Contribution to the theory of stimulated Raman emission in an optical resonator

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Equations are derived which describe the dynamics of stimulated Raman emission in a resonator inhomogeneously filled with matter and excited by an external light beam. Cases of traversely inhomogeneous and "weak" longitudinal filling are investigated. It is shown that in the case of transversely inhomogeneous filling the stationary generation of an arbitrary number of Stokes components possesses the same features as in the case of homogeneous filling (the difference is that the parameters of the system may change). A principally different pattern of the process emerges in the case of weak longitudinal filling. A number of stable regimes of stationary generation arise and the one that is realized depends on initial conditions in the system. General expressions are obtained for the fields of all stimulated Raman emission components in the regimes mentioned; the dependence of the number of generated components, of their intensities, and of their frequencies on the intensity of the incident external beam is investigated. The existence of regimes with similar properties in two-level optical lasers is established for weak longitudinal filling of the cavity by the active component of the medium.

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INTRODUCTION

This paper is devoted to the development of a theory of stimulated Raman scattering in an optical resonator excited by an external light beam. The beam is incident along the resonator axis on one of the partly-transparent mirrors, and excites in the resonator one of its modes. In turn, the light field of the natural oscillations of the resonator causes, when definite threshold values are exceeded, generation of light at the first, second, etc. Stokes frequencies. At typical values of the distance between the mirrors, i.e., at resonator lengths $L \sim 0.03-3$ cm, and at spontaneous Raman scattering line widths $\Delta\omega \sim 0.1-10\ \text{cm}^{-1}$, a large number of axial (longitudinal) modes can take part in each of the Stokes components in the process under consideration. As to the transverse modes, it is assumed below that they can be selected in the resonator, so that only natural oscillations with the smallest transverse numbers, which differ only in the values of the longitudinal index, can be excited. The process of stimulated Raman scattering for the first Stokes component under the indicated conditions was first considered by the present author[1]. Later on the results were generalized to the case of an arbitrary number of generated Stokes components[2,3]. The analysis in this paper was limited to the case of uniform filling of the resonator with the active medium.

In this paper, the theory is developed for the case of longitudinally and transversely non-uniform filling. In analogy with[1-3], the investigation consists in the derivation of equations describing the dynamics of the process of stimulated Raman emission under the considered conditions and then an analysis of these equations by the methods of oscillation theory. The treatment presented below shows that the transverse inhomogeneity of the filling of the resonator under the considered conditions does not change in principle the picture of the generation in comparison with the case of homogeneous filling in the sense that the system of equations describing the process of stimulated Raman emission will have in this case the same form as in the case of homogeneous filling. At the same time, in the case of longitudinally-inhomogeneous filling by the active component of the medium, the generation of the Stokes components of the stimulated Raman emission acquires fundamentally a new character, which is determined by a radically new topology of the phase space of the system.

We consider below in detail the case of "weak" longitudinal filling which can be regarded as the opposite limiting case of homogeneous filling (i.e., filling "to the brim"). One of the features of the process of stimulated Raman emission in the case of weak longitudinal filling is the existence of a stable regime of monochromatic generation on one of the resonator modes, in spite of the fact that the threshold of the self-excitation at other modes, when taken separately, can be lower. This phenomenon can be called self-capture of the mode. We obtain below general expressions for the fields of all the components of the stimulated Raman emission in stationary regimes of monochromatic generation, and investigate the dependence of the number of stable regimes, of the number of generated components, of their intensities, and of their frequencies on the intensity of the incident external beam.

1. DERIVATION OF INITIAL EQUATIONS

Inasmuch as we consider below the interaction of an arbitrary number of Stokes components of stimulated Raman emission and bear it in mind that the field of each of these components is itself made up of an arbitrary number of resonator modes (whose natural frequencies are close to the frequency of the given component), it is necessary to retain all the terms in the expansion of the electric field in the resonator in the oscillation modes:

$$\mathbf{E} = \sum_l \mathbf{E}^{(l)}, \quad \mathbf{E}^{(l)} = \sum_s \mathscr{E}_{ls}(t)\mathbf{E}_{ls}(\mathbf{r}) \quad . \qquad (1)$$

Here l is the order of the component of the stimulated Raman emission, s is the number of the axial mode with natural frequency ω_{ls} close to the frequency of the component of order l ($\omega_l = p + l\omega_r$, p is the "central" frequency of the exciting beam, ω_r is the frequency of the vibrational transition of the medium), $\mathbf{E}_{ls}$ is the axial mode of the oscillations of natural frequency ω_{ls}, the spectrum of the function $\mathbf{E}^{(l)}$ is concentrated near

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the frequency ω_l (we describe the exciting radiation here as a component with index $l = 0$).

