Thermonuclear "drift" instabilities

A. B. Mikhaĭlovskiĭ

I. V. Kurchatov Institute of Atomic Energy
(Submitted November 20, 1974)
Zh. Eksp. Teor. Fiz. **68**, 1772–1782 (May 1975)

We study theoretically the instability of a plasma caused by α particles produced in thermonuclear reactions. We show that an earlier developed theory of thermonuclear instabilities was incomplete and that it did not take into account instabilities which are the most important ones from the point of view of the theory of the turbulent transfer of plasma across the magnetic field in Tokomak type toroidal traps. Such instabilities which are studied in the present paper are caused by the spatial inhomogeneity in the α-particle distribution and by the curvature of the magnetic field lines. We consider the instabilities produced by either untrapped α particles or by trapped α particles.

PACS numbers: 52.35.E, 52.55.F

1. INTRODUCTION

Some time ago Korablev[1] and Kolesnichenko and Oraevskiĭ[2] drew attention to the fact that charged products of thermonuclear reactions, such as α-particles, can be the cause of plasma instabilities. Such instabilities were later called thermonuclear instabilities. Korablev[1] and other authors[3-7] have studied in detail the thermonuclear instabilities of a uniform plasma in a uniform magnetic field and Kolesnichenko and Oraevskiĭ[2] and particularly Belikov, Kolesnichenko, and Oraevskiĭ[4] have also studied thermonuclear "drift" instabilities, i.e., those which are caused by the spatial inhomogeneity of the α-particles.

It would seem that all basic types of thermonuclear instabilities have been found and that macroscopic effects caused by the α-particles must be determined by the instabilities which have already been studied. This is essentially the point of view on which the recent paper by Belikov, Kolesnichenko, and Oraevskiĭ[8] is based, which is devoted to an evaluation of the turbulent coefficients for the transfer of a plasma due to the thermonuclear Alfvén instability of a uniform plasma pointed out by the same authors.[4]

However, we shall show in the present paper that the existing linear theory of thermonuclear instabilities is incomplete. It does not take into account a certain family of drift (gradient) instabilities which must lead to transfer coefficients larger than the ones discussed in[8] as will become clear from estimates in what follows.

The thermonuclear drift instabilities which have been considered before[2,4] have in fact no direct relation to the thermonuclear problem. The fact is that the authors of these papers studied a class of so-called potential drift instabilities. The assumption that they have a potential character is valid only when the velocity of the α-particles is small compared to the Alfvén velocity and this is not the case for a thermonuclear plasma as was first noted by Karablev and Rudakov.[3]

In the present paper we study non-potential (electromagnetic) drift instabilities. We shall analyze several kinds of instabilities and we shall mainly consider instabilities which develop only if there is spatial inhomogeneity in the α-particles and curvature of the magnetic field.

To elucidate the role of the curvature in thermonuclear instabilities of a tokamak we remind ourselves of some results of an earlier paper by us.[9]

A particle moving in a magnetic field with an alternating curvature and in the field of a perturbation with a longitudinal wavenumber $k_{\parallel}$ behaves as if it moved in a straight magnetic field and in the field of a wave with an effective longitudinal wavenumber $k_{\parallel\text{eff}}$ which differs from $k_{\parallel}$.[9] In particular, in the case of the tokamak geometry $k_{\parallel\text{eff}} = k_{\parallel} \pm 1/qR$, where R is the radius of curvature of the magnetic axis of the tokamak, $q \equiv aB_S/RB_\theta$ is the so-called safety factor of the tokamak, B_S, B_θ are the toroidal and the poloidal parts of the stationary magnetic field, and a is the radial coordinate in the vicinity of which the perturbation is localized. As a consequence of this particles with a longitudinal velocity $v_{\parallel}$ which satisfies the relation

$$\omega=v_{\parallel}(k_{\parallel}\pm 1/qR). \tag{1.1}$$

will turn out to be in resonance.

This kind of resonance is to some extent analogous to the cyclotron resonance for which

$$\omega=k_{\parallel}v_{\parallel}\pm\omega_B, \tag{1.2}$$

where ω_B is the cyclotron frequency of the corresponding kind of particle.

It is well known[5] that in a plasma in a magnetic field there exist kinds of waves such that for them the usual, Cerenkov resonance, $\omega = k_{\parallel}v_{\parallel}$, is ineffective, i.e., the contribution of such a resonance to the dispersion equation for the corresponding type of wave is very small, or is completely absent. Alfvén waves are an example of this. For a study of Alfvén waves which are weakly sensitive to Cerenkov resonance we must thus take into account other kinds of resonance.

In the uniform plasma and uniform magnetic field approximation the sole addition to the Cerenkov resonance is the cyclotron resonance (1.2). This resonance is just the basis of the mechanism for the thermonuclear Alfvén instability considered in the second part of the paper by Belikov, Kolesnichenko, and Oraevskiĭ.[4] In this sense the gist of this second part consists in the fact that its authors applied the respresentation of a theory of a uniform plasma for the build-up of Alfvên waves by fast particles to the concrete case of a velocity distribution of the fast particles, assuming that distribution to be isotropic and mono-energetic.

