Dynamics and statistics of a system of vortices

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We consider a system of linear vortices as a model of two-dimensional turbulence. We find an exact solution of the problem of the interaction of three vortices—the first elementary interaction process in isotropic turbulence kinetics. This problem is used as an example to trace the tendency for energy transfer along the spectrum from small to large scale lengths in two-dimensional flow. We obtain a stationary solution of the equation for the generating functional for a system of vortices which corresponds to an energy spectrum $E(k) = \kappa^2 \sigma/4\pi k$ (κ is the intensity and σ the density of the vortex distribution). We derive from an analysis of the hierarchy of equations for s-vortex distributions a number of properties of the fine structure of nonuniform turbulent flow.

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In two-dimensional turbulent flow the energy has, in contrast to the three-dimensional case, a tendency to be transferred from small to large scale lengths. A number of theoretical papers[1-4] noted the possibility of such an effect. This effect was directly observed in numerical experiments on two-dimensional turbulence [5] and in laboratory studies with a conducting liquid in a strong magnetic field. [6] A similar effect in meteorology is sometimes called "negative viscosity" [7] and it manifests itself, roughly speaking, in the fact that cyclones feed the general circulation of the atmosphere.

The phenomenon of a directed transfer of energy along the spectra of three- and two-dimensional turbulence (in the opposite direction of the wavenumber axis) was explained earlier [8,9] from a unified point of view as the consequence of the statistical irreversibility of turbulent flow. The main role was there played by the averaging over an invariant manifold, which is determined for an ideal incompressible liquid by the conservation law for the circulation of the velocity along a liquid contour. This manifold was used in the theory of hydrodynamic instabilities [10] and was called the "equivorticity field sheet." The averaging over the sheet was performed in[8,9] by means of equi-probable arrangements of a large number n of fluid elements of the same volume, into which the flow broke up.

A natural means for studying irreversible processes in turbulent flow is the hierarchy of equations for finite-dimensional distribution functions for the vortex field. [11] However, these equations are rather complicated and it becomes necessary to find simplified models. Some very simple spectral models have been given earlier. [12,13] In the present paper we consider as a model of two-dimensional turbulence a system of infinitely thin identical linear vortices. The field arguments of the distribution functions then drop out and we can use the principle of identical particles (vortices). In the model there occur not only operators describing collective effects, but also the operators for the interaction of a finite number of vortices. The consideration of elementary processes, which is of interest in itself, helps us in the understanding of the collective interaction. Moreover, on a scale larger than the average distance between vortices the model considered (as compared to the case of a continuous vortex field) does not change the basic nonlinear effects connected with the transfer of energy along the spectrum.[1)]

One shows easily that for a system of linear vortices averaging over a sheet [8,9] as $n \to \infty$ leads to a Poisson distribution of the vortices. It is natural to consider this distribution to be an equilibrium one in the case of unlimited flow when the excess energy already escapes to the region of very large scale lengths while there are for the phase trajectory no other essential limitations than that it must pertain to the sheet. Below we shall obtain the Poisson distribution as an exact stationary solution of the problem.

1. THE SYSTEM OF VORTICES

We consider N point vortices with intensities (circulations of the velocity round the vortex) κ_α (α = 1, ..., N) on an unbounded plane. The vortex coordinates $x_i^{(\alpha)}(t)$ (i = 1, 2) satisfy the Hamiltonian set of equations [14] which we write in the form

$$\varkappa_\alpha \frac{dx_i^{(\alpha)}}{dt} = \varepsilon_{ij} \frac{\partial H}{\partial x_j^{(\alpha)}} = -\frac{\varepsilon_{ij}}{2\pi} \sum_\beta{}' \frac{\varkappa_\alpha \varkappa_\beta (x_j^{(\alpha)} - x_j^{(\beta)})}{l_{\alpha\beta}^2}, \tag{1.1}$$

$$H = -\frac{1}{2\pi} \sum_{\alpha<\beta} \varkappa_\alpha \varkappa_\beta \ln l_{\alpha\beta} = \text{const.} \tag{1.2}$$

Here ϵ_{ij} is an antisymmetric tensor ($\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$), we sum from 1 to 2 over repeated Latin indices, $l_{\alpha\beta}$ is the distance between vortices, and the prime on the summation sign means that the terms with $\alpha = \beta$ are dropped. The quantity H has the meaning of the energy of the interaction between the vortices and is an integral of motion. From the fact that H is invariant under translations it follows that

$$\sum_\alpha \varkappa_\alpha x_i^{(\alpha)} = \text{const}_i. \tag{1.3}$$

Expressing H in terms of polar coordinates for the vortices (ρ_α, φ_α) we can rewrite (1.1) in the form

$$\varkappa_\alpha \rho_\alpha \frac{d\rho_\alpha}{dt} = \frac{\partial H}{\partial \varphi_\alpha}, \quad \varkappa_\alpha \rho_\alpha \frac{\partial \varphi_\alpha}{\partial t} = -\frac{\partial H}{\partial \rho_\alpha}. \tag{1.4}$$

From the rotational invariance of H it follows that

$$\sum_\alpha \varkappa_\alpha \rho_\alpha^2 = \text{const.} \tag{1.5}$$

Using (1.3) we can transform the integral of motion (1.5) to the form

$$\sum_{\alpha,\beta} \varkappa_\alpha \varkappa_\beta l_{\alpha\beta}^2 = \text{const.} \tag{1.6}$$

In the present paper we shall basically consider a system of identical vortices: $\kappa_\alpha \equiv \kappa$. However, it is useful in what follows to discuss also other variants, for instance, a "vortex plasma" consisting of an equal number of vortices with intensities which have the same absolute magnitude, but opposite signs.

It follows from (1.2) and (1.6) that the motion of a system of identical vortices is finite: $0 < l_0 < l_{\alpha\beta} < L_0 < \infty$, where l_0 and L_0 are scale lengths determined by the initial conditions. If we choose the origin to be in the center of mass of the system of vortices, so that the invariant (1.3) vanishes, an estimate of the area S_N occupied by the system of vortices gives the invariant (1.5) divided by $\varkappa N$. It is also useful to bear in mind that a group of vortices positioned sufficiently near one another acts at large distances as a single vortex with the sum intensity.