For the sake of argument, we consider a resonator with plane-parallel mirrors, bearing in mind here that the linear component of the medium filling the space between the mirrors can also be homogeneous in the transverse direction with respect to the resonator axis z. If the zero value of z corresponds to the surface of one of the mirrors, then the expression for the axial modes can be written in the form (see[4,5])

$$\mathbf{E}_{ls}(\mathbf{r}) = g_{ls}(\mathbf{r}_\perp)\sin k_{ls}z, \quad k_{ls} = \pi m_{ls}/L, \tag{2}$$

where L is the distance between mirrors, m_{ls} is an arbitrary (large) integer. The natural frequency corresponding to a mode of this type is defined by

$$\omega_{ls} = \pi c m_{ls}/L n_{\text{eff}}(\omega_{ls}), \tag{3}$$

where c is the speed of light in vacuum, the function $n_{\text{eff}}(\omega)$ is a specified function of ω for a given type of transverse inhomogeneity of the linear component and is determined by the dispersion of the substance, i.e., by the frequency dependence of its refractive index $n(\omega)$, and by the concrete form of the indicated transverse inhomogeneity.

In the case of greatest practical interest, when the characteristic scale of variation of $\mathbf{g}_{ls}$ as a function of $\mathbf{r}_\perp$ is much larger than the length $2\pi/k_{ls}$ of the light wave, and when the values of ω_{ls} correspond to regions of transparency of the linear component of the medium, the $n_{\text{eff}}(\omega)$ dependence in the vicinity of the considered frequencies is usually weak. As to the numbering of the modes in (1), it can obviously be arbitrary. For convenience in the exposition, however, we adopt here a special numbering such that the following inequalities hold for l, s, l', and s'

$$m_{l's'} - m_{ls} = (l' - l)\Delta m + s' - s, \tag{4}$$

where Δm is an integer that does not depend on l, s, l', and s' (convenient values are $\Delta m \sim (\pi\omega_r/cL)n_{\text{eff}}(p)$).

The sought system of equations describing the change of the quantities $\mathscr{E}_{ls}$ should be determined from the equations of the field in the resonator and from the material equation of the medium. The field equations can be written in the form of the following system of ordinary differential equations for the coefficients $\mathscr{E}_{q\tau}$ from the expansion of this field in the modes $\mathbf{E}_{q\tau}$ and $\mathbf{H}_{q\tau}$ (see[4,6]):

$$\ddot{\mathscr{E}}_{q\tau} + \frac{\omega_{q\tau}}{Q_{q\tau}}\dot{\mathscr{E}}_{q\tau} + \omega_{q\tau}^2\mathscr{E}_{q\tau} = \frac{\omega_{q\tau}^2}{N_{q\tau}}\int(\mathbf{P}\mathbf{E}_{q\tau} - \mathbf{M}\mathbf{H}_{q\tau})\,dV, \tag{5}$$

where

$$N_{q\tau} = \frac{1}{4\pi}\int \varepsilon \mathbf{E}_{q\tau}^2\,dV = -\frac{1}{4\pi}\int \mu \mathbf{H}_{q\tau}^2\,dV$$

is the norm of the $(\mathbf{E}_{q\tau}, \mathbf{H}_{q\tau})$ mode; $Q_{q\tau}$ is its figure of merit; ϵ and μ are the dielectric constant and magnetic permeability of the linear component of the medium; $\mathbf{P} = \mathbf{P}(\mathbf{r}, t)$ and $\mathbf{M} = \mathbf{M}(\mathbf{r}, t)$ are the additional parts of the polarization and magnetization of the medium with respect to those principal parts which are taken into account in the definition of the modes. In our case, $\mathbf{P} = \mathbf{P}_{\text{nl}} + \mathbf{P}_{\text{ext}}$, where $\mathbf{P}_{\text{nl}}$ is the nonlinear form of the polarization, determined by the active component of the medium in the resonator; $\mathbf{P}_{\text{ext}} = \text{Re}(\mathbf{P}_0 e^{-ipt})$ and $\mathbf{M}_{\text{ext}} = \text{Re}(\mathbf{M}_0 e^{-ipt})$ are the specified extraneous polarization and magnetization, and are determined by the exciting beam.

The equation for the nonlinear part of the polarization, i.e., the material equation of the medium that is active in the Raman spectrum, is (see, e.g.,[7,8])

$$\mathbf{P}_{\text{nl}} = Nx\frac{d\alpha}{dx}\mathbf{E}, \quad \ddot{x} + 2h\dot{x} + \omega_r^2 x = \frac{1}{m}\frac{d\alpha}{dx}E^2. \tag{6}$$

The constant coefficient $d\alpha/dx$ here is the derivative of the polarizability of the molecule of the medium with respect to the normal coordinate x of the displacements of the nuclei, m is the reduced mass corresponding to this normal coordinate, h is the half-width of the line of the spontaneous Raman scattering ($h \ll \omega_r$), and N is the density of the molecules of the medium. In the case of inhomogeneous filling of the resonator by the active component of the medium, the density N depends on $\mathbf{r}$.