We consider, like Belikov, Kolesnichenko, and Ostraevskiĭ[4] the build-up of Alfvén waves, but in contrast to[4], we assume that the main mechanism for the build-up is not the cyclotron resonance (1.2), but the modified Cerenkov resonance (1.1), which comes into play only when the curvature of the magnetic field is taken into account.

It will be clear from what follows that the role of a resonance such as (1.1) consists, in particular, in that, when it is present, the contribution of the resonance particles to the dispersion equation for Alfvén waves is appreciably increased. However, on top of this, an important fact is that in contrast to the usual Cerenkov resonance in the case considered by us particles with a longitudinal velocity different from the longitudinal phase velocity of the wave $\omega/k_{\parallel}$ turn out to be in resonance. As the longitudinal phase velocity of the Alfvén waves is nothing but the Alfvén velocity c_A, the condition (1.1) means that both slow particles $v_{\parallel} < c_A$, and fast particles $v_{\parallel} > c_A$ can interact in resonance with Alfvén waves.

We have earlier[9] considered the interaction between Alfvén waves and slow (thermal) ions, $v_{\parallel} < c_A$. According to (1.1) a resonance interaction with such particles occurs when $|k_{\parallel}| \ll 1/qR$ which corresponds to perturbations which are strongly stretched along the magnetic field lines.

In the case of thermonuclear α particles their velocity is large compared to the Alfvén velocity, $v_{\parallel} > c_A$. Equation (1.1) is then satisfied provided

$$|k_{\parallel}| \approx 1/qR, \quad |k_{\parallel} \pm 1/qR| \ll 1/qR. \tag{1.3}$$

For such $k_{\parallel}$ the frequency of the Alfvén waves is approximately equal to

$$\omega = c_A/qR = B_\theta/a(4\pi\rho_0)^{1/2}, \tag{1.4}$$

where ρ_0 is the plasma mass density. Frequencies of this order of magnitude occur in the theory of the helical instability.[10]

It is well known[11] that drift effects are important when $\omega \lesssim \omega_*$, where $\omega_* \approx m v_\alpha \rho_\alpha / a^2$ is a characteristic drift frequency, v_α, ρ_α are the velocity and Larmor radius of an α particle, a is the small radius of the torus which is also of the same order of magnitude as the characteristic dimension of the α particles inhomogeneity, and m is the azimuthal wavenumber along the small azimuth of the torus. The condition $\omega \lesssim \omega_*$ for the frequencies (1.4) means that the wavelength of the perturbation along the small azimuth of the torus must be such that

$$k_\theta \rho_\alpha \gtrsim a c_A / v_\alpha q R, \tag{1.5}$$

where $k_\theta \equiv m/a$.

Bearing in mind that the magnetic field of the tokamak has a field line shear Θ of the order $\Theta \approx a/R$, we find, using (1.1), (1.5) and the order of magnitude relation $|k_{\parallel} \pm 1/qR| \gtrsim k_\theta \Theta x/a$, where x is the radial dimension of the localization of the perturbation, the estimate

$$x \approx \rho_0 R/a. \tag{1.6}$$

In accordance with the general statements developed earlier[9] one should expect that the growth rate of the Alfvén waves must also contain the parameter β_α, the ratio of the α-particle pressure to the magnetic field pressure,

$$\gamma \approx \beta_\alpha \operatorname{Re} \omega. \tag{1.7}$$

Substituting the above-mentioned values for ω, x, and γ into the approximate formula[12] for the coefficients for spatial transfer $D \approx \gamma^2 x^2/\omega$ shows that for α-particle pressures comparable with the plasma pressure and for reasonable values for the other parameters of the plasma and the magnetic field the coefficients for the spatial transfer from thermonuclear drift instabilities can only be an order of magnitude less than the Bohm diffusion coefficient.[12]

The instability which we have here discussed qualitatively will be considered quantitatively in Sec. 2. It is caused by the interaction between the Alfvén waves and α particles in flight and can thus be called the drift instability due to α particles in flight.

Knowing from the paper by Kadomtsev[13] that not only α particles in flight, but also trapped α particles can lead to instabilities we also consider the possibility of a build-up of Alfvén waves due to trapped α particles. This is the subject of Sec. 3 of our paper. It will be shown in Sec. 3 that for the same Re ω and x the growth rate of the oscillations caused by trapped α particles is somewhat larger than the growth rate due to particles in flight. The instability due to trapped α particles must thus play a somewhat larger part than the instability due to α-particles in flight.

Apart from Alfvén waves we consider also fast magnetosonic waves in Sec. 4. We shall show that such waves can grow when the ratio of the Larmor radius of the α particles to the transverse dimensions of the plasma is finite.