If there is only one vortex, it will remain at rest by virtue of (1.3). Two identical vortices will, according to the classical Helmholtz result, rotate around the center of mass along a circle with constant angular velocity $\varkappa/\pi l_{12}^2$. As the distance between the vortices is an integral of motion, (1.6), in the description of uniform and isotropic turbulence (Sec. 3) the two-vortex operator gives zero when acting upon a two-vortex distribution function. There occurs thus as the first non-trivial interaction process in the kinetics of uniform turbulence the problem of three vortices; the solution of this is given in Sec. 2.

We introduce the energy spectrum for a system of point vortices. We have

$$\Omega(\mathbf{x}) = \sum_\alpha \varkappa_\alpha \delta(\mathbf{x}-\mathbf{x}^{(\alpha)}), \tag{1.7}$$

$$\begin{gathered}\tilde{\Omega}(\mathbf{k}) = (2\pi)^{-1}\int \Omega(\mathbf{x})e^{-i\mathbf{kx}}d^2x = (2\pi)^{-1}\sum_\alpha \varkappa_\alpha e^{-i\mathbf{kx}^{(\alpha)}},\\ |\tilde{\Omega}(\mathbf{k})|^2 = (2\pi)^{-2}\Big[\sum_\alpha \varkappa_\alpha^2 + 2\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta \cos(\mathbf{k}(\mathbf{x}^{(\alpha)}-\mathbf{x}^{(\beta)}))\Big],\end{gathered} \tag{1.8}$$

where $\Omega(\mathbf{x})$ is the vortex field and $\tilde{\Omega}(\mathbf{k})$ its Fourier transform.

Using (1.8) we get for the energy spectral density

$$\begin{aligned}E_N(k) &= \pi k\,\overline{|\tilde{\mathbf{v}}(\mathbf{k})|^2} = \pi k^{-1}\,\overline{|\tilde{\Omega}(\mathbf{k})|^2}\\ &= (4\pi k)^{-1}\Big[\sum_\alpha \varkappa_\alpha^2 + 2\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta J_0(kl_{\alpha\beta})\Big],\end{aligned} \tag{1.9}$$

where $\tilde{\mathbf{v}}(\mathbf{k})$ is the Fourier transform of the velocity field, the bar indicates averaging over angles, and J_0 is a Bessel function. For identical vortices

$$E_N(k) = \varkappa^2(4\pi k)^{-1}\Big[N + 2\sum_{\alpha<\beta}J_0(kl_{\alpha\beta})\Big]. \tag{1.10}$$

The first terms in (1.9) and (1.10) correspond to the vortex self energy and the second ones to the energy of the interaction determined by the set of distances apart of the vortices.

We assume that we can indicate for a system of identical vortices the above mentioned external and internal scale lengths (L_0 and l_0), at least for a finite time interval. For a system with vortices of one sign one can easily indicate these scale lengths using (1.2) and (1.6), in that case, for an unlimited time interval. From (1.9) we have

$$\int_{L^{-1}}^{l^{-1}} E_N(k)\,dk = \frac{1}{4\pi}\ln\frac{L}{l}\sum_\alpha \varkappa_\alpha^2 + \frac{1}{2\pi}\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta \ln\frac{LC_1}{l_{\alpha\beta}}, \tag{1.11}$$

where we have assumed that $l \ll l_0 < L_0 \ll L$; $C_1 = 2e^{-C}$, $C \approx 0.577$ is Euler's constant. Apart from an additive constant (1.11) is the same as (1.2). Moreover, using the identity

$$\int_0^\infty J_0(x)\,dx = 1$$

we find

$$\int_{L^{-1}}^{l^{-1}} kE_N(k)\,dk = \frac{l^{-1}-L^{-1}}{4\pi}\sum_\alpha \varkappa_\alpha^2 + \frac{1}{2\pi}\sum_{\alpha<\beta}\varkappa_\alpha\varkappa_\beta l_{\alpha\beta}^{-1}. \tag{1.12}$$

The ratio of expressions (1.12) to (1.11) gives us the average wavenumber $k_v(t)$ which characterizes the distribution of the energy over the Fourier components of the velocity field. A decrease in k_v means a transfer of energy from small to large scales, and vice versa. The quantity k_v can change only when we take into account the change in the second of the sums occurring in (1.12). In the case of identical vortices that sum is proportional to the average absolute magnitude of the relative velocities (1.1) produced by the vortices. The evolution of the relative configuration of the vortices which determine the energy spectrum is thus, generally speaking, slowed down, i.e., the system spends a longer time in states with small k_v. This also means statistically an irreversible tendency to transfer energy from small to large scale lengths. In what follows this tendency will be traced already for N = 3.

2. THE INTERACTION OF THREE VORTICES

Kelvin [15] was one of the first to turn to the three-vortice problem. He showed that a uniform rotation of three identical vortices, positioned at the vertices of an equilateral triangle, is stable while the rotation of the same vortices, distributed along one line with one vortex at equal distances from the others, is unstable. Zhukovskiĭ [16] in his lecture for the 70th birthday of Helmholtz gave (without comment) a picture of the motion of three vortices for the case when two identical vortices are close to one another and at some distance there is a vortex with the opposite sign (and, evidently, the same absolute magnitude).

Although the integrals of motion (1.2), (1.3), and (1.5) have been known for a long time, the solution of the problem of three (identical or non-identical) vortices has, apparently, not been published. Batchelor [17] notes specifically in his recently published detailed monograph that the details of the motion in the case of three vortices are not clear. We consider this problem in rather a lot of detail, taking into account its connection with the theory of turbulence and also its uniqueness as a "three-body problem" allowing an exact solution.