Substituting (1) in (6) and omitting the terms that are nonresonant with respect to the oscillator in the left-hand side, we can easily verify that x can be represented in the form

$$x = \sum_{lss'} x_{lss'}(\mathbf{E}_{ls}\mathbf{E}_{l+1,s'}), \tag{7}$$

where the functions $x_{lss'}$ satisfy the equations

$$\ddot{x}_{lss'} + 2h\dot{x}_{lss'} + \omega_r^2 x_{lss'} = \frac{1}{m}\frac{d\alpha}{dx}\mathscr{E}_{ls}\mathscr{E}_{l+1,s'}. \tag{8}$$

Substituting further (1) and (7) in the first equation of (6), we easily obtain an expression for $\mathbf{P}_{\text{nl}}$, which, taking (5) into account, yields the following system:

$$\begin{aligned}&\ddot{\mathscr{E}}_{q\tau} + \frac{\omega_{q\tau}}{Q_{q\tau}}\dot{\mathscr{E}}_{q\tau} + \omega_{q\tau}^2\mathscr{E}_{q\tau} = \frac{\omega_{q\tau}^2}{N_{q\tau}}\text{Re}(f_{q\tau}e^{-ipt})\\ &+\frac{d\alpha}{dx}\frac{\omega_{q\tau}^2}{N_{q\tau}}\sum_{l's'lss''}\mathscr{E}_{l's'}x_{lss''}\int N(\mathbf{r})(\mathbf{E}_{q\tau}\mathbf{E}_{l's'})(\mathbf{E}_{ls}\mathbf{E}_{l+1,s''})\,dV,\end{aligned} \tag{9}$$

which forms together with (8) a closed system of equations for the temporal functions $\mathscr{E}_{q\tau}$ and $x_{lss''}$. We have put here

$$f_{q\tau} = \int(\mathbf{P}_0\mathbf{E}_{q\tau} - \mathbf{M}_0\mathbf{H}_{q\tau})\,dV = \frac{ic}{4\pi p}\int(\mathbf{E}_0\mathbf{H}_{q\tau})\,dS, \tag{10}$$

$\mathbf{E}_0$ is the complex amplitude of the electric field vector of the exciting beam passing through one of the slightly transparent mirrors (on which this beam is directly incident), under the condition that there is no second mirror (and by the same token no resonator); the integration in the last expression of (10) is with respect to the internal (relative to the resonator) surface of the first mirror. However, for an arbitrary form of the inhomogeneity of the active component of the medium, i.e., for an arbitrary form of the function $N(\mathbf{r})$, the system (8), (9) is quite complicated. We therefore confine ourselves below to two cases: 1) transverse inhomogeneity ($N = N(\mathbf{r}_\perp)$) and 2) inhomogeneity corresponding to weak longitudinal filling of the resonator by the active component of the medium (see (12)).

In the first case, the system (9) is simplified because some of the coefficients that are contained in it (in the form of volume integrals) vanish. Starting from (2) and (4), we can verify that in this case Eqs. (8) and (9) reduce to the system considered in[2], in which it is only necessary to replace formally the parameters in the following manner:

$$\begin{aligned}&n(\omega) \to n_{\text{eff}}(\omega),\\ &N\int(\mathbf{E}_{q\tau}\mathbf{E}_{l's'})(\mathbf{E}_{ls}\mathbf{E}_{q'\tau'})\,dV \to \int N(\mathbf{r}_\perp)(\mathbf{E}_{q\tau}\mathbf{E}_{l's'})(\mathbf{E}_{ls}\mathbf{E}_{q'\tau'})\,dV.\end{aligned} \tag{11}$$

All the subsequent transformations of the so-obtained system, and the conclusions of[1-3], with allowance for (11), are directly applicable also to the considered case of transversely inhomogeneous filling (both with respect to the linear and with respect to the active components of the medium).

The regimes of stationary oscillations of the field and their stability were investigated in[3], on the basis of the system of[2], for the case of a sufficient dispersion of the refractive index $n(\omega)$, for example such as the dispersion of the refractive index of a liquid or a solid. Therefore the results of[3] are valid also for the case of a transversely-inhomogeneous filling of the resonator, provided that the dispersion of $n_{\text{eff}}(\omega)$ is relatively large (for example, of the same order as the dispersion of liquids or solids). In[2], in addition, a system of equations was obtained, valid in the case of very weak dispersion of $n(\omega)$, for example such as the dispersion of the refractive index of low-pressure gases. The corresponding system was considered in the subsequent paper[9], where it was shown that in the case of stimulated Raman emission in gases, various field components are in general synchronized with one another, so that the total output radiation from the resonator constitutes a sequence of ultrashort (subpicosecond) pulses. In accordance with the foregoing, the results of[9] are valid also in the case of transversely-inhomogeneous filling, including liquid or a solid, if the dispersion of the quantity $n_{\text{eff}}(\omega)$ is small enough (a quantitative criterion is given[2], for Eq. (11) must also be taken into account). Thus, we arrive at the conclusion that in the case of sufficiently low dispersion of $n_{\text{eff}}(\omega)$, the use of a transversely-inhomogeneous medium (for example, fiber optics[5]) will make feasible the generation of subpicosecond pulses by stimulated Raman emission in condensed media.