2. THE BUILD-UP OF ALFVÉN WAVES BY α PARTICLES IN FLIGHT

We reduce the linearized set of hydromagnetic equations to the equation (cf.[9])

$$\mathbf{B}_0 \nabla \tilde{\alpha} + \frac{c \nabla \tilde{p}}{\mathbf{B}_0^4} [\mathbf{B}_0 \nabla (\mathbf{B}_0^2 + 8\pi p_0)] - \operatorname{div}\left[\frac{c\mathbf{B}_0}{\mathbf{B}_0^2}, \rho_0 \frac{d\tilde{\mathbf{V}}}{dt}\right] = 0. \tag{2.1}$$

Here B_0, p_0, and ρ_0 are the equilibrium values of the magnetic field, the pressure, and the plasma density, $\tilde{p}$, $\tilde{V}$ are the pressure and plasma velocity perturbations. The quantity $\tilde{\alpha}$ indicates the perturbation of the electric current along the magnetic field and is determined by the relation

$$\tilde{\alpha} = \frac{c}{4\pi \mathbf{B}_0^2} \mathbf{B}_0 \operatorname{rot} \tilde{\mathbf{B}}, \tag{2.2}$$

where $\tilde{B}$ is the magnetic field perturbation.

In the case of an axisymmetric tokamak with a circular cross section and for oscillation frequencies appreciably higher than the characteristic growth rate for the flute instability we get from Eq. (2.1) (for details see[9]):

$$(\omega^2 - c_A^2 k_{\parallel}^2)\xi_m - \frac{m^2}{a^2 k_\perp^2 R \rho_0}\left[\left(1 - i\frac{ak_a}{m}\right)\tilde{p}_{m-1} + \left(1 + i\frac{ak_a}{m}\right)\tilde{p}_{m+1}\right] = 0. \tag{2.3}$$

Here ξ_m is the m-th harmonic of the radial component of the plasma displacement ξ, $\xi = e^{im\theta - in\varphi}\xi_m$, where m and n are integers, and θ and φ cyclic coordinates along the small and the large azimuth of the torus, the displacement vector ξ is connected with the velocity $\tilde{V}$ through the relation $-i\omega\xi = \tilde{V}$, and with the perturbed magnetic field through $\tilde{B} = \operatorname{curl}[B_0 \times \xi]$. We take the time-dependence of the perturbation in the standard form $e^{-i\omega t}$. We choose the pressure perturbation in the form

$$\tilde{p} = e^{im\theta - in\varphi}(\tilde{p}_m + \tilde{p}_{m-1}e^{-i\theta} + \tilde{p}_{m+1}e^{i\theta}),$$

where the terms $\tilde{p}_{m\mp1}$ take the toroidality into account. We take the radial dependence of the perturbations in the form $\exp(ik_a a)$, where a is the radial coordinate; $k_\perp^2 \equiv k_a^2 + (m/a)^2$. The other quantities in (2.3) are: $c_A^2 = B_S^2/4\pi\rho_0$ is the square of the Alfvén velocity, $k_\parallel = (m - nq)/qR$ is the longitudinal component of the wavevector, R, q, B_S are the radius of curvature of the magnetic axis, the safety factor of the stability, and the longitudinal magnetic field (see Sec. 1).

To find the connection between $\tilde{p}_{m\mp1}$ and ξ_m we start from the expression for the perturbed distribution function in the form of an integral over trajectories (cf.[9]):

$$\tilde{f}=\frac{i\omega}{R}\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega\omega_B}\frac{\partial F}{\partial a}\right)\int_{-\infty}\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\left(\cos\theta+\frac{ak_a}{m}\sin\theta\right)\xi\,dt'. \quad (2.4)$$

Here $\epsilon = v^2/2$, v, $v_\perp$, and $v_\parallel$ are the total, the transverse, and the longitudinal (with respect to the direction of the magnetic field) velocities of the particle, $\omega_B = eB_S/Mc$ is the cyclotron frequency of the particle.

We assume that the particles in flight, which are the ones which interest us, move along the magnetic field lines with unchanged velocity. We then find from the drift equations[14] $\theta(t') = v_\parallel(t' - t)/qR$. Using that relation we can integrate over t' in (2.4) and we get

$$\tilde{f}=\frac{i\omega}{2R}\xi\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega\omega_B}\frac{\partial F}{\partial a}\right)\times\left[\frac{(ak_a/m-i)e^{i\theta}}{\omega-v_\parallel(k_\parallel+1/qR)}-\frac{(ak_a/m+i)e^{-i\theta}}{\omega-v_\parallel(k_\parallel-1/qR)}\right]. \quad (2.5)$$

Using

$$\tilde{p}=\int M(v_\perp^2/2+v_\parallel^2)\tilde{f}\,d\mathbf{v},$$

we evaluate $\tilde{p}_{m\mp1}$. We substitute the result into (2.3) after which we obtain the dispersion equation

$$\omega^2-c_A^2k_\parallel^2+\frac{\omega}{4R^2\rho_0}\int M\left(\frac{v_\perp^2}{2}+v_\parallel^2\right)^2\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega\omega_B}\frac{\partial F}{\partial a}\right)\left[\frac{1}{\omega-v_\parallel(k_\parallel-1/qR)}+\frac{1}{\omega-v_\parallel(k_\parallel+1/qR)}\right]d\mathbf{v}=0. \quad (2.6)$$

