_0$, the spectrum of the "oscillator" with fixed value of k is discrete if $|\epsilon_h - \hbar\omega/2| > |V_k|$, and continuous if $|\epsilon_k - \hbar\omega/2| \le |V_k|$. The wave-vector-space region delineated by the second inequality is precisely the region of the parametric excitation, in which the operators

$$a_k(t)=\exp\left(\frac{i}{\hbar}\tilde{\mathscr{H}}_0 t\right)a_k\exp\left(-\frac{i}{\hbar}\tilde{\mathscr{H}}_0 t\right)$$

and $a_k^+(t)$ in the Heisenberg representation increase exponentially with time. This causes also the experimental growth of the pair correlators $n_k = \langle a_k^+a_k\rangle$ and $\sigma_k = \langle a_k a_{-k}\rangle$, where the averaging is carried out with the density matrix $\tilde\rho$. From (4), using (6) and (7), we get

$$i\hbar\dot n_k=V_k^*\sigma_k^*-V_k\sigma_k+\sum_{123}[\Phi_{k,1;2,3}\langle a_k^+a_1^+a_2a_3\rangle-\text{c.c.}],$$

$$i\hbar\dot\sigma_k=(2\varepsilon_k-\hbar\omega)\sigma_k+V_k^*(2n_k+1)+\sum_{123}[\Phi_{k,1;2,3}\langle a_1^+a_2a_3a_{-k}\rangle+\Phi_{-k,1;2,3}\langle a_k a_1^+a_2a_3\rangle]. \tag{8}$$

The smallness of the magnon interaction makes it possible to split up the higher correlators into pair correlators, taking into account at the same time that not only the n_k but also the σ_k differ from zero. After splitting the quaternary operators in (8), we obtain

$$i\hbar\dot{n}_k=\Delta_k^*\sigma_k^*-\Delta_k\sigma_k+2\sum_{123}{}' [\Phi_{k,1;2,3}\langle a_k^+a_1^+a_2a_3\rangle-\text{c.c.}],$$

$$i\hbar\dot{\sigma}_k=(2\varepsilon_k-\hbar\omega+2\Lambda_k)\sigma_k+\Delta_k^*(2n_k+1)$$
$$+2\sum_{123}{}' [\Phi_{k,1;2,3}\langle a_1^+a_2a_3a_{-k}\rangle+\Phi_{-k,1;2,3}\langle a_ka_1^+a_2a_3\rangle], \qquad (8')$$

where

$$\Delta_k=V_k+2\sum_{k'}\Phi_{k,-k;k',-k'}\sigma_{k'}^*, \quad \Lambda_k=2\sum_{k'}\Phi_{k,k';k,k'}n_{k'}, \qquad (9)$$

and the sums with the primes denote the remaining parts of the corresponding sums after the splitting in the first stage. These parts make a contribution linear in the interaction to the equations for n_k and σ_k, and play the role of the collision integral. To find its explicit form we can use, for example, a method similar to the Bogolyubov method for deriving the kinetic equation. The resultant equations have the following structure:

$$i\hbar\dot{n}_k=\Delta_k^*\sigma_k^*-\Delta_k\sigma_k+J_n\{n,\sigma\},$$
$$i\hbar\dot{\sigma}_k=(2\varepsilon_k-\hbar\omega+2\Lambda_k)\sigma_k+\Delta_k^*(2n_k+1)+J_\sigma\{n,\sigma\}, \qquad (10)$$

where J_n and J_σ are the analogs of the collision integral, and nonlinear functionals of n_k and σ_k, and are very complicated and cumbersome in form.

Equations analogous to (10) were considered by Zakharov, L'vov, and Starobinets.[5] The dynamic parts in their equations and in (10) coincide. As to the collision terms, in the cited papers they are represented by expressions $\gamma_k(n_k-n_k^0)$ and $\gamma_k\sigma_k$ respectively in the first and second equation of (10) (n_k^0 is the equilibrium Bose distribution of the magnons). This representation corresponds to linearization of the collision integrals in the deviation from the equilibrium distribution, and naturally requires that $n_k-n_k^0$ and σ_k be small. In the case when the amplitude of the high-frequency field does not exceed the threshold value, the quantities $n_k-n_k^0$ and σ_k are small to the extent that the amplitude is small. Then the collision integrals J_n and J_σ are reduced (neglecting the arrival terms) to the relaxation terms $\gamma_k(n_k-n_k^0)$ and $\gamma_k\sigma_k$, respectively.

