Dynamic spontaneous breaking of gauge invariance in asymptotically free theories

A. A. Ansel'm and D. I. D'yakonov

Leningrad Institute of Nuclear Physics, USSR Academy of Sciences
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We consider the mechanism of dynamical spontaneous breaking of gauge invariance suggested by Coleman and Weinberg. In this mechanism scalar fields acquire nonzero vacuum expectation values due to quantum effects. With the help of renormalization-group technique we study scalar electrodynamics and gauge theories of the Yang-Mills type. We find that for the latter it is possible to reconcile the dynamical symmetry breaking with the asymptotic freedom of the theory, provided the coupling constants are appropriately chosen. The occurrence of symmetry breaking turns out to be closely associated with the existence of infrared poles in the coupling constants in the asymptotically free theories. The arising masses of vector mesons automatically screen these undesirable poles. It is shown that the physical results are independent (at least to the first nontrivial order of the perturbation-theory expansion) of the initial gauge of the vector fields.

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1. INTRODUCTION

The discovery of "asymptotic freedom," i.e., the vanishing of the effective charges at short distances in gauge theories of the Yang-Mills type,[1] has raised the hope of using quantum field theory to describe real phenomena. Even for strong interactions, where the use of field theory usually encounters serious difficulties due to inapplicability of perturbation theory, it may turn out that at very high energies the effective interaction becomes weak and perturbation theory calculations become really meaningful. If this were indeed to be found correct, then in strong interactions we would encounter for the first time a situation in which it is possible to use not only the general amplitude properties that follow from the basic principles of quantum field theory (analyticity, unitarity, etc.), but also carry out actual calculations. It must also be added that although the asymptotic freedom which means vanishing of the interaction at short distances, does not provide literally a field-theoretical foundation for the parton idea, owing to the excessively slow decrease of the interaction, it nevertheless leads to predictions that do not differ greatly from the parton predictions.

The main theoretical problem encountered at the present time by the asymptotically-free theories is how to get rid of massless gauge vector mesons without losing at the same time the asymptotic freedom. The usual mechanism with the aid of which gauge mesons can acquire a mass is the mechanism of spontaneous symmetry breaking of the Higgs type,[2] in which the mass is turned on "softly" enough not to change the asymptotic character of the theory.

Spontaneous symmetry breaking after Higgs has, however, a number of shortcomings. First, introduction of terms with negative mass squared into the Lagrangian is not particularly attractive. In addition, in asymptotically free theories in which there are no unphysical poles in the effective coupling constants in the ultraviolet region, poles do appear in the infrared region. Although the Higgs mechanism, which leads to the acquisition of mass by the particles, can in principle eliminate these infrared poles, the resultant particle masses are not connected in any way with the positions of the infrared poles.

In this paper we want to avoid these shortcomings of the Higgs model and, in particular, consider a model in which dynamic symmetry breaking would itself be due to the existence of the infrared poles, and the resultant masses would automatically cover up these poles so as to make them fictitious. By way of such a model we use the mechanism proposed by Coleman and Weinberg.[3] They considered the possibility of dynamic symmetry breaking as a result of interaction in higher orders of perturbation theory in the case when the scalar-field mass is chosen equal to zero. The last requirement calls for a clear-cut definition: what is assumed to be equal to zero is the renormalized mass of the scalar field. This condition, which must be imposed artificially in each succeeding order of perturbation theory, seems somewhat unnatural at first glance. And in general, fixation of a renormalized particle mass, which as a result of symmetry breaking vanishes from the theory in a certain sense (the true particles turn out to be quanta of the field differing from the initial value by a constant) may appear to be unphysical. It seems to us that it is precisely this aspect which is the starting point for the criticism of [3] on the part of Ioffe, Novikov, and Shifman.[4] We, however, cannot agree with this criticism. An attempt to formulate the theory immediately in terms of new physical particles does not answer the question of the stability of the initial field system. The Goldstone-Higgs phenomenon consists in the fact that if the potential energy (which does not depend on the derivative parts of the Lagrangian) takes as a function of the field φ the form shown in Fig. 1a, with a negative second derivative $d^2V/d\varphi^2 < 0$ at $\varphi = 0$, then spontaneous symmetry breaking takes place. If the curve takes the form shown in Fig. 1b, with $(d^2V/d\varphi^2)_{\varphi=0} > 0$, there is no symmetry breaking. As formulated by Coleman and Weinberg,[3] the problem is in essence to determine what takes place, in the sense of the stability of the system, if $(d^2V/d\varphi^2)_{\varphi=0} = 0$, and quantum effects are turned on. Then, $V(\varphi)$ is replaced by a certain effective potential V_{eff} (to be defined exactly later on), and the condition $(d^2V/d\varphi^2)_{\varphi=0} = 0$ is artificially imposed in each succeeding order of perturbation theory with the aid of mass renormalization. It seems to us that the Coleman and Weinberg formulation of the problem is quite valid and perhaps brings us closer to the under-