We consider now the second case, corresponding to longitudinal inhomogeneity of the active component of the medium, i.e., we put $N = N(\mathbf{r}_\perp, z)$, and bear in mind the fact that the values of N differ from zero only in the interval $0 < z < \Delta L$ (or $L - \Delta L < z < L$), and that the longitudinal dimension ΔL satisfies the conditions

$$\Delta L \ll L, \quad \frac{h}{\omega_l} k_l \Delta L \ll 1, \quad (l=-1, -2\ldots), \tag{12}$$

where $k_l = c^{-1}\omega_l n_{\text{eff}}(\omega)$; the linear component of the medium can in this case be transversely inhomogeneous. The inequalities (12) will be called the conditions of weak longitudinal filling. We fix one each of the axial modes of the resonator oscillations in the vicinity of each of the frequencies ω_l within the limits of the line width of the spontaneous Raman emission, and label these arbitrarily chosen modes with the index ν_l. The conditions (12) mean that in the region occupied by the active medium we can put $\sin k_{ls} z \approx \sin k_{l\nu_l} z$. Equations (8) and (9) then simplify because all the nonlinear terms in them can be represented in the form of the products $\mathscr{E}_l \mathscr{E}_{l+1}$ and linear combinations of the products $\mathscr{E}_q x_l$, where

$$\mathscr{E}_l = \sum_s \mathscr{E}_{ls}, \quad x_l = \sum_{ss'} x_{lss'}, \tag{13}$$

With the quantities $x_{lss'}$ entering in this system only via x_l. The system obtained in this manner (which we do not write out here for lack of space) differs significantly in form from the equations describing the considered process of stimulated Raman emission for the case of a homogeneous medium (see[2]), and this in turn leads to principal differences in the dynamics of the process. Our problem is to investigate the solutions corresponding to harmonic (in each of the components) oscillations of the field and their stability.

We note first that in all the cases of practical interest the obtained system is close to conservative and its solutions can be sought in the form

$$\begin{gathered}\mathscr{E}_{q\tau} = {}^1/_2 Y_{q\tau} \exp[-i(\Omega_{q\nu_q} - \Delta_q)t] + \text{c.c.},\\ x_l = {}^1/_2 X_l \exp[-i(\Omega_{l+1,\nu_{l+1}} - \Omega_{l\nu_l} + \Delta_l - \Delta_{l+1})t] + \text{c.c.},\end{gathered} \tag{14}$$

With the quantities $Y_{q\tau}$ and X_l regarded here as "slowly varying amplitudes." Here t means the dimensionless time pt, and we have introduced the symbol

$$\Omega_{ls} = \omega_{ls}/p. \tag{15}$$

Substituting the expressions (14) in the system under consideration and averaging, as is customary in the Van der Pol method over the fast oscillations, we can obtain a system of equations for the quantities $Y_{q\tau}$ and X_l; this system, in turn, can be written in the form

$$\dot{Y}_{q\tau} = G(Y_{q's}, X_l), \quad \beta \dot{X}_l = H(Y_{q''}, X_l), \tag{16}$$

where

$$Y_q = \sum_{s'} Y_{qs'}, \tag{17}$$

$G \sim H$; the parameter β is small under the conditions

$$\mu_{ls} \ll \mu, \quad |\Delta_q| \ll \mu \tag{18}$$

and at not too large excess over the generation threshold. We have put here

$$\mu_{ls} = \omega_{ls}/2pQ_{ls}, \quad \mu = h/p. \tag{19}$$

The first condition of (18) is of greatest practical interest and will henceforth be assumed satisfied. It means that the width of the resonance curve of each of the considered resonator modes is much smaller than the width of the spontaneous Raman scattering line. The second condition, as will be seen below, is also satisfied if the values of $\Omega_{q\nu_q}$ are correctly chosen or, upon correct choice of the "zeroth approximation" eigenfunctions numbered by the indices $q_\nu q$.

Bearing in mind (18) and assuming here that $\beta \dot{X}_l = 0$, we can write in explicit form the system of equations for the quantities Y_{ls}:

$$\dot{Y}_{q\tau} = G(Y_{q's}, X_l(Y_{q''})); \tag{20}$$

The nonlinear terms of the right-hand side take here the form of linear combinations of the products $Y_l^* Y_{l+1} Y_{q-1}$, $Y_l Y_{l+1}^* Y_{q+1}$, the sum of these terms having factors that depend explicitly on the time

$$\exp[i(\Omega_{ql}^{(1,2)} - \Delta_{ql}^{(1,2)})t], \tag{21}$$

where

$$\Delta_{ql}^{(1)} = \Delta_q + \Delta_l - \Delta_{q-1} - \Delta_{l+1}, \quad \Delta_{ql}^{(2)} = \Delta_q + \Delta_{l+1} - \Delta_l - \Delta_{q+1}. \tag{22}$$

The values of $\Omega_{ql}^{(1,2)}$ (the expressions for which we do not write out), are determined by the function $n_{\text{eff}}(\omega)$. At $dn_{\text{eff}}/d\omega \equiv 0$, these quantities are equal to zero at all values of q and l. On the other hand, if $dn_{\text{eff}}/d\omega \equiv 0$, then they are in general different from zero (with the exception of $\Omega_{q,q-1}^{(1)}$ and $\Omega_{qq}^{(2)}$, which vanish identically). At a sufficient dispersion of $n_{\text{eff}}(\omega)$, for example on the order of that of the refractive index of liquids or solids, i.e., at $dn_{\text{eff}}/d\omega \gtrsim 10^{-17}$ sec/rad, the nonzero quantities $|\Omega_{ql}^{(1,2)}|$ greatly exceed the typical values of $\mu_{q\tau}$, which determine in turn, at not too large excesses above the generation threshold, the characteristic times of variation of the amplitudes Y_q. If it is recognized at the same time that corrections Δ_q to the correctly chosen frequencies $\Omega_{q\nu_q}$ are usually smaller than or do not exceed significantly the values $\mu_{q\tau}$ (see (33)), then we see that in the case of sufficient dispersion of $n_{eff}(\omega)$ (it is precisely this case which will be considered below), some of the indicated terms in the right-hand side of (20) are, on account of the factors (21), rapidly oscillating in time and can be left out. As a result, Eqs. (20) simplify and take the form