Hence we find that perturbations with $\mathrm{Re}\,\omega = c_A/qR$ have a growth rate

$$\mathrm{Im}\,\omega=\frac{\pi q}{8\rho_0 R}\frac{1}{|qRk_\parallel-1|}\int_{\varepsilon_\parallel}^{\infty}M(\varepsilon+\varepsilon_\parallel)^2\left(\frac{\partial F}{\partial\varepsilon}+\frac{mqR}{ac_A\omega_B}\frac{\partial F}{\partial a}\right)(2\varepsilon)^{1/2}d\varepsilon, \quad (2.7)$$

where the quantity $\epsilon_\parallel = c_A^2/2(qRk_\parallel - 1)^2$ indicates the longitudinal energy of the resonance particles per unit mass, $\epsilon_\parallel = v_{\parallel\mathrm{res}}^2/2$. We have assumed in Eq. (2.7) that the α-particle velocity distribution is isotropic so that $F = F(\epsilon, a)$ with the normalization

$$\int_0^\infty F(2\varepsilon)^{1/2}d\varepsilon=N_\alpha,$$

where N_α is the α-particle density.

Integrating by part we can verify that when $\partial F/\partial a = 0$ and $F(\epsilon)$ arbitrary the imaginary part $\mathrm{Im}\,\omega < 0$ (the oscillations are damped). The instability is thus only possible thanks to the gradient (drift) effect. If the α-particle energy distribution has a δ-function form, $F = \delta(\epsilon - \epsilon_\alpha)N_\alpha(a)/(2\epsilon_\alpha)^{1/2}$, we get from Eq. (2.7)

$$\mathrm{Im}\,\omega=-\frac{\pi q}{4\rho_0 R}\frac{N_\alpha M_\alpha(\varepsilon_\alpha+\varepsilon_\parallel)}{|qRk_\parallel-1|}\left[1-\frac{mqR\varkappa_n(\varepsilon_\alpha+\varepsilon_\parallel)}{2ac_A\omega_{B\alpha}}\right],\quad \varkappa_n\equiv\partial\ln N_\alpha/\partial a. \quad (2.8)$$

The condition for instability ($\mathrm{Im}\,\omega > 0$) which follows from this means qualitatively the same as the estimate (1.5) given earlier.

The growth rate (2.8) of the perturbations increases with decreasing $|qRk_\parallel = 1|$, but together with a decrease in $|qRk_\parallel - 1|$ the characteristic radial dimension of the perturbation also decreases. The combination $D \approx (\mathrm{Im}\,\omega)^2x^2/\mathrm{Re}\,\omega$, which characterizes the transfer coefficients remains therefore unchanged when $|qRk_\parallel = 1|$ changes.

Putting $\epsilon_\parallel \approx \epsilon_\alpha$ we get from (2.8) the estimate

$$\mathrm{Im}\,\omega\approx{}^3/_8\pi\beta_\alpha\,\mathrm{Re}\,\omega, \quad (2.9)$$

where $\beta_\alpha = {}^{16}/_3\,\epsilon_\alpha M_\alpha N_\alpha/B_S^2$ is the ratio of the α-particle pressure to the magnetic field pressure. We also used this estimate in Sec. 1.

Results close to (2.8) and (2.9) are also obtained for different velocity distributions. For instance, for a Maxwellian distribution, $F \propto \exp(-M_\alpha\epsilon/T_\alpha)$

$$\mathrm{Im}\,\omega=-\frac{\sqrt{\pi}}{4}\frac{N_\alpha T_\alpha q(1-\omega_*/\omega)\exp\{-M_\alpha\varepsilon_\parallel/T_\alpha\}}{\rho_0 Rv_{T\alpha}|k_\parallel qR-1|}\left(1+\frac{M_\alpha\varepsilon_\parallel}{T_\alpha}+\frac{M_\alpha^2\varepsilon_\parallel^2}{T_\alpha^2}\right),$$
$$v_{T\alpha}=(2T_\alpha/M_\alpha)^{1/2},\quad \omega_*=mT_\alpha\varkappa_n/aM_\alpha\omega_{B\alpha}. \quad (2.10)$$

When $|9Rk_\parallel - 1| \approx c_A/v_{T\alpha}$ we get from this an estimate of the form (2.9) but with a different, somewhat smaller, numerical coefficient.