First of all we shall be interested in the relative motion of three identical vortices. We write the integrals of motion (1.2) and (1.6) in the form

$$a_1a_2a_3 = \lambda^3, \tag{2.1}$$

$$a_1^2 + a_2^2 + a_3^2 = 3. \tag{2.2}$$

Here the a_m are the dimensionless lengths of the sides of the triangle in the vertices of which the vortices are situated (Fig. 1); as the unit length we take the root mean square distance r between the vortices which is an integral of motion. As the basic dimensionless parameter in the problem we have chosen the integral of motion λ which is the ratio of the geometric average of the distance apart of the vortices to the root mean square distance.

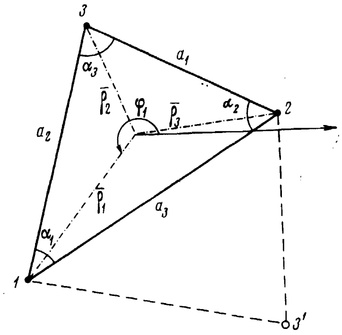

FIG. 1. The vortex configuration.

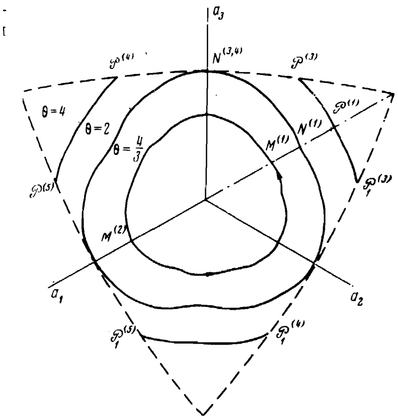

FIG. 2. Phase trajectories of the relative motion of the vortices.

The phase trajectories corresponding to different values of λ lie on the intersection of the integral surfaces (2.1) and (2.2). We must then bear in mind that the quantities a_m are positive, as they are the lengths of the sides of a triangle and each of them can be at most the sum of the other two. The phase trajectories are thus confined to the first quadrant and bounded by the planes

$$a_1=a_2+a_3, \quad a_2=a_3+a_1, \quad a_3=a_1+a_2. \tag{2.3}$$

To investigate the behavior of the phase trajectories in the simplest way it is convenient to introduce cylindrical coordinates (p, q, ψ) with the p-axis along the bisectrix of the trihedral axis formed by the old a_m—axes. We have

$$a_m=3^{-1/2}[p+2^{1/2}q\cos(\psi+\delta_m)], \quad \delta_1=0, \quad \delta_{2,3}=\mp\frac{2\pi}{3}, \tag{2.4}$$

where by definition the angle ψ is reckoned from the projection of the a_1-axis on the p = const plane. In the new variables Eqs. (2.1) to (2.3) become

$$p^3-\frac{3}{2}pq^2+2^{-1/2}q^3\cos 3\psi=\lambda^3 3^{3/2}, \tag{2.5}$$

$$p^2+q^2=3, \tag{2.6}$$

$$p=2^{1/2}q\cos(\psi+\delta_m), \quad m=1,\ 2,\ 3. \tag{2.7}$$

We show in Fig. 2 the projections on the p = const plane: the a_m—axes, the intersection curves of the planes (2.7) with the sphere (2.6) (indicated by dashed lines) and the phase trajectories—the lines of the intersection of the surface (2.5) with the sphere (2.6) for the parameter values $\theta \equiv \lambda^{-6} = 4/3$, 2, 4. The parameter θ is connected with the interaction energy (1.2). For arbitrary N

$$\theta=\prod_{\alpha<\beta}(l_{\alpha\beta}/r)^{-2}, \quad N(N-1)r^2=2\sum_{\alpha<\beta}l^2_{\alpha\beta}, \tag{2.8}$$

$$4\pi H=\varkappa^2[\ln\theta-N(N-1)\ln r].$$

As the geometric mean of positive quantities cannot exceed their root mean square, we have $\theta \geq 1$. The case $\theta = 1$ (the central point in Fig. 2) corresponds to the uniform rotation of the vortices when they are at the vertices of an equilateral triangle. The relative motion of the vortices is then according to Lyapunov clearly stable.

When $1 < \theta < 2$ the phase trajectories are closed concentric curves. The relative motion of the vortices is periodic and the phase trajectories depend on θ in a continuous way. Moreover, in the range $1 \leq \theta < 2$ the vortices cannot go through states where they lie on one line. The orientation of the vortices γ—which is equal to 1 when the sequence of the vortices with numbers 1, 2, 3 goes counterclockwise around the triangle and -1 when it goes round clockwise (the case $\gamma = -1$ is shown in Fig. 1 by dashed lines)—therefore does not change with time. We note also that when $1 < \theta < 2$ each of the sides of the triangle successively goes through all stages from the smallest to the largest value.

In the range $2 < \theta < \infty$ the phase trajectory consists of three pieces. When the phase point reaches one of the dashed lines (Fig. 2) the vortices fall on a single straight line, after which the orientation γ of the vortices changes. The smallest side of the triangle remains all the time the smallest. As in the first case, the relative motion of the vortices is periodic and the phase trajectory depends in a continuous way on θ. As $\theta \to \infty$ two vortices approach each other so closely that they rotate around one another independent of the third vortex. The effect of the third vortex shows up only in the total rotation of the whole system around the center of mass.

A peculiar case is $\theta = 2$. The phase trajectory is split into three parts by the points of contact with the dashed lines. These points of contact are stationary points for the relative motion of the vortices, corresponding to a symmetric configuration of the vortices on a single straight line. The absolute motion consists in a rigid rotation of the system around the central vortex. Such a state is clearly unstable. If a perturbation changes the magnitude of θ, depending on the sign of this change the system changes to one of the regimes considered above. If the perturbation does not change θ the system may jump past the stationary point, reaching the neighboring section. On each of the sections of the phase trajectory corresponding to $\theta = 2$ the motion is aperiodic—the phase point tends asymptotically to the appropriate stationary point (see the calculations which follow below) while the orientation of the vortices does not change. The phase trajectory as a whole has clearly as function of the parameter θ a discontinuity at $\theta = 2$.