1. If the collisions cause a stationary distribution to be established in the system of magnons described by the equations in (10), i.e., as $t\to\infty$, the derivatives n_k and σ_k tend to zero, then the stationary values of n_k and σ_k are determined from the equations

$$\Delta_k^*\sigma_k^*-\Delta_k\sigma_k+J_n\{n,\sigma\}=0,$$
$$(2\varepsilon_k-\hbar\omega+2\Lambda_k)\sigma_k+\Delta_k^*(2n_k+1)+J_\sigma\{n,\sigma\}=0. \qquad (10')$$

Zakharov et al.[5] retained the expressions $\gamma_k(n_k-n_k^0)$ and $\gamma_k\sigma_k$ for the collision terms when they derived the stationary distribution. The stationary distribution obtained in this manner differs appreciably from the equilibrium distribution with $n_k=n_k^0$ and $\sigma_k=0$. Consequently, the values of $n_k-n_k^0$ and σ_k for this distribution are not small, so that a phenomenological account of the relaxation is not suitable for the determination of the stationary distribution.

Equations (10) can be obtained from the effective Hamiltonian corresponding to the self-consistent field approximation in the system with the Hamiltonian (5):

$$\mathcal{H}_{\text{eff}}=\sum_k\left[\xi_ka_k^+a_k+\frac{1}{2}(\Delta_ka_ka_{-k}+\Delta_k^*a_k^+a_{-k}^+)\right] \qquad (11)$$
$$+\sum_{1234}\Phi_{1,2;3,4}a_1^+a_2^+a_3a_4, \quad \xi_k=\varepsilon_k-\hbar\omega/2+\Lambda_k.$$

Here Λ_k and Δ_k are determined by formulas (9), in which n_k and σ_k are equal to their stationary values. The stationary distributions coincide for the systems with the Hamiltonians (5) and (11). The constancy of Λ_k and Δ_k makes it possible to diagonalize with the aid of the canonical uv transformation

$$a_k=u_kb_k+v_k^*b_{-k}^+, \quad a_{-k}=u_kb_{-k}+v_k^*b_k^+, \quad |u_k|^2-|v_k|^2=1 \qquad (12)$$

the quadratic part of the Hamiltonian (11). As a result we have

$$\tilde{\mathcal{H}}_{\text{eff}}=\sum_k\tilde{\varepsilon}_kb_k^+b_k+\tilde{\mathcal{H}}_4, \qquad (11')$$

where

$$\tilde{\varepsilon}_k=(\xi_k^2-|\Delta_k|^2)^{1/2}, \qquad (13)$$

and $\tilde{\mathcal{H}}_4$ is obtained from the interaction Hamiltonian of (11) by changing from the operators a_k to the operators b_k with the aid of (12). The coefficients u_k and v_k are defined as follows:

$$u_k=\left(\frac{\xi_k+\tilde{\varepsilon}_k}{2\tilde{\varepsilon}_k}\right)^{1/2}, \quad v_k=-\frac{\Delta_k}{|\Delta_k|}\left(\frac{\xi_k-\tilde{\varepsilon}_k}{2\tilde{\varepsilon}_k}\right)^{1/2}. \qquad (14)$$

It is obvious that diagonalization of the Hamiltonian (11′) is possible only under the condition

$$|\xi_k|>|\Delta_k| \qquad (15)$$

for all $\mathbf{k}$. In addition, from the fact that $|u_k|^2$ is positive it follows that the sign of $\tilde{\epsilon}_k$ should coincide with the sign of the quantity ξ_k. On the other hand, the sign of ξ_k should be the same for all $\mathbf{k}$, for otherwise ξ_k should vanish at certain values of $\mathbf{k}$, by virtue of the continuity, and this would contradict the condition (15). The sign in (13) is chosen under the assumption that $\xi_k>0$.