$$\dot{Y}_{q\tau}=-[\mu_{q\tau}+i(\Delta_q+\Omega_{q\tau}-\Omega_{q\nu_q})]Y_{q\tau}+(\delta_{q\tau}|Y_{q+1}|^2-\rho_{q\tau}|Y_{q-1}|^2)Y_q +iF_{0\tau}\delta_{0q}\exp[i(\Omega_{0\nu_0}-\Delta_0-1)t], \quad (23)$$

where

$$\delta_{q\tau}=\frac{2\eta\gamma_{q\tau}a_q}{\mu-i(\Theta_q+\Delta_{q+1}-\Delta_q)}, \quad \rho_{q\tau}=\frac{2\eta\gamma_{q\tau}a_{q-1}}{\mu+i(\Theta_{q-1}+\Delta_q-\Delta_{q-1})},$$

$$a_q=\int N(\mathbf{r})(\mathbf{E}_{q\nu_q},\mathbf{E}_{q+1,\nu_{q+1}})^2\,dV, \quad \Theta_q=\Omega_{q\nu_q}+\Omega-\Omega_{q+1,\nu_{q+1}}$$

$$\eta=\frac{1}{4mp^2\Omega}\frac{d\alpha}{dx}, \quad \Omega=\frac{\omega_r}{p}, \quad \gamma_{q\tau}=\frac{1}{4}\frac{d\alpha}{dx}\frac{\Omega_{q\tau}}{N_{q\tau}}, \quad (24)$$

$$F_{0\tau}=\frac{\Omega_{0\tau}^2}{2N_{0\tau}}f_{0\tau},$$

δ_{0q} are Kronecker symbols. Equations (17) compliment Eqs. (23) to form a closed system. The values of $Y_{q\tau}$ at $q > 0$ in (23) must be set equal to zero, since the excitation of the corresponding (anti-Stokes) components generally takes place as a result of terms with factors (21). Omitting the corresponding terms, we neglect by the same token, in particular, the fields of the indicated components.

It should be noted that the coefficients in (23), just as in (16) and (20), depend in fact on prescribed values of ν_q. By the same token, each of the indicated systems actually is an aggregate of systems corresponding to different sets of ν_q. The investigation of the phase space of the system (17) and (23), at a fixed series of values of ν_q, would not give correct results on the whole, since this space does not correspond on the whole to the phase space of the initial system. The latter, in turn, is connected with the fact that on going from Eqs. (16) and (20) to Eqs. (23) we have assumed that the frequency corrections Δ_q to the values of $\Omega_{q\nu_q}$ are relatively small (at a given scale of the time variation of the amplitudes Y_q), which in turn can be satisfied for the vicinities of various regimes only by specifying different sets of ν_q.

In accordance with the foregoing, under the considered conditions (12), at a sufficient dispersion of $n_{eff}(\omega)$, the problem reduces to an investigation of the equilibrium states of the system (17) and (23) for different sets of ν_q. These states, by virtue of (14) correspond to harmonic oscillations (in each of the components) of the field in the initial system.

2. HARMONIC OSCILLATIONS

Let the exciting beam be monochromatic and let its frequency p be close to one of the natural frequencies $\omega_{0\nu_0}$. We assume that Δ_0 is equal to

$$\Delta_0=\Omega_{0\nu_0}-1. \quad (25)$$

The equilibrium positions of the system (17), (23) are determined from the condition $\mathring{Y}_{q\tau} \equiv 0$, which yields a system of algebraic equations for the time-independent quantities $Y_{q\tau} \equiv \tilde{Y}_{q\tau}$. The algebraic system of equations obtained in this manner for $\tilde{Y}_{q\tau}$ can be easily reduced to a system of equations for the quantities

$$\tilde{Y}_q=\sum_\tau \tilde{Y}_{q\tau}.$$

We have

$$\tilde{Y}_q[1-G_q(\Delta_q)(\delta_q|\tilde{Y}_{q+1}|^2-\rho_q|\tilde{Y}_{q-1}|^2)]=Z_0\delta_{0q}, \quad (26)$$

where we have introduced the notation

$$Z_0=iG_0(\Delta_0)F_{0\nu_0}, \quad G_q(\Delta_q)=-i\sum_s\frac{1}{\Delta_q-i\mu_{qs}+s\Delta\Omega_{ax}^{(q)}},$$

$$\Delta\Omega_{ax}^{(q)}=\frac{\pi c}{pLn_{eff}(\omega_q)}, \quad \delta_q=\delta_{q\nu_q}, \quad \rho_q=\rho_{q\nu_q} \quad (27)$$

and we have put $\delta_{q\tau} = \delta_{q\nu_q}$ and $\rho_{q\tau} = \rho_{q\nu_q}$, since usually the indicated quantities δ and ρ are slow functions of τ. By virtue of (10) and (24), the quantity G_0 is determined by the complex amplitude of the exciting beam, so that the values of $|Z_0|^2$ are proportional to the intensity of this beam. At a given set of values of linear losses μ_{qs} of each of the modes, the quantity G_q is a specified function of Δ_q. For example, if these losses do not depend on the index s ($\mu_{qs} \equiv \mu_q$), then we have

$$G_q(\Delta_q)=\frac{\pi}{i\Delta\Omega_{ax}^{(q)}}\operatorname{ctg}\left[\frac{\pi}{\Delta\Omega_{ax}^{(q)}}(\Delta_q-i\mu_q)\right].$$