We construct the equation for one of the sides, a_1, of the triangle. To fix the ideas we take $\varkappa > 0$, which corresponds to a counterclockwise rotation of the liquid close to the vortex. Projecting the velocities produced by vortex 1 in the points where the vortices 2 and 3 are situated onto the side a_1 we get (Fig. 1):

$$\frac{da_1}{dt}=\frac{\gamma\varkappa}{2\pi r^2}\left(\frac{\sin\alpha_3}{a_2}-\frac{\sin\alpha_2}{a_3}\right). \tag{2.9}$$

We have here taken into account the orientation of the vortices and the fact that the sides are normalized by the root mean square value r. After simple transformations and using (2.1), (2.2), and the properties of a triangle, Eq. (2.9) becomes

$$\frac{db_1}{d\tau}=\gamma\,\mathrm{sign}\{a_3-a_2\}\,[4-\theta b_1(2b_1-3)^2]^{1/2}[\theta b_1(3-b_1)^2-4]^{1/2},$$
$$b_1=a_1^2,\quad \tau=t\varkappa(2\pi r^2)^{-1}. \tag{2.10}$$

Equations (2.1) and (2.2) in conjunction with the condition $a_3 = a_2$ give

$$\theta b_1(3-b_1)^2=4, \tag{2.11}$$

and in conjunction of any of the conditions (2.3)

$$\theta b_1(2b_1-3)^2=4. \tag{2.12}$$

We see that the factor $\gamma\,\mathrm{sign}(a_3 - a_2)$, which takes the values ± 1, does not vary in the range of changes of b_1 in which the expression which stands as its factor in (2.10) does not vanish.

Equation (2.11) has the roots

$$b_1^{(1)}=4\cos^2\left(\frac{\pi}{3}+\frac{\beta_1}{3}\right),\quad b_1^{(2)}=4\cos^2\left(\frac{\pi}{3}-\frac{\beta_1}{3}\right),\quad \cos\beta_1=\theta^{-1/2}. \tag{2.13}$$

The third root which is larger than 3 is not considered because of (2.2). Expressions (2.13) give the smallest and largest values of b_1, compatible with (2.1) and (2.2). These values correspond in Fig. 2 to the points of intersection of the phase trajectories with the a_1 axis (for $\theta = 4/3$ these are the points $M^{(1)}$ and $M^{(2)}$).

The roots of Eq. (2.12)

$$b_1^{(3)}=2\cos^2\left(\frac{\pi}{3}+\frac{\beta_2}{3}\right),\quad b_1^{(4)}=2\cos^2\left(\frac{\pi}{3}-\frac{\beta_2}{3}\right),\quad b_1^{(5)}=2\cos^2\frac{\beta_2}{3},$$
$$\cos\beta_2=(2/\theta)^{1/2},\quad \theta\geqslant 2,\quad b_1^{(3)}\leqslant b_1^{(4)}\leqslant b_1^{(5)} \tag{2.14}$$

correspond to the points of intersection of the phase trajectories with the planes (2.3). These points are indicated in Fig. 2 for $\theta = 4$; the superscript denotes the number of the root.

To fix the ideas we shall assume that at $\tau = 0$ we have $\gamma = 1$, $b_1 = b_1^{(1)}$. One sees easily that the phase point starts to move along the phase trajectory in the counterclockwise direction. We have the quadrature

$$\tau=\int_{b_1^{(1)}}^{b_1(\tau)}\chi(b)\,db,\quad \chi(b)=[4-\theta b(2b-3)^2]^{-1/2}[\theta b(3-b)^2-4]^{-1/2}. \tag{2.15}$$

When $1 < \theta < 2$ the phase point will all the time move in the counterclockwise direction, describing a closed curve. For the period we have the expression

$$T(\theta)=2\int_{b_1^{(1)}}^{b_1^{(2)}}\chi(b)\,db,\quad 1<\theta<2. \tag{2.16}$$

One can easily show that as $\theta \to 1$, $T(\theta) \to 2\pi/3$.

In the range $2 < \theta < \infty$ the phase point reaching $\mathscr{P}^{(3)}$ (to illustrate this point we use the curve for $\theta = 4$ shown in Fig. 2) turns back, and the sign of γ changes. The next turn occurs in the point $\mathscr{P}_1^{(3)}$. For the period we get

$$T(\theta)=4\int_{b_1^{(1)}}^{b_1^{(3)}}\chi(b)\,db,\quad 2<\theta<\infty. \tag{2.17}$$

As $\theta \to \infty$ we have the asymptotic behavior $T(\theta) \approx 4\pi/9\theta$.

Finally, for $\theta = 2$ we have $b_1^{(3)} = b_1^{(4)} = 1/2$, and there appears in (2.15) a non-integrable singularity. The phase point will thus asymptotically tend to $N^{(3,4)}$. Linearizing Eq. (2.10) near this point we get

$$b_1(\tau)={}^1/_2-[{}^1/_2-b_1(\tau_0)]\exp\{-3^{3/2}(\tau-\tau_0)\},$$
$$b_1(\tau_0)\leqslant{}^1/_2,\quad \theta=2. \tag{2.18}$$

In Fig. 3 we show the function $T(\theta)$, given by Eqs. (2.16)

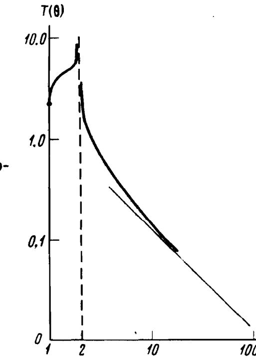

FIG. 3. The period of the relative motion of the vortices as function of the interaction energy.

and (2.17), which becomes infinite when $\theta = 2$.

As the relative motion of the system is slowed down when the critical point $N^{(3,4)}$ is approached, it is of interest to trace how the average wavenumber k_V, introduced in Sec. 1, changes. The change in k_V is (with a positive coefficient) proportional to the change in

$$k_*=b_1^{-1/2}+b_2^{-1/2}+b_3^{-1/2},\quad b_m=a_m^2. \tag{2.19}$$

As independent variable we use the quantity b_1 which increases monotonically when the critical point is approached; see (2.18). Differentiating (2.19), and using (2.1), (2.2), and (2.14) we find in the critical point

$$k_*'=0,\quad k_*''=4\sqrt{2}/3$$

(we indicate differentiation with respect to b_1 by a prime). The quantity k_*, and therefore also k_V, therefore reaches a minimum in the critical point when the relative motion of the system is slowed down in accordance with the general conclusions of the end of Sec. 1. It is interesting to note that the quantity k_* also reaches a local minimum in the point $N^{(1)}$ where the relative motion of the system is retarded since $db_1/d\tau = 0$ in this case.