The "new" quasiparticles are described by the kinetic equation for the distribution function $\tilde{n}_k=\langle b_k^+b_k\rangle$:

$$\dot{\tilde{n}}_k=I_{\tilde{n}}.$$

The collision integral $I_{\tilde{n}}$ corresponding to the Hamiltonian $\tilde{\mathcal{H}}_4$ takes the usual form (the difference between the departure and arrival terms) and can be obtained, for example, by calculating the transition probabilities in the quasiparticle system. The equilibrium distribution of the new quasiparticles, which corresponds to the stationary magnon distribution defined by Eqs. (10′), is determined from the equation

$$I_{\tilde{n}}=0. \qquad (10'')$$

Following the transformation (12), the interaction Hamiltonian $\tilde{H}_4$ and the collision integral contain terms with and without conservation of the number of quasiparticles. Therefore the equilibrium distribution satisfying Eq. (10″) is described by a Bose function with zero chemical potential:

$$\tilde{n}_k=[\exp(\tilde{\varepsilon}_k/T)-1]^{-1}. \qquad (16)$$

For the correlators n_k and σ_k we obtain with the aid of (12) and (14) the following stationary values:

$$n_k+\frac{1}{2}=\frac{\xi_k}{\tilde{\varepsilon}_k}\left(\tilde{n}_k+\frac{1}{2}\right), \quad \sigma_k=-\frac{\Delta_k^*}{\tilde{\varepsilon}_k}\left(\tilde{n}_k+\frac{1}{2}\right), \qquad (17)$$

and from (9) we obtain, in analogy with the BCS model in superconductivity theory, the self-consistency constitutions[6]:

$$\Lambda_k=\sum_{k'}\psi_{kk'}\left[\frac{\xi_{k'}}{\tilde{\varepsilon}_{k'}}\left(\tilde{n}_{k'}+\frac{1}{2}\right)-\frac{1}{2}\right], \qquad (18)$$

$$\Delta_k=V_k-\sum_{k'}\chi_{kk'}\frac{\Delta_{k'}}{\tilde{\varepsilon}_{k'}}\left(\tilde{n}_{k'}+\frac{1}{2}\right); \qquad (18')$$

$$\psi_{kk'}=\Phi_{k,k';k,k'},\quad \chi_{kk'}=\Phi_{k,-k;k',-k'}.$$

Thus, the magnon system goes over under conditions of parametric excitation into a stationary state if the self-consistency conditions (18) and (18′) have a solution. The stationary distribution is determined in this case by formulas (16) and (17), in which T is the temperature of the thermostat. The role of the thermostat can be played, for example, by a phonon system that is in thermodynamic equilibrium.[1)] The corresponding magnon-phonon interaction is due to magnetostriction energy of exchange origin and therefore conserves the number of magnons.

The distribution (16) changes if account is taken of relativistic magnon interactions that do not conserve the number of magnons. However, in view of the smallness of the relativistic interactions in comparison with the exchange interactions, they result in small corrections to the distribution (16). Formulas (17), which express n_k and σ_k in terms of n_k in the stationary state, remain the same as before. It is easy to verify that the distributions (17) cause the vanishing of the dynamic parts of Eqs. (10′). The same distributions (17), in which n_k satisfies Eq. (10″), cause also the vanishing of the collision integrals J_n and J_σ. From this it follows, in particular, that a magnon system in the stationary state does not absorb the energy of the pump field.

2. In view of the complexity of the system (18) and (18′), we consider a very simple model, in which the interactions are constant and real: $\psi_{kk'}=\psi$, $\chi_{kk'}=\chi$ and $V_k = V$. Here Λ_k and Δ_k are also real and constant. By going from summation to integration in (18) and (18′) we obtain transcendental equations for the self-consistency parameters Λ and Δ:

$$\Lambda=\frac{\psi a^3}{2\pi^2}\int_0^{k_m}\left\{\frac{(\theta_c a^2k^2+\xi)(\tilde n_k+{}^1/_2)}{[(\theta_c a^2k^2+\xi)^2-|\Delta|^2]^{1/2}}-\frac{1}{2}\right\}k^2\,dk, \tag{19}$$

$$\Delta=V-\frac{\chi a^3}{2\pi^2}\Delta\int_0^{k_m}\frac{(\tilde n_k+{}^1/_2)k^2\,dk}{[(\theta_c a^2k^2+\xi)^2-|\Delta|^2]^{1/2}} \tag{20}$$

where k_m is the value of k on the boundary of the Brillouin zone, and

$$\xi=\Lambda+\varepsilon_0-\hbar\omega/2. \tag{21}$$

The condition (15) reduces in this case to the inequality[2)]

$$\xi\geqslant|\Delta|. \tag{22}$$

For simplicity we assume a quadratic dispersion of the magnons up to the boundary of the Brillouin zone. This does not affect qualitatively the conditions for the existence of the stationary distribution.