The system (26) with respect to the Stokes components determines the values of $|\tilde{Y}_q|^2$, Δ_q ($q = -1, -2, \ldots$). The phase shifts of the oscillations of each of the Stokes components are arbitrary, i.e., the initial system is autonomous with respect to each of these components. We see that Eqs. (26), for any set of values of ν_q, is a system of infinite order.

Assuming that the mirror reflection coefficients at the frequencies of the considered components are close to unity, we find that all the resonator modes that take part in the stimulated Raman emission satisfy the conditions

$$\mu_{qs} \ll \Delta\Omega_{ax}^{(q)} \quad (28)$$

Recognizing also that in this case $|\Delta_q| \ll \Delta\Omega_{ax}^{(q)}$ (see (33)), we can put

$$G_q(\Delta_q)\approx 1/(\mu_{q\nu_q}+i\Delta_q). \quad (29)$$

In addition, taking (18) into account, the values of Δ_q in the expressions for δ_q and ρ_q can be set equal to zero.

We present the solution of the system (26), which is valid for the case when $G_q(\Delta_q)$ is determined by (29). The form of this solution depends on the values of $|Z_0|^2$, so that the straight line $|Z_0|^2 \geq 0$ can be broken up into a number of intervals

$$|Z_0|_m^2(\{\nu_q\})<|Z_0|^2<|Z_0|_{m-1}^2(\{\nu_q\})$$

(m is an arbitrary negative integer), in which this form remains unchanged. Each interval of the values of $|Z_0|^2$, corresponding to a given value of m, is characterized by the fact that $|m|$ Stokes components are simultaneously generated in it (i.e., $\tilde{Y}_q \neq 0$ at $q = -1, -2, \ldots m$ and $\tilde{Y}_q = 0$ at $q < m$). In the corresponding interval, we have for odd m

$$|\tilde{Y}_q|^2=\begin{cases}\dfrac{1}{\alpha^{(q)}}\left[\Psi\left(\dfrac{|Z_0|^2}{\beta^{(m-1)}}\right)-\beta^{(q)}\right], & q=-1,-3,\ldots \geqslant m\\[2ex] \dfrac{1}{\alpha^{(q)}}(\beta^{(m-1)}-\beta^{(q)}), & q=0,-2,-4,\ldots > m\end{cases} \quad (30)$$

For even m, other expressions are valid:

$$|\tilde{Y}_q|^2 = \begin{cases} \dfrac{1}{\alpha^{(q)}}(\beta^{(m-1)}-\beta^{(q)}), & q=-1,-3,\ldots > m \\ \dfrac{1}{\alpha^{(q)}}[|Z_0|^2 D(\beta^{(m-1)})-\beta^{(q)}], & q=0,-2,-4,\ldots \geqslant m \end{cases}, \qquad (31)$$

where

$$\beta^{(0)}=\beta^{(-1)}=0, \quad \beta^{(-2)}=\beta_{-1}, \quad \beta^{(-3)}=\beta_{-2}, \quad \alpha^{(0)}=\alpha^{(-1)}=1,$$

$$\beta^{(q)} = \begin{cases} \beta\left(\dfrac{|q|}{2}, q\right), & q=-4,-6\ldots \\ \beta\left(\dfrac{|q|-1}{2}, q\right), & q=-5,-7\ldots \end{cases}, \quad \alpha^{(q)} = \begin{cases} \alpha\left(\dfrac{|q|}{2}, q\right), & q=-2,-4\ldots \\ \alpha\left(\dfrac{|q|-1}{2}, q\right), & q=-3,-5.. \end{cases}$$

$$\beta(k,q)=\sum_{l=1}^{k-1}\left(\prod_{l'=l+1}^{k}\alpha_{q+2l'-1}\right)\beta_{q+2l-1}+\beta_{q+2k-1},$$

$$\alpha(k,q)=\prod_{l=1}^{k}\alpha_{q+2l-1}, \qquad (32)$$

$$\Psi(y)=|\rho_0|^{-2}\{-(\mu_0\rho_0'+\Delta_0\rho_0'')+[|\rho_0|^2(\mu_0^2+\Delta_0^2)(y-1)+ +(\mu_0\rho_0'+\Delta_0\rho_0'')^2]^{1/2}\},$$

$$D(y)=\frac{\mu_0^2+\Delta_0^2}{(\mu_0+\rho_0'y)^2+(\Delta_0+\rho_0''y)^2},$$

$$\beta_q=\mu_q/\delta_q', \quad \alpha_q=\rho_q'/\delta_q', \quad \mu_q=\mu_{q\nu_q}$$