Going over to the absolute motion of the vortices we use the second of Eqs. (1.4) for the first vortex, writing this equation in dimensionless form:

$$\bar\rho_1\frac{d\varphi_1}{d\tau}=\frac{\partial}{\partial\bar\rho_1}\ln(a_1a_2a_3),\quad \bar\rho_m=\rho_m r^{-1}, \tag{2.20}$$

where the quantities a_m must be taken to be expressed in terms of $\bar\rho_m$ and φ_m; in particular (Fig. 1)

$$a_3^2=\bar\rho_1^2+\bar\rho_2^2-2\bar\rho_1\bar\rho_2\cos(\varphi_1-\varphi_2). \tag{2.21}$$

From the condition that the origin is at the center of mass of the system of vortices we find relations of the form

$$\bar\rho_3^2=\bar\rho_1^2+\bar\rho_2^2+2\bar\rho_1\bar\rho_2\cos(\varphi_1-\varphi_2). \tag{2.22}$$

Using (2.2) we get

$$\bar\rho_m^2=\frac{2}{3}-\frac{1}{3}a_m^2. \tag{2.23}$$

Using (2.1), (2.2), and (2.21) to (2.23) we change Eq. (2.20) to read

$$\frac{d\varphi_1}{d\tau}=\frac{4+\theta b_1(3-b_1)(3-2b_1)}{2(2-b_1)}\equiv\omega_1(b_1). \tag{2.24}$$

The angular velocity (2.24) is bounded as the quantity $b_1 < 2$ when $\theta \neq 2$, while for $\theta = 2$ we have $\omega_1(b_1) = 1 + 5b_1 - 2b_1^2$. Using (2.15) we have the quadrature

$$\varphi_1(\tau)-\varphi_1(0)=\int_{b_1^{(1)}}^{b_1(\tau)}\chi(b)\,\omega_1(b)\,db.$$

As the angle of rotation of the system after a time $T(\theta)$ (for $\theta \neq 2$) is not necessarily a rational fraction of

π, the absolute motion of the vortices is in the general case not periodic.

We dwell briefly on the problem of the interaction of three non-identical vortices. Each of these vortices can in turn consist of a group of (possibly identical) vortices; it is important only that the size of the groups is small compared to the distances between their centers of mass. The hierarchy of such vortex structures is a particular class of analytically soluble problems.

After substituting κ_1 for κ Eq. (2.9) retains the same form (apart from the normalization of the side). Equations (1.2) and (1.6) give two relations between the three sides of the triangle in the vertices of which the vortices are positioned. In particular, we can express l_{12} and l_{13} in square roots in terms of l_{23}, when κ_2/κ_3 equals $-1/4$, $\pm 1/3$, $\pm 1/2$, $-2/3$, ± 1, $-3/2$, ± 2, ± 3, -4. In the case of arbitrary κ_m the problem reduces to a quadrature of an implicit function.

The problem of the interaction of four vertices which have pairwise equal intensities and which are distributed symmetrically with respect to the center of mass can be solved similarly. One can also have a fifth vortex of arbitrary intensity situated in the center of mass. In all these cases the problem reduces to the dynamics of a triangle with two links on its sides.

3. STATISTICAL DESCRIPTION OF THE FIELD OF A VORTEX

We consider the s-vortex distribution density for a system of N identical vortices

$$f_s^{(N)}(t, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(s)}) = \left\langle \prod_{m=1}^{s} \delta(\mathbf{x}^{(m)}(t) - \mathbf{r}^{(m)}) \right\rangle, \quad s \leq N, \tag{3.1}$$

where the pointed brackets indicate a probability average, and $\mathbf{x}^{(m)}(t)$ is the position of the m-th vortex. The conditions for the compatibility of the distribution densities have the form

$$f_s^{(N)} = \int f_{s+1}^{(N)} d^2 r^{(s+1)}, \quad f_0^{(N)} = 1. \tag{3.2}$$

We can obtain the equations for the $f_s^{(N)}$ as an appropriate particular case from the equations for the finite-dimensional distributions of the field of a vortex.[11] However, we can start directly from Eqs. (1.1). We shall assume that the distribution densities (3.1) are symmetric in their spatial arguments. It is sufficient that this condition be satisfied at the initial time as the dynamics cannot introduce an asymmetry because the vortices are identical. Differentiating (3.1) with respect to the time and using (1.1) and the identity of the vortices we get after some simple transformations

$$\frac{\partial f_s^{(N)}}{\partial t} = A_s f_s^{(N)} + (N-s) B_s f_{s+1}^{(N)}, \quad (s=1, \ldots, N), \tag{3.3}$$

$$A_s = \frac{\varkappa \varepsilon_{ik}}{2\pi} \sum_{m=1}^{s} {\sum_{p=1}^{s}}' \frac{r_k^{(m,p)}}{r_{m,p}^2} \frac{\partial}{\partial r_i^{(m)}}, \quad A_1 = 0,$$
$$(r_k^{(m,p)} \equiv r_k^{(m)} - r_k^{(p)}), \tag{3.4}$$

$$B_s = \frac{\varkappa \varepsilon_{ik}}{2\pi} \sum_{m=1}^{s} \int d^2 r^{(s+1)} \frac{r_k^{(m,s+1)}}{r_{m,s+1}^2} \frac{\partial}{\partial r_i^{(m)}}$$
$$r_{m,p}^2 \equiv (r_k^{(m,p)})^2. \tag{3.5}$$

The operator A_s describes the interaction of s vortices. The operator B_s takes the influence of any of the remaining N−s vortices on the chosen s vortices statistically into account.