We consider first Eqs. (19) and (20) at zero temperature. Introducing the dimensionless integration variable $x = (\theta_c/\xi)^{1/2}ak$, we obtain

$$\frac{\hbar\omega/2-\varepsilon_0}{|\Delta|}+\eta=\frac{1}{4\pi^2}\frac{\psi}{\theta_c}\left(\frac{|\Delta|}{\theta_c}\right)^{1/2}\eta^{3/2}\int_0^{x_m}\left[\frac{1+x^2}{[(1+x^2)^2-\eta^{-2}]^{1/2}}-1\right]x^2\,dx, \tag{23}$$

$$\Delta=V-\frac{\Delta}{4\pi^2}\frac{\chi}{\theta_c}\left(\frac{\xi}{\theta_c}\right)^{1/2}\int_0^{x_m}\frac{x^2\,dx}{[(1+x^2)^2-\eta^{-2}]^{1/2}},$$

$$\eta=\xi/|\Delta|,\quad x_m=(\theta_c/\xi)^{1/2}ak_m. \tag{24}$$

It follows from the inequality (22) that the left-hand side of (23) cannot be less than unity. Therefore the necessary condition for the existence of a solution of (23) is the inequality

$$\frac{1}{4\pi^2}\frac{\psi}{\theta_c}\left(\frac{|\Delta|}{\theta_c}\right)^{1/2}f(\eta)\geqslant1,\quad f(\eta)=\eta^{3/2}\int_0^\infty\left[\frac{1+x^2}{[(1+x^2)^2-\eta^{-2}]^{1/2}}-1\right]x^2\,dx$$

(we have increased the right-hand side, replacing the upper limit by infinity).

The function $f(\eta)$ is finite at all η in the interval $(1,\infty)$ (with $f(\infty) = 0$), and its maximum value is $\infty 1$. For the existence of the solution there should therefore be satisfied the inequality

$$\frac{\psi}{\theta_c}\left(\frac{|\Delta|}{\theta_c}\right)^{1/2}>K,$$

or

$$|\Delta|>\theta_c\left(\frac{\theta_c}{\psi}\right)^2K^2,\quad K=\frac{4\pi^2}{\max f(\eta)}. \tag{25}$$

On the other hand, it is seen from (24) that at positive χ we have $|\Delta| < |V|$, and according to (25), for a solution to exist we must satisfy the inequality

$$\psi>K\theta_c(\theta_c/|V|)^{1/2},$$

which is impossible.

Since the integrand in (24) is always smaller than unity, the integral does not exceed $x_m \sim (\theta_c/\xi)^{1/2}$. Therefore at negative χ it follows from (24) that

$$\Delta<V/(1-|\chi|/\theta_c).$$

The inequality (25) leads in this case to the following necessary condition:

$$1-|\chi|/\theta_c<|V|\psi^2/\theta_c^3,$$

i.e., the amplitude of the interaction should be close to θ_c, which contradicts the assumption that the interaction is small.

Thus, the self-consistency equations (23) and (24) have no solution at zero temperature for reasonable values of the parameters V, ψ, and χ.