We now discuss the role of the resonance electrons which, generally speaking, favor a damping of the oscillations. As a consequence of the fact that under thermonuclear conditions the thermal velocity of the electrons is large compared to the α-particle velocity (when $T_e \approx 15$ keV and $T_\alpha \approx 3.5$ MeV, we have $v_{Te}/v_{T\alpha} \approx 4$), only those electrons with a longitudinal velocity small compared to their thermal velocity can be in resonance. However, there are few electrons in flight with small longitudinal velocities due to the suppression of electrons with small $v_\parallel/v_\perp$ between the toroidal magnetic field mirrors. According to Coppi et al.[15] the suppression effect must lead to an additional factor in the electron damping rate of the order of $(\omega/|k_\parallel - 1/qR|v_{Te})^2$, i.e., in our case of the order of $(v_{T\alpha}/v_{Te})^2$. Taking this into account and also the fact that when there is no suppression effect the electron damping rate would have a form, similar to (2.10), we find that the damping by the electrons can be neglected, provided

$$\beta_\alpha/\beta_e>(v_{T\alpha}/v_{Te})^3, \quad (2.11)$$

where β_e is the ratio of the electron pressure to the magnetic field pressure. In the cases of practical interest this condition must be assumed to be satisfied.

Trapped electrons can also lead to a damping of the oscillations. The negative contribution of the trapped electrons to the growth rate of the oscillations is caused by the collisions of the trapped electrons with the other particles (ions and electrons in flight). According to[14] such a collisional damping of the oscillations by the trapped electrons is a maximum when $\nu_{e\,\mathrm{eff}}/\omega \approx 1$ which in our case means $\nu_e R^2/ac_A \approx 1$ ($\nu_{e\,\mathrm{eff}} \approx (R/a)\nu_e$, where ν_e is the electron collision frequency). However, for parameters of the plasma and the magnetic field typical for a thermonuclear reactor[16] (plasma density $n_0 \approx 3.10^{14}$ cm^{-3}, plasma temperature $T_0 \approx 15$ keV, magnetic field $B_0 \approx 4 \times 10^4$ gauss, $R \approx 10^3$ cm, $R/a \approx {}^7/_5$) the quantity $\nu_e R^2/ac_A$ is about $^1/_{20}$. The electron damping rate must therefore contain, apart from the usual factors β_α and $(a/R)^{1/2}$—the relative fraction of trapped electrons—also the small factor $\nu_e R^2/ac_A$. Taking this into account we can write the condition that we can neglect the damping by trapped electrons in the form

$$\beta_\alpha/\beta_e>(R/a)^{1/2}\nu_e R/c_A. \quad (2.12)$$

3. DRIFT BUILD-UP OF ALFVÉN WAVES BY TRAPPED α PARTICLES

We shall obtain the dispersion equation describing the build-up of Alfvén waves by trapped α-particles similarly as in Sec. 2, viz., by evaluating the perturbed pressure of the trapped α particles as function of the displacement ξ and substituting the result into the small oscillations Eq. (2.3).

The initial Eq. (2.4) for the perturbed distribution function is valid also in the case of trapped particles. We simplify this equation by noting that for trapped particles $v_\parallel^2 \ll v_\perp^2$, $v_\perp^2/2 \approx \epsilon = \text{const}$. We write the integral over t' in (2.4) as an average over the closed trajectory of the trapped particles (see, e.g.,[11]), so that

$$\tilde{f}=-\frac{\omega\varepsilon e^{i[(m-s)\theta-n\varphi]}\xi_m}{2R(\omega-\langle\omega_D\rangle)}\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega\omega_B}\frac{\partial F}{\partial a}\right)\left\langle\left[\left(\cos\theta+\frac{ak_a}{m}\sin\theta\right)e^{is\theta}\right]\right\rangle, \tag{3.1}$$

where $\langle\omega_D\rangle \equiv (\epsilon m/a\omega_B R)\langle\cos\theta\rangle$ is the average magnetic drift frequency, while the symbol $\langle\ldots\rangle$ means

$$\langle\ldots\rangle=\oint\ldots\frac{d\theta}{v_\parallel}\Big/\oint\frac{d\theta}{v_\parallel}. \tag{3.2}$$

In deriving (3.1) we have split off in the expression for $\xi \equiv \xi_m e^{i(m\theta-n\varphi)}$ the "flute" factor $e^{i(m-s)\theta-in\varphi}$ and taken it outside the integral sign, assuming that the perturbation is localized around a rational magnetic surface such that $m-s-nq\approx 0$. The number $s = 0, \pm 1, \pm 2$, and so on indicates the number of waves fitted into the length of the torus. The case $s = 0$ corresponds to an almost flute-like Alfvén wave with frequency $\omega \ll c_A/qR$. The case $s = 1$ corresponds to a wave of frequency $\omega \approx c_A/qR$ considered in the preceding section. When $s \geq 1$ the number s for the perturbations considered by us which are localized around rational magnetic surfaces is, apart from a factor, the same as the longitudinal wavenumber,

$$s=k_\parallel qR. \tag{3.3}$$

Using (3.1) we calculate the pressure perturbation, and after that we get from (2.3) the dispersion equation

$$\omega^2-c_A^2k_\parallel^2+\frac{\omega}{8\pi R^2\rho_0}\int\frac{B_s d\mu d\varepsilon M\varepsilon^2}{\omega-\langle\omega_D\rangle}\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega\omega_B}\frac{\partial F}{\partial a}\right)\times\oint\frac{d\theta}{v}|\langle[\sin(\lambda+\theta)e^{is\theta}]\rangle|^2=0, \tag{3.4}$$

where $\lambda = \arctan(m/ak_a)$.