The hierarchy of Eqs. (3.3) can be replaced by a single equation for the generating functional

$$L^{(N)}[t; h] = \left\langle \prod_{\alpha=1}^{N} [1 + h(\mathbf{x}^{(\alpha)}(t))] \right\rangle, \tag{3.6}$$

similar to what is done in molecular kinetic theory.[18] Differentiating (3.6) with respect to time and using (1.1) and the identity of the vortices we get after some transformations

$$\frac{\partial L^{(N)}}{\partial t} = Q L^{(N)}, \quad Q = \frac{\varkappa}{2\pi} \varepsilon_{ij} \iint d^2x\, d^2x' [1 + h(\mathbf{x})]$$
$$\times [1 + h(\mathbf{x}')] \frac{x_h - x_k'}{|\mathbf{x} - \mathbf{x}'|^2} \frac{\partial}{\partial x_i} \frac{\delta^2}{\delta h(\mathbf{x}) \delta h(\mathbf{x}')}. \tag{3.7}$$

The distribution densities (3.1) can be expressed in terms of variational derivatives of the functional (3.6):

$$\left. \frac{\delta^s L^{(N)}}{\delta h(\mathbf{r}^{(1)}) \ldots \delta h(\mathbf{r}^{(s)})} \right|_{h=0} = \frac{N!}{(N-s)!} f_s^{(N)} \equiv F_s^{(N)} \quad (s = 0, 1, \ldots, N). \tag{3.8}$$

The characteristic functional of the field of a vortex, given by (1.7) with $\kappa_\alpha \equiv \kappa$, is simply connected with the functional (3.6):

$$\Phi^{(N)}[t; z] = \left\langle \exp\left\{ i \int z(\mathbf{x}) \Omega(t, \mathbf{x}) d^2x \right\} \right\rangle = L^{(N)}[t; e^{i\varkappa z} - 1]. \tag{3.9}$$

We consider the limit as $N \to \infty$. We note that the operator Q of (3.7) is independent of N although it is obtained from the N-vortex problem. It is obvious to assume that Eq. (3.7) retains its form in the limit and that limits exist for the expressions occurring on the right-hand side of (3.8). This condition is equivalent to the existence for statistical characteristics of the field of a vortex, as the connection between the functionals (3.9) is independent of N. This is immediately clear from the expressions for the moments of the field of a vortex (we do not indicate the time dependence)

$$\langle \Omega(\mathbf{x}) \rangle = \varkappa F_1(\mathbf{x}), \tag{3.10}$$

$$\langle \Omega(\mathbf{x}) \Omega(\mathbf{x}') \rangle = \varkappa^2 \delta(\mathbf{x} - \mathbf{x}') F_1(\mathbf{x}) + \varkappa^2 F_2(\mathbf{x}, \mathbf{x}'), \tag{3.11}$$

and similar expressions for higher order moments (we drop the index N for the limiting quantities).

Averaging (1.10) we get the limiting expression for the energy spectrum per vortex:

$$E_1(k) = \lim_{N \to \infty} N^{-1} \langle E_N(k) \rangle = \frac{\varkappa^2}{4\pi k} \left[1 + \int J_0(kr) P(\mathbf{r}) d^2 r \right], \tag{3.12}$$

$$P(\mathbf{r}) = \lim_{N \to \infty} (N-1) \int f_2^{(N)}(\mathbf{x} - \mathbf{r}, \mathbf{x}) d^2 x$$
$$= \lim_{N \to \infty} \left(\int F_1^{(N)}(\mathbf{x}) d^2 x \right)^{-1} \int F_2^{(N)}(\mathbf{x} - \mathbf{r}, \mathbf{x}) d^2 x, \tag{3.13}$$

where $P(\mathbf{r})$ is the limiting density of vortices at a distance $\mathbf{r}$ from a chosen vortex.

As $N \to \infty$ the hierarchy of Eqs. (3.3) becomes

$$\partial F_s / \partial t = A_s F_s + B_s F_{s+1} \quad (s = 1, 2, \ldots). \tag{3.14}$$

In the case of uniform flow we have, using (3.10) and (3.13)

$$F_1 = \sigma, \ \langle \Omega \rangle = \varkappa \sigma, \ F_s \equiv \sigma^s (1 + g_s), \ P(\mathbf{r}) = \sigma [1 + g_2(\mathbf{r})], \tag{3.15}$$

$$E(k) = \sigma [E_1(k) - \varkappa_2 k^{-2} \delta(k)] = \frac{\varkappa^2 \sigma}{4\pi k} \left[1 + \sigma \int J_0(kr) g_2(\mathbf{r}) d^2 r \right], \tag{3.16}$$

where σ is the constant density of vortices, E(k) is the normally used spectral distribution of energy for velocity fluctuations per unit area (the singular part subtracted from E_1 corresponds to the energy of the average motion). We can also obtain Eq. (3.16) from (3.11) after subtracting $\langle\Omega\rangle^2$ and Fourier-transforming.

The hierarchy of Eqs. (3.14) starts for uniform flow with s = 2, while when there is isotropy $A_2F_2 = 0$ so that the first elementary interaction process, which follows from these equations, is the interaction of three vortices considered above.

4. POISSON DISTRIBUTION OF VORTICES. SIMILARITY REGIME

The equation

$$\frac{\partial L}{\partial t}=QL,$$

where Q is the operator determined by Eq. (3.7), admits a stationary solution

$$L[h]=\exp\left\{\sigma\int h(\mathbf{x})\,d^2x\right\}. \tag{4.1}$$

This solution corresponds to a Poisson distribution of vortices in which the positions of the vortices are independent and uniformly distributed over the whole of space.[2)] Substituting (4.1) into (3.8) (with N = ∞) and using (3.15) we get

$$F_s=\sigma^s,\ g_s=0,\ P=\sigma. \tag{4.2}$$

One sees easily that (4.2) satisfies the hierarchy of Eqs. (3.14).