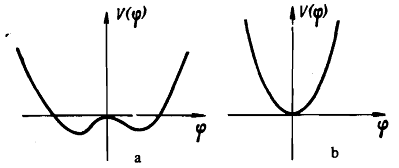

FIG. 1

standing of the situation in more realistic models, where the vanishing of all the masses (before symmetry breaking) is ensured by the symmetry itself and therefore does not require artificial renormalization in each succeeding order of perturbation theory, We emphasize finally that the condition that the renormalized mass of the field φ vanish is not obligatory. Spontaneous symmetry breaking takes place also at a positive square of this mass, so long as the mass does not reach a certain critical value.

The question of dynamic symmetry breaking in gauge theories of the Yang-Mills type, just as in quantum electrodynamics of massless charged scalar particles, was considered already by Coleman and Weinberg.[3] They, however, investigated in detail only the case when the scalar-field interaction constant λ is of order $\lambda \sim g^4$, where g is the constant of the interaction with the gauge fields. Yet from the point of view of asymptotic freedom, the critical value is $\lambda \sim g^2$; in this case the "single-loop" approximation, to which the analysis in [3] is principally confined, no longer holds. Indeed, the criterion for the probability of the single-loop approximation are the inequalities

$$\lambda \ll 1, \quad g^2 \ll 1, \quad \lambda \ln(\varphi/M) \ll 1, \quad g^2 \ln(\varphi/M) \ll 1,$$

where φ is the value of the scalar field and M is a certain normalization point (or cutoff constant). The expression for the effective potential in the single-loop approximation is of the form [3]

$$V_{eff} = \varphi^4[a\lambda + (b\lambda^2 + cg^4)\ln(\varphi/M)], \tag{1}$$

where $a > 0$, $b > 0$, and $c > 0$ are numerical constants. In order for the minimum of this expression to be in the region of applicability of the single-loop approximation, it is necessary to have $\lambda \ll g^2$. In the case $\lambda \sim g^2$ of interest to us, it is necessary to extend the region of applicability of the formulas to fields φ such that $\lambda \ln(\varphi/M) \sim 1$ (retaining the condition $\lambda \ll 1$). The renormalization group technique, also formulated in [3], is helpful here.

The main result of the present paper is the following: We show that the appearance of particle masses due to dynamic symmetry breaking is connected with the existence of an infrared pole of the effective charge $\lambda = \lambda(\ln(P/M))$ of the interaction of the scalar particles.

In many variants of the gauge theories, $\lambda(\ln(P/M))$ tends to infinity from the side of negative values, i.e., it has an infrared pole at a certain value of the momentum (see Fig. 3 and curve 3 of Fig. 4 below). The presence of this pole guarantees the passage through zero of the effective coupling constant $\lambda(\ln(P/M))$ at a certain value of the momentum larger than the pole value. From the physical point of view it is quite natural to expect the system to cease to be stable, owing to the sign reversal, and spontaneous symmetry breaking to take place. Our analysis confirms this expectation.

Spontaneous symmetry breaking, from the point of view of the effective potential,[3] corresponds to the presence of a minimum of V_{eff} at $\varphi \neq 0$. We shall show that the equation for the determination of this minimum (at small coupling constants) is

$$\lambda(\ln(\varphi/M)) \approx 0, \tag{2}$$

where λ is the same function as above, but of a different argument (for a more accurate equation see (37)).

If λ as a function of its argument has an infrared pole $\lambda \to -\infty$ (but not $+\infty$), then Eq. (2) always has a solution; spontaneous symmetry breaking takes place in such theories. The masses that result from this phenomenon turn out to be larger than the pole infrared value of the momentum, so that the poles are indeed offset by the particles masses. Thus, if spontaneous symmetry breaking of this kind takes place in a theory with asymptotic freedom, we are almost certain to arrive at a theory in which the effective charges have no unphysical poles in either the ultraviolet or the infrared region.[1)]

We note also that it is fortunate that symmetry breaking occurs not at large values of λ, close to the pole, where even the renormalization-group methods become in fact unsuitable, but at $\