We now take into account that the motion of the particles along the magnetic field lines of the Tokamak $B_0 = B_S\{1 + (a/R)\cos\theta\}$ proceeds with a velocity

$$v_\parallel=(2\varepsilon)^{1/2}\left[1-\frac{\mu B_s}{\varepsilon}\left(1+\frac{a}{R}\cos\theta\right)\right]^{1/2}. \tag{3.5}$$

Instead of the variable μ we introduce a variable κ, defining it by the relation (cf.[14])

$$\varkappa^2=\frac{1}{2}\left[1+\frac{R}{a}\left(1-\frac{\mu B_s}{\varepsilon}\right)\right]. \tag{3.6}$$

We can then write Eq. (3.4) in the form

$$\omega^2-k_\parallel^2c_A^2+\left(\frac{a}{R}\right)^{1/2}\frac{\omega}{4\pi R^2\rho_0}\times\int\frac{M\varepsilon^{5/2}d\varepsilon\,d\varkappa^2}{K(\varkappa)(\omega-\langle\omega_D\rangle)}\left(\frac{\partial F}{\partial\varepsilon}+\frac{m}{a\omega_B\omega}\frac{\partial F}{\partial a}\right)Q_s(\lambda,\varkappa), \tag{3.7}$$

$$Q_s(\lambda,\varkappa)=\left|\int_{\theta_0}\frac{d\theta\sin(\lambda+\theta)e^{is\theta}}{(2\varkappa^2-1-\cos\theta)^{1/2}}\right|^2. \tag{3.8}$$

Here $K(\kappa)$ is the complete elliptical integral of the first kind; θ_0 is defined by the relation $1 + \cos\theta_0 = 2\kappa^2$; the integration over κ is between the limits 0 and 1—this range of κ corresponds to trapped particles. The quantity $\langle\omega_D\rangle$ as function of κ takes the form

$$\langle\omega_D\rangle=-\frac{\varepsilon m}{a\omega_B R}\left[\frac{2E(\varkappa)}{K(\varkappa)}-1\right], \tag{3.9}$$

where $E(\kappa)$ is a complete elliptical integral of the second kind.

According to (3.7) resonance between particles and the wave occurs at frequencies of the order of the magnetic drift frequency, $\omega \approx -\epsilon m/\omega_B Ra$. Under those conditions the quantity $\partial F/\partial\epsilon$ is small compared to $(m/a\omega_B)\partial F/\partial a$ like a/R. We can thus neglect it on the left-hand side of Eq. (3.7). We then get from (3.7)

$$\mathrm{Re}\,\omega=k_\parallel c_A,$$
$$\mathrm{Im}\,\omega=\left(\frac{a}{R}\right)^{1/2}\frac{1}{8R^2\rho_0}\frac{mM}{a\omega\omega_B}\int\frac{\varepsilon^{5/2}d\varepsilon\,d\varkappa^2}{K(\varkappa)}\frac{\partial F}{\partial a}Q_s(\lambda,\varkappa)\delta(\omega-\langle\omega_D\rangle). \tag{3.10}$$

For an isotropic monoenergetic α-particle distribution this expression reduces to the following one:

$$\mathrm{Im}\,\omega=\sigma\left(\frac{R}{a}\right)^{1/2}\beta_\alpha\frac{c_A^2}{R^2\omega}, \tag{3.11}$$

$$\sigma=\frac{3}{16\sqrt{2}}\frac{\partial\ln N_\alpha}{\partial\ln a^2}\left[\frac{\varkappa}{K(\varkappa)}\frac{Q(\lambda,\varkappa)}{|\partial(E/K)/\partial\varkappa|}\right]_{\varkappa=\varkappa_0}, \tag{3.12}$$

where the quantity κ_0 satisfies the relation

$$2E(\varkappa_0)/K(\varkappa_0)-1=\omega\omega_B Ra/m\varepsilon_\alpha. \tag{3.13}$$

In deriving (3.11) we used the normalization condition

$$\int_0^\infty F(\varepsilon)(2\varepsilon)^{1/2}d\varepsilon=N_\alpha,$$

which follows for an isotropic $F(\epsilon,\mu)$ from the general normalization condition

$$\int\frac{F(\varepsilon,\mu)B_0\,d\eta\,d\varepsilon}{|v_\parallel|}=N_\alpha.$$

The perturbation which is built up has, according to (3.13), for a given frequency ω the longer a wavelength, i.e., it possesses a smaller m the larger the left-hand side of (3.13). The maximum of the left-hand side of Eq. (3.13) is reached for small κ_0, $\kappa_0 \ll 1$, which corresponds to resonance α particles with small $v_\parallel/v_\perp$. For small κ the function $Q_S(\lambda,\kappa)$ is independent of s and κ and given by the expression

$$Q_s(\lambda,\varkappa)={}^1/_2\pi^2\sin^2\lambda, \tag{3.14}$$

while the quantity σ takes the form

$$\sigma=\frac{3\pi^2}{32\sqrt{2}}\frac{\partial\ln N_\alpha}{\partial\ln a^2}\sin^2\lambda. \tag{3.15}$$

The resonance condition (3.13) then becomes simple:

$$\omega=m\varepsilon_\alpha/a\omega_B R. \tag{3.16}$$

We get from (3.16) and the first Eq. (3.10) for $k_\parallel = 1/qR$ an estimate for the characteristic wavenumber for the small azimuth of the torus:

$$m\rho_\alpha/a\approx c_A/v_\alpha \tag{3.17}$$

(here and in what follows the estimates are given for $q \approx 1$).