Substituting (4.1) into (3.9) (for N = ∞) gives

$$\Phi[z]=\exp\left\{\sigma\int\left[e^{i\varkappa z(\mathbf{x})}-1\right]d^2x\right\}.$$

The cumulants of the field of a vortex take the form

$$K_s(\mathbf{r}^{(1)},\ldots,\mathbf{r}^{(s)})=(-i)^s\frac{\delta^s\ln\Phi}{\delta z(\mathbf{r}^{(1)})\ldots\delta z(\mathbf{r}^{(s)})}\Bigg|_{z=0}=\sigma\varkappa^s\prod_{p=2}\delta(\mathbf{r}^{(p)}-\mathbf{r}^{(1)}),\quad s\geqslant2.$$

Using (4.2) we get for the energy spectrum from (3.16)

$$E(k)=\varkappa^2\sigma/4\pi k. \tag{4.3}$$

When energy is transferred from small to large scale lengths a similarity regime may be established in which, in particular,

$$g_2(\tau,r)=g(\tau)\ (r\sigma^{1/2})^{-\alpha},\ \tau=\varkappa\sigma t.$$

The condition that the integral in (3.16) converges gives

$$1/2<\alpha<2. \tag{4.4}$$

Then

$$E(\tau,k)=\frac{\varkappa^2\sigma}{4\pi}k^{-1}+\varkappa^2\sigma^{2-\alpha/2}\Gamma\left(1-\frac{\alpha}{2}\right)\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1}g(\tau)k^{-3+\alpha} \tag{4.5}$$

where Γ(x) is the gamma function. By virtue of (4.4) the second term in (4.5) is the determining one in the small wavenumber region. One sees easily that the transfer of energy from small to large scale lengths corresponds to a decrease of g(τ). As g(τ) → 0 the spectrum (4.5) goes over into (4.3). The value $\alpha = 4/3$, corresponding to the "5/3-law" for the energy spectrum, lies in the range (4.4). Unfortunately, so far we have not succeeded to narrow that range down and to determine the nature of the behavior of g(τ), without resorting to a formal hypothesis about closure or an arbitrary cut-off in the series expansions in a parameter which is not small in any variant of perturbation theory. We note that owing to the phenomenon of the intermittent nature of turbulent flow (see [19], where we refer to older papers) the behavior of the energy spectrum cannot have a strictly universal nature.

5. THE FINE STRUCTURE OF TURBULENT FLOW

The ratio of the second term on the right-hand side of (3.14) to the first one is for s ≥ 2, when we use (3.4) and (3.5), of the order

$$\eta=\sigma l_*r_{min},$$

where σ is the density of vortices, l_* some integral correlation scale length, and r_{min} the minimum distance apart in the group of s vortices. Without specifying l_* more precisely we can state that η becomes small as $r_{min}\to 0$.

For the analysis of the hierarchy of Eqs. (3.14) with $\eta \ll 1$ we use Bogolyubov's method.[18] Assuming that the scale of the inhomogeneity of the turbulent flow is of the same order as l_* and initially considering the case where all distances apart between the s vortices are of the same order of magnitude we rewrite (3.14) in the form

$$\frac{\partial F_1}{\partial t}=B_1F_2, \tag{5.1}$$

$$\frac{\partial F_s}{\partial t}=\eta^{-1}A_sF_s+B_sF_{s+1}\qquad(s\geqslant2). \tag{5.2}$$

We have here split off the parameter η. This can be done by going over to the appropriate dimensionless variables or by considering η to be a formal expansion parameter.

We assume that F_s (s ≥ 2) depends on t only through the functional dependence on F_1. Expanding these functionals in series and using (5.1) we get

$$F_s=\sum_{m=0}^{\infty}\eta^mF_s^{(m)}, \tag{5.3}$$

$$\frac{\partial F_s}{\partial t}=\sum_{m=0}^{\infty}\eta^m\left\{\frac{\delta F_s^{(m)}}{\delta F_1}\frac{\partial F_1}{\partial t}\right\}=\sum_{m,p=0}^{\infty}\eta^{m+p}\left\{\frac{\delta F_s^{(m)}}{\delta F_1}B_1F_2^{(p)}\right\}, \tag{5.4}$$

where the braces indicate integration over the point in which we take the variational derivative.[3)] On the other hand, the time derivative of F_s can be evaluated by substituting (5.3) into (5.2). Equating the expression obtained in that way to (5.4) we get

$$\sum_{m=0}^{\infty}\eta^mA_sF_s^{(m)}=\sum_{m,p=0}\eta^{1+m+p}\left\{\frac{\delta F_s^{(m)}}{\delta F_1}B_1F_2^{(p)}\right\}-\sum_{m=0}^{\infty}\eta^{1+m}B_sF_{s+1}^{(m)}. \tag{5.5}$$

In zeroth and higher orders in η we get, respectively, from (5.5)

$$A_sF_s^{(0)}=0, \tag{5.6}$$

$$A_sF_s^{(m)}=\sum_{q+p=m-1}\left\{\frac{\delta F_s}{\delta F}B_1F_2^{(p)}\right\}-B_sF_{s+1}^{(m-1)}\qquad(m\geqslant1). \tag{5.7}$$

We see from (5.6) that $F_s^{(0)}$ is a function only of the invariants of motion of a system of s vortices. For s = 2 this gives the condition for local isotropy

$$F_2^{(0)}(\mathbf{r}^{(1)},\mathbf{r}^{(2)})=\mathscr{F}_2(1/2(\mathbf{r}^{(1)}+\mathbf{r}^{(2)});\ r_{12}). \tag{5.8}$$

Starting with Kolmogorov's classical paper [20] one always presupposed the fundamental property of local isotropy of turbulent flow, which had been verified experimentally but had not been given a theoretical foundation. The physical mechanism for the local isotropization is in the model considered extremely simple. When the vortices approach each other very closely they start to rotate fast around the center of mass and this leads to an averaging of the correlations over directions.