(δ', ρ', and δ'', ρ'' are the real and imaginary parts of the numbers δ and ρ; here and below it is implied that the expressions (24) and (27) for these quantities are taken formally at $\Delta_{q+1} = \Delta_q = \Delta_{q-1} = 0$). The frequency corrections Δ_q for arbitrary m are given by the formulas

$$\Delta_q=\frac{\Theta_q}{\mu}\mu_q+\frac{\Theta_q+\Theta_{q-1}}{\mu}\rho_q'|\tilde{Y}_{q-1}|^2 \qquad (33)$$

and the initial amplitudes $\tilde{Y}_{q\tau}$ are determined by the expressions

$$\tilde{Y}_{q\tau}=G_q^{-1}(\Delta_q)[\mu_{q\tau}+i(\Delta_q+\Omega_{q\tau}-\Omega_{q\nu_q})]^{-1}\tilde{Y}_q. \qquad (34)$$

We present also expressions, which follow from (31) and (30), for the boundary conditions $|Z_0|_m^2(\{\nu_q\})$ that determine the excitation thresholds of the generation regimes $|m|$ of the Stokes components:

$$|Z_0|_m^2(\{\nu_q\})=\begin{cases}\beta^{(m-1)}/D(\beta^{(m)}), & m=-1,-3,-5,\ldots \\ \beta^{(m)}/D(\beta^{(m-1)}), & m=-2,-4,-6,\ldots\end{cases}. \qquad (35)$$

We see that according to (24), (27), and (30)—(34), all the considered regimes that are possible in the initial system at fixed values of its parameters have different sets of values of ν_q, and these values of ν_q can be specified arbitrarily. Expression (34) is exact in this case. Expressions (30)—(33) and (35) are the lowest approximation in terms of the parameter $\mu_q/\Delta\Omega_{ax}^{(q)}$ (see (28)). The corresponding higher-order approximations will not be written out here. The presented expressions make it possible, in particular, to estimate the total number of the possible regimes in the initial system at given values of its parameters, and to classify these regimes in accordance with the number of the generated components, by starting from the fact that the oscillation regimes with participation of $|m|$ Stokes components correspond only to those sets of ν_q for which

$$|Z_0|_m^2(\{\nu_q\})<|Z_0|^2.$$

It is interesting to note that according to (1), (14), and (34), the spatial structure of the field in each of the considered regimes is determined for each of the components by a linear combination of the coordinate parts of $E_{q\tau}$ of different resonator modes. Nonetheless, we shall refer to these regimes subsequently (although conditionally) as single-mode, bearing in mind the fact that when (28) is taken into account the dominant role is assumed in the indicated combination, for a given value of ν_q, by the mode $E_{q\nu_q}$. No such situation arises in the case of homogeneous filling of the resonator by the active component of the medium, when the regimes of the monochromatic generations are strictly single-mode (see[1,3]).

We consider now the general character of the dependence of the intensities of the oscillations $|\tilde{Y}_q|^2$ on the excitation intensity. With the aid of (30) and (31) we can easily see that under the conditions (28), when an odd number $|m|$ of Stokes components is generated, the intensity of the oscillations of all the even components, including the exciting one, is practically independent of the intensity of the beam incident from the outside. The intensities of the oscillations of all the odd components increase monotonically with increasing incident intensity. At the same time, when an even total number of Stokes components is generated, it is the intensities of the oscillations of all the odd components which turn out to be practically independent of the intensity of the light incident from the outside; the intensities of the oscillations of all the even components, including the exciting one, increase monotonically with increasing intensity of the exciting beam.

As to the corrections Δ_q to the natural frequencies of the resonator oscillations, they are due according to (33) to a change in the refractive index of the medium in the strong field, and at not too large excesses over threshold, i.e., at not too large $|\tilde{Y}_{q-1}|^2$ they can be seen to be usually less than or not much larger than the values of μ_q. We see also that at $|\Theta_q|$, $|\Theta_{q-1}| \ll \mu_q$ or else at $|\Theta_q + \Theta_{q-1}| \ll \mu_q$ the values of $|\Delta_q|$ cannot exceed the values of μ_q even at very large values of $|\tilde{Y}_{q-1}|^2$.

3. STABILITY OF HARMONIC OSCILLATIONS

In the case of homogeneous filling of the resonator, the only stable stationary regime is the one characterized by the fact that in each Stokes component there is excited a mode with the lowest self-excitation threshold (see[1,3]). In the considered case of weak longitudinal filling, such a regime can be set in correspondence only to one of all the possible sets of ν_q. The remaining aggregate $\{\nu_q\}$ corresponds to regimes in which the principal role is played by the oscillation modes with higher self-excitation thresholds. We shall show that these regimes, in the case of a weak longitudinal filling, turn out also to be stable, and furthermore in a wide range of the system parameters. For simplicity we confine ourselves to the regime in which one first Stokes component is generated, with allowance for the two modes in it ($\tau = \nu_{-1}$, 0). We assume also that

$$\Delta=0; \quad |\Omega_{-1\nu_{-1}}-1|, |\Omega_{-10}-1| \ll \mu; \qquad (36)$$