Let now $T \neq 0$ and let the terms containing the temperature be large in comparison with those considered above (otherwise the conditions for the existence of the stationary state remain the same as before). We then obtain from (19) and (20), introducing the dimensionless variable $x = \tilde\epsilon_k/|\Delta|$,

$$\Lambda=\frac{\psi}{4\pi^2}\left(\frac{|\Delta|}{\theta_c}\right)^{1/2}\int_{\sqrt{\eta^2-1}}^\infty\frac{(\sqrt{x^2+1}-\eta)^{1/2}}{\exp(x|\Delta|/T)-1}dx, \tag{26}$$

$$\Delta\left\{1+\frac{\chi}{4\pi^2}\left(\frac{|\Delta|}{\theta_c}\right)^{1/2}\int_{\sqrt{\eta^2-1}}^\infty\frac{(\sqrt{x^2+1}-\eta)^{1/2}\,dx}{\sqrt{x^2+1}[\exp(x|\Delta|/T)-1]}\right\}=V. \tag{27}$$

The integral in (26) has the largest value at $\eta = 1$, and the obtained integral is smaller than

$$J=\int_0^\infty\frac{\sqrt{x}\,dx}{\exp(x|\Delta|/T)-1}=\left(\frac{T}{|\Delta|}\right)^{3/2}\frac{\sqrt{\pi}}{2}\zeta({}^3/_2),$$

where $\zeta(z)$ is the Riemann zeta function.

This leads to the following necessary condition:

$$\Lambda<\frac{\zeta({}^3/_2)}{8\pi^{3/2}}\psi\left(\frac{T}{\theta_c}\right)^{3/2}. \tag{28}$$

On the other hand, it follows from the inequality (22) that $\Lambda > \hbar\omega/2 - \epsilon_0$. A comparison of the last two inequalities shows that in the considered model (ψ and χ constant) the necessary conditions for the existence of a stationary state at $T \neq 0$ also turn out to be quite stringent. This is the consequence of the replacement of the amplitudes $\psi_{kk'}$ and $\chi_{kk'}$ by constants, as a result of which the dangerous region shifts towards lower k, where the convergence of the integrals in (19) and (20) is ensured by the smallness of the density of states.

We call attention in this connection to the fact that in the two-dimensional model, even within the framework of the considered crude model, the self-consistency conditions are always satisfied at $T \neq 0$. Indeed, the integrals in the right-hand sides of the equations that replace (19) and (20) diverge logarithmically in the two-dimensional case as $\xi \to |\Delta|$, and this ensures the existence of a solution without restrictions on the parameters. Assuming, e.g., $\chi = 0$, i.e., $\Delta = V$, we obtain the equation

$$-\Lambda = \frac{\psi T}{4\pi\theta_c} \ln\left\{1 - \exp\left[-\frac{\sqrt{\xi^2 - V^2}}{T}\right]\right\}$$

($\xi = \Lambda + \epsilon_0 - \hbar\omega/2$), which, as can be readily seen, has a solution at all values of ψ, V, ω, and T.

3. In the nonstationary case, the time behavior of the system should be described by the equations for the correlators n_k and σ_k:

$$i\hbar\dot{n}_k = \Delta_k^*\sigma_k^* - \Delta_k\sigma_k,$$

$$i\hbar\dot{\sigma}_k = (2\varepsilon_k - \hbar\omega + 2\Lambda_k)\sigma_k + \Delta_k^*(2n_k + 1); \quad (29)$$

Λ_k and Δ_k are defined by relations (9).

Since the collisions in the system of magnons do not lead in this case to establishment of a stationary state, allowance for the collision integrals in Eqs. (29) would lead apparently to quantitative but not to qualitative changes. In linearized form ($\Lambda_k = 0$, $\Delta_k = V_k$), this system describes an exponential growth of the correlators n_k and σ_k, with a growth rate

$$\eta_k = [|V_k|^2 - (\varepsilon_k - \hbar\omega/2)^2]^{1/2}.$$

Since this system is inhomogeneous, it has nonzero solutions also at the zero initial conditions $n_k(0) = \sigma_k(0) = 0$.

Let us examine the influence of the anharmonicities that lead to a limitation on the growth of the correlators, under the following simplifying assumptions:

$$\psi_{kk'} = \psi, \quad \chi_{kk'} = \chi, \quad V_k = V_k^* = V, \quad 2\varepsilon_k = \hbar\omega.$$