We find an estimate for $k_x \approx 1/x$ from the condition that the change in $k_\parallel c_A$ over a length of the order of x caused by the shear of the magnetic field lines must not exceed the magnitude c_A/R (cf. Sec. 1). This estimate gives

$$x\approx 1/k_x\approx a/m\approx\rho_\alpha v_\alpha/c_A, \tag{3.18}$$

so that we must assume that $\sin^2\lambda \approx 1$ in (3.15).

A general qualitative conclusion following from the analysis given above consists in the statement that trapped α particles may lead to the build-up of Alfvén waves with a frequency $\mathrm{Re}\,\omega \approx c_A/R$, transverse wavelengths of order $\lambda_\perp \approx \rho_\alpha v_\alpha/c_A$, and a growth rate $\mathrm{Im}\,\omega$ such that

$$\mathrm{Im}\,\omega/\mathrm{Re}\,\omega \approx (R/a)^{1/2}\beta_\alpha. \tag{3.19}$$

The condition that we can neglect the damping of the oscillations by trapped electrons is approximately of the same form as (2.12), but without the factor $(R/a)^{1/2}$ on the right-hand side,

$$\beta_\alpha/\beta_e > \nu_e R/c_A. \tag{3.20}$$

Such a relative decrease in the role of the trapped electrons is caused by the fact that when $\omega \approx \omega_D$ the growth rate due to trapped α particles as compared to the growth rate due to the particles in flight has a large factor of order $\omega_*/\omega_0 \approx R/a$ and a small factor $(a/R)^{1/2}$ —the fraction of trapped α-particles.

4. DRIFT BUILD-UP OF MAGNETOSONIC OSCILLATIONS

If the drift velocity of the α particles $V_{dr} \approx \rho_\alpha v_\alpha/a$ is comparable with the Alfvén velocity, i.e., if

$$\rho_\alpha/a \gtrsim c_A/v_\alpha, \tag{4.1}$$

it is necessary to take drift effects into account not only when considering Alfvén waves, but also for the case of fast magnetosonic waves.[17]

For a small plasma pressure, $\beta \ll 1$, the magnetosonic waves are described by a dispersion equation of a well known form (see, e.g.,[18])

$$c^2k^2/\omega^2 - \varepsilon_{22} = 0, \tag{4.2}$$

where k is the total wavenumber, ϵ_{22} is a component of the dielectric permittivity tensor in the system of coordinates where the 1 axis is directed along the component of the wavevector at right angles to the magnetic field B_0 and the 3 axis along B_0.

Both the ions from the basic component of the plasma and α particles contribute to ϵ_{22} so that $\epsilon_{22} = \epsilon_{22}^{(0)} + \epsilon_{22}^{(\alpha)}$. When $\omega \ll \omega_{Bi}$, $k_\perp \rho_\alpha \ll 1$—and these are just the perturbations which are of interest to us—the quantity $\epsilon_{22}^{(0)}$ has a well known form, $\epsilon_{22}^{(0)} = c^2/c_A^2$ (see, e.g.,[18]). Assuming the α-particle distribution to be Maxwellian, which—as was shown above—does not essentially limit the general nature of the results, we can use the expression for $\epsilon_{22}^{(\alpha)}$ given in a paper by Fridman and the present author,[19]

$$\varepsilon_{22}^{(\alpha)} = -\beta_\alpha \frac{c^2k_\perp^2}{\omega}\left(1 - \frac{\omega_*}{\omega}\right)\int \frac{F(v_\parallel)\,dv_\parallel}{\omega - k_\parallel v_\parallel}, \tag{4.3}$$

where

$$\beta_\alpha = 8\pi N_\alpha T_\alpha/B_s^2, \quad \omega_* = \varkappa_n m T_\alpha/\omega_{B\alpha} a. \tag{4.3$'$}$$

Under those assumptions it follows from (4.2) that

$$\mathrm{Re}\,\omega = kc_A,$$
$$\mathrm{Im}\,\omega = -\frac{\sqrt{\pi}}{2}\beta_\alpha \frac{c_A^2k_\perp^2 \exp\{-(kc_A/k_\parallel v_{T\alpha})^2\}}{|k_\parallel| v_{T\alpha}}\left(1 - \frac{\omega_*}{kc_A}\right). \tag{4.4}$$

It is clear that the maximum growth rate is reached when $k_\parallel/k_\perp \approx c_A/v_{T\alpha}$. The easiest perturbations to build up are those with $k_a \lesssim m/a$, as then $m/ak_\perp \approx 1$. The condition for the instability has the form (4.1) and the maximum growth rate is of the order of

$$\mathrm{Im}\,\omega \approx \beta_\alpha\,\mathrm{Re}\,\omega. \tag{4.5}$$

As regards the transverse wavelength, there is no lower bound for it, except that it must not be large compared to the transverse dimensions of the plasma, i.e., $m \approx 1$, $k_\perp \approx 1/a$.