$$\mu_{-1\tau} \ll |\Omega_{-1\nu_{-1}}-\Omega_{-10}|, \mu_0.$$

This means that the frequency of the optical oscillations in the exciting beam coincides exactly with the natural frequency $\Omega_{0\nu_0}$ of the resonator. Both natural frequencies of the oscillations in the Stokes component lie near the center of the line of the spontaneous Raman scattering; the resonance curves of the corresponding modes are much narrower than the frequency interval between them and than the frequency width of the mode at the exciting-beam frequency. Under these conditions, Eqs. (23) reduce to the system

$$\dot{Y}_{-1\tau}=-[\mu_{-1\tau}+i(\Delta_{-1}+\Omega_{-1\tau}-\Omega_{-1\nu_{-1}})]Y_{-1\tau}+\delta'_{-1}\frac{Y_{-1}}{(1+\alpha|Y_{-1}|^2)^{\varkappa}}, \quad (37)$$

where $\alpha = G_0(0)\rho_0' > 0$; $\tau = \nu_{-1}, 0$; $\kappa = 2$. We shall regard the case $\mu_{-1\nu_{-1}} \neq \mu_{-10}$, as possible, so that the self-excitation thresholds of the considered modes, when taken separately, can differ significantly.

Linearizing the system (17), (37) in the vicinity of the considered equilibrium position, we obtain the following expression for the characteristic polynomial $D(\lambda)$ that determines the stability of the corresponding generation regime:

$$D(\lambda)=\lambda^3+2\lambda^2\left(\frac{4\alpha\mu_{-1\nu_{-1}}|\bar{Y}_{-1}|^2}{1+\alpha|\bar{Y}_{-1}|^2}+\mu_{-10}-\mu_{-1\nu_{-1}}\right)+\lambda(\Omega_{-1\nu_{-1}}-\Omega_{-10})^2+4(\Omega_{-1\nu_{-1}}-\Omega_{-10})^2\frac{\alpha\mu_{-1\nu_{-1}}|\bar{Y}_{-1}|^2}{1+\alpha|\bar{Y}_{-1}|^2}. \quad (38)$$

Applying next the Routh-Hurwitz criterion, we obtain the stability condition

$$\mu_{-10}>\mu_{-1\nu_{-1}}(1-\alpha|\bar{Y}_{-1}|^2)/(1+\alpha|\bar{Y}_{-1}|^2). \quad (39)$$

We see that if the intensity of the oscillations $|\tilde{Y}_{-1}|^2$ exceeds the value of α^{-1} (this corresponds to an approximate two fold excess above the generation threshold for the mode ν_{-1}), then the condition (39) is satisfied at any ratio of $\mu_{-1\nu_{-1}}$ and μ_{-10}, i.e., for any ratio of the self-excitation thresholds of the considered mode. This means, in particular, that the generation regime in which the predominent role is played by the mode with the higher self-excitation threshold turns out to be stable, in spite of the presence, within the limits of the line width, of spontaneous Raman scattering of the type of oscillations with a smaller (or even much smaller) self-excitation threshold. In the opposite case $|\tilde{Y}_{-1}|^2 < \alpha^{-1}$, the stability conditions of the similar regime, according to (39), is determined by the ratio of the excess above the generation thresholds at the mode ν_{-1} and the ratio of both thresholds. We see that the described regime, which we shall call the "mode self-capture" regime, is observed in a wide range of variation of system parameters and its degree of stability is larger the larger the excess above the generation threshold.

Thus, in the considered case of weak longitudinal filling of the resonator, it becomes possible to observe a large number of stable oscillation regimes obtained at fixed values of the system parameters and at various sets of the values of ν_q. The realization of any particular regime will be determined by the initial conditions in the system. If the number of generating components is $|m|$ and the number of axial modes within the limit of the line width of the spontaneous Raman scattering is n, then the number of such regimes can be estimated at $n^{|m|}$, i.e., this number can be quite large. For example, at $n = |m| = 10$ it amounts to 10^{10}. Because of this, such systems can find use as elements of optical memory, optical logical devices, etc.

We note in conclusion that equations of the type (37) can be obtained also for weak longitudinal filling of the resonator by a two-level active medium under the condition that the width of the resonance curve of each of the resonator modes is much smaller than the values of $T_\perp^{-1}$ and $T_\parallel^{-1}$, where $T_\perp$ and $T_\parallel$ are the transverse and longitudinal relaxation times for the considered transition of the substance. In this case it is necessary to put $\kappa = 1$ in (37). As applied to the case of a two-level medium, the conditions of weak longitudinal filling will take the form (12), where ω_l and h must be taken to mean respectively the frequency (ω_L) and the half-width ($T_\perp^{-1}$) of the luminescence line. By direct calculations on the basis of Eq. (37) with $\kappa = 1$ we can verify that the mode self-capture regime is here likewise stable in a wide range of system parameters. Thus, the considered regime should be observed also in the case of a two-level weak longitudinal filling of the resonator, i.e., in the corresponding two-level quantum generator. It is clear also that a stable mode self-capture regime should be observed also in a Raman laser with a traveling pump wave in the case of weak longitudinal filling of the resonator. The appearance of this regime is possible also in a homogeneously-filled Raman laser, if the excess above the generation threshold is large enough. In the latter case, the role of the required inhomogeneity of the active medium can be played by the longitudinal distribution of the intensity of the exciting beam.

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