The latter condition denotes neglect of the dependence of the energy ϵ_k on the wave vector $\mathbf{k}$, corresponding to a pump-field frequency ω close to $2\epsilon_0/\hbar$. Under these assumptions, the system (29) reduces to a system of nonlinear differential equations for the quantities $n = \sum_k n_k/N$, $\sigma = \sum_k \sigma_k/N$:

$$\begin{aligned} i\hbar\dot{n} &= V(\sigma^* - \sigma), \\ i\hbar\dot{\sigma} &= 2\psi n\sigma + (V + \chi\sigma)(2n+1), \\ i\hbar\dot{\sigma}^* &= 2\psi n\sigma^* + (V + \chi\sigma^*)(2n+1). \end{aligned} \quad (30)$$

The system (30) reduces to a differential equation of second order for the quantities n, which has the following first integral at zero initial conditions:

$$\hbar^2\dot{n}^2 = n[4V^2 + n(4V^2 - \chi^2) - 2n^2\chi(\psi + \chi) - n^3(\psi + \chi)^2], \quad (31)$$

so that the solution can be expressed with the aid of the elliptic integral

$$\int_0^{n_0} \frac{dn}{[nP_3(n)]^{1/2}} = \frac{1}{\hbar}\int dt, \quad (32)$$

where $P_3(n)$ is the polynomial in the square brackets in (31), and n_0 is the smallest positive root of the cubic equation $P_3(n) = 0$. At least one positive root of this equation exists, since $P_3(0) > 0$ at $n = 0$ and $P_3(n) \to -\infty$ as $n \to \infty$. If χ and ψ have the same sign and $|\chi| > 2|V|$, then there is only one positive root.

Thus, periodic oscillations are produced in the system, and their period and the maximum value of n depend on the ratio of the field amplitude V to the amplitudes χ and ψ of the interactions between the magnons. Thus, at $\chi = 0$, if $|V| \ll |\psi|$, then the maximum value of the increment to the magnetization is $\Delta M \approx \mu |V/\psi|^{2/3}$, and the period of the oscillations is $T \approx \hbar(V^2|\psi|)^{-1/2}$. If the inverse inequality $|V| \gg |\psi|$ is satisfied, then the amplitude is $\Delta M \approx \mu |V|/|\psi|$, and the period is $T \approx \hbar/|V|$.

The absorbed power (per particle) is expressed in terms of the derivative $\dot{n}$:

$$\left\langle \frac{\partial \mathscr{H}}{\partial t} \right\rangle = \frac{\hbar\omega}{2}\dot{n}$$

and also varies periodically with amplitude $\frac{1}{2}\omega(V^4/|\psi|)^{1/3}$, if $|V| \ll |\psi|$. In the other limiting case, the amplitude of the absorbed power is equal to $\omega V^2/2|\psi|$.

We take the opportunity to thank I. M. Lifshitz, A. F. Andreev, M. I. Kaganov, and L. P. Pitaevskiĭ for a useful discussion.

[1)] A similar analysis for a semiconductor in the field of a strong electromagnetic wave is carried out in [8].

[2)] In a preceding paper [7] we have assumed not that ψ and χ are constant, but that they are different from zero only near the surface $2\epsilon_k = \hbar\omega$ in a layer of width V. This assumption, however, is not valid, since the magnon interaction region is not connected with the magnitude of the pump field.

[1] H. Suhl, J. Phys. Chem. Sol., 1, 209 (1957).
[2] E. Schlömann, I. I. Green, and V. Milano, J. Appl. Phys., 31, 386 (1960).
[3] Ya. A. Monosov, Nelineinyĭ magnitnyĭ rezonans (Nonlinear Magnetic Resonance), Nauka, 1971.
[4] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves), Nauka, 1967.
[5] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 59, 1200 (1970) [Sov. Phys.-JETP 32, 656 (1971)]; V. E. Zakharov and V. S. L'vov, Zh. Eksp. Teor. Fiz. 60, 2066 (1971) [Sov. Phys.-JETP 33, 1133 (1971)].
[6] J. R. Schrieffer, The Theory of Superconductivity, Benjamin, 1964.
[7] V. M. Tsukernik and R. P. Yankelevich, ZhETF Pis. Red. 17, 590 (1973) [JETP Lett. 17, 419 (1973)].
[8] V. M. Galitskiĭ, S. P. Goreslavskiĭ, and V. F. Elesin, Zh. Eksp. Teor. Fiz. 57, 207 (1969) [Sov. Phys.-JETP 30, 117 (1970)].

Translated by J. G. Adashko
226