The frequency of the oscillations $\mathrm{Re}\,\omega = kc_A$ is for $k \approx 1/a$ of the order of c_A/a, i.e., it is large compared to the frequencies of the Alfvén waves considered in Secs. 2, 3. If we take into account what we said in Sec. 2 about trapped particles we can thus conclude that their role in the damping of the oscillations will be negligibly small. As far as the damping of the magnetosonic oscillations by electrons in flight is concerned, this can be neglected, if condition (2.11) is satisfied.

5. DISCUSSION OF THE RESULTS

We have shown that there exists a class of thermonuclear drift instabilities with growth rates which depend on the spatial gradients of the α particles and on the curvature of the magnetic field lines (Secs. 2 and 3). Such instabilities are connected with the build-up of Alfvén waves by α particles. If the ratio of the Larmor radius of the α particles to the dimensions of the plasma inhomogeneity is finite, fast magnetosonic waves can also be built up by drift effects.

The coefficients for a turbulent transfer across the magnetic field caused by the instabilities studied by us turn out, as we estimate them, to be appreciably larger than the corresponding coefficients due to earlier studied instabilities, and, generally speaking, to be comparable to the Bohm diffusion coefficient. It seems thus that the instabilities considered above must be taken into account in the theory of reactors and tokamaks.

[1] L. V. Korablev, Zh. Eksp. Teor. Fiz. 53, 1600 (1967) [Sov. Phys.-JETP 26, 922 (1968)].

[2] Ya. I. Kolesnichenko and V. N. Oraevskiĭ, Atomnaya énergiya 23, 289 (1967).

[3] L. V. Korablev and L. I. Rudakov, Zh. Eksp. Teor. Fiz. 54, 818 (1968) [Sov. Phys.-JETP 27, 439 (1968)].

[4] V. S. Belikov, Ya. I. Kolesnichenko, and V. N. Oraevskiĭ, Zh. Eksp. Teor. Fiz. 55, 2210 (1968) [Sov. Phys.-JETP 28, 1172 (1969)].

[5] A. B. Mikhaĭlovskiĭ, Vopr. Teor. Pl. (M. A. Leontovich, Ed.) 6, 70 (1970).

[6] T. D. Kaladze, D. G. Lominadze, and K. N. Stepanov, Zh. Tekh. Fiz. 43, 2517 (1973).[Sov. Phys.-Tech. Phys. 18, 1583 (1974)].

[7] T. D. Kaladze, D. G. Lominadze, and K. N. Stepanov, Zh. Tekh. Fiz. 44, 273 (1974) [Sov. Phys.-Tech. Phys. 19, 175 (1974)].

[8] V. S. Belikov, Ya. I. Kolesnichenko, and V. N. Oraevskiĭ, Zh. Eksp. Teor. Fiz. 66, 1686 (1974) [Sov. Phys.-JETP 39, 828 (1974)].

[9] A. B. Mikailovsky, Nucl. Fusion 13, 259 (1973).

[10] B. B. Kadomtsev, Vopr. Teor. Pl. 2, 132 (1963) [Plasma Turbulence Academic Press, 1965].

[11] A. B. Mikhaĭlovskiĭ, Teoriya plazmennykh neustoĭchivosteĭ (Theory of Plasma Instabilities) Vol. 2, Atomizdat, 1971 [English translation published by Plenum].

[12] B. B. Kadomtsev, Vopr. Teor. Pl. 4, 188 (1965).

[13] B. B. Kadomtsev, ZhETF Pis. Red. 4, 15 (1966) [JETP Lett. 4, 10 (1966)].

[14] B. B. Kadomtsev and O. P. Pogutse, Zh. Eksp. Teor. Fiz. 51, 1734 (1966) [Sov. Phys.-JETP 24, 1172 (1967)].

[15] B. Coppi, M. N. Rosenbluth, and P. Rutherford, Phys. Rev. Lett. 21, 1055 (1968).

[16] I. N. Golovin, Yu. N. Dnestrovsky, and D. P. Kostomarov, Proc. Nucl. Fusion Reactor Conf., Culham, 1970, p. 194.

[17] A. Hasegawa, Phys. Rev. Lett. 27, 11 (1971).

[18] V. D. Shafranov, Vopr. Teor. Pl. 3, 3 (1963) [Rev. Plasma Phys. 3, 1 (1967)].

[19] A. B. Mikhaĭlovskiĭ and A. M. Fridman, Zh. Eksp. Teor. Fiz. 51, 1430 (1966) [Sov. Phys.-JETP 24, 965 (1967)].

Translated by D. ter Haar
191