For $s \geq 3$ Eq. (5.6) gives, apart from the local isotropy condition, some more important additional information. We showed in Sec. 2 that when the position of the center of mass is fixed the dynamics of a system of three vortices is completely determined by the invariants of motion (2.1) and (2.2). When $\theta < 2$ the orientation of the vortices γ is also conserved. However, the γ-dependence drops out since we stipulated in Sec. 3 that we look for solutions symmetric under a permutation of identical vortices. Thus, for s = 3

$$F_3^{(0)}(\mathbf{r}^{(1)}, \mathbf{r}^{(2)}, \mathbf{r}^{(3)}) = \mathcal{F}_3({}^1/_3(\mathbf{r}^{(1)}+\mathbf{r}^{(2)}+\mathbf{r}^{(3)});\ r_{1,2}^2+r_{2,3}^2+r_{3,1}^2;\ r_{1,2}r_{2,3}r_{3,1}). \quad (5.9)$$

The function (5.9) has one argument less than would follow from merely the condition of local isotropy.

For turbulent flow which from the beginning is uniform and isotropic all discussions and calculations are carried out analogously if we choose for the function determining the time-dependence of the s-vortex distributions g_2 instead of F_1. The dependence on the first argument then, of course, drops out in Eqs. (5.8) and (5.9).

In the case when not all distances apart between the s vortices satisfy the condition $\sigma l_* r_{\alpha,\beta} \ll 1$, but only a part, referring to a subgroup of p vortices, Eq. (5.6) must be replaced by

$$A_p F_s^{(0)} = 0 \qquad (p<s). \qquad (5.10)$$

The solution of this equation is a function of the invariants of motion for a system of p vortices and, generally speaking, an arbitrary function of the coordinates of the remaining s−p vortices.

It is natural to introduce the condition

$$F_s \to F_p F_{s-p}\ (p \geq 2)\ \text{as}$$
$$\max_{\alpha,\beta=1,\ldots,p} \{r_{\alpha,\beta}\} / \min_{\substack{\alpha=1,\ldots,p \\ \gamma=p+1,\ldots,s}} \{r_{\alpha,\gamma}\} \to 0. \qquad (5.11)$$

It then follows from (5.10) that $F_p^{(0)}$ is a function of the invariants of a system of p vortices.

We note that the condition for the compatibility of the distributions follows from (5.11) for p = s − 1

$$F_{s-1} = \lim_{R\to\infty} \left[\int_{|r^{(1)}|\leq R} F_1 d^2 r^{(1)} \right]^{-1} \int_{|r^{(s)}|\leq R} F_s\, d^2 r^{(s)},$$

which is the analog of (3.2).

In general, it is impossible to use the solution of Eq. (5.7) obtained for small distances to evaluate the integral terms occurring in (5.1), (5.2), and (5.7). If we nevertheless use (5.6) for all distances and use the identity

$$B_s = \int d^2 r^{(s+1)} \left[A_{s+1} - A_s - \frac{\varkappa}{2\pi} \varepsilon_{ik} \sum_{p=1}^{s} \frac{r_k^{(s+1,p)}}{r_{s+1,p}^2} \frac{\partial}{\partial r_i^{(s+1)}} \right]$$

we can show that

$$A_s F_s^{(m)} = 0, \quad B_s F_{s+1}^{(m)} = 0, \qquad (s=1, 2, \ldots;\ m=0, 1, \ldots).$$

The integral terms therefore drop out in all orders of perturbation theory. This fact may serve as an indication that the series (5.3) has a finite radius of convergence.

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1) The given model describes also the dynamics of the quantum vortices of the superfluid component of helium II. Recent experiments [21] which made it possible to visualize such vortices show that vortices do not form a simple lattice (Tkachenko [22] has considered the stability and small oscillations of simple vortex lattices) and that their motion may have a turbulent character.

2) A more general class of stationary solutions can be obtained by including the invariants of motion given in Sec. 1. In particular, the invariant (1.5) corresponds to a Gaussian distribution of the vortex positions.

3) To avoid misunderstandings we point out that the superscript of the F_s in the present section indicate the order in the expansion in η for a system with an infinite number of vortices in contrast to Sec. 3 where the superscript indicated the total number of vortices in the system.

1 L. Onsager, Nuovo Cim., Suppl. **6**, 279 (1949).
2 R. Fjørtoft, Tellus **5**, 225 (1953).
3 R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
4 G. K. Batchelor, Phys. Fluids **12**, Suppl. 233 (1969).
5 D. K. Lilly, Phys. Fluids **12**, Suppl. 240 (1969).
6 Yu. B. Kolesnikov and A. B. Tsinober, Magn. grid. **3**, 23 (1972); Mekh. zhidk. i gaza **4**, 146 (1974).
7 V. P. Starr, Physics of Negative Viscosity Phenomena, McGraw-Hill, New York, 1968.
8 E. A. Novikov, Arch. Mech. **4**, 741 (1974).
9 E. A. Novikov, in Turbulentnye techeniya (Turbulent Flow) Nauka, 1974, p. 85.
10 V. I. Arnol'd, Prik. Mat. Mekh. **29**, 846 (1965).
11 E. A. Novikov, Dokl. Akad. Nauk SSSR **177**, 299 (1967) [Sov. Phys.-Doklady **12**, 1006 (1968)].
12 V. N. Desnyanskii and E. A. Novikov, Fiz. Atm. i okeana **10**, 127 (1974).
13 V. N. Desnyanskii and E. A. Novikov, Fiz. Atm. i okeana **10**, 127 (1974).
14 H. Lamb, Hydrodynamics, Cambridge University Press, 1932.
15 Lord Kelvin, Mathematical and Physical Papers, Cambridge University Press, Vol. 4, 1910, p. 135.
16 N. E. Zhukovskiĭ, Sobr. soch. (Collected Papers) Vol. 7 (1950).
17 G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.
18 N. N. Bogolyubov, Problemy dinamicheskoĭ teorii v statisticheskoĭ fizike (Problems of a Dynamical Theory in Statistical Physics) Gostekhizdat, 1946 [Studies in Stat. Mech. **1**, 5 (1962)].
19 E. A. Novikov, Prik. Mat. Mekh. **35**, 266 (1971).
20 A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941) [reprinted in Usp. Fiz. Nauk **93**, 476 (1967); Sov. Phys.-Uspekhi **10**, 734 (1968)].
21 G. A. Williams and R. A. Packard, Phys. Rev. Lett. **33**, 280 (1974).
22 V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **50**, 1573 (1966) [Sov. Phys.-JETP **23**, 1049 (1966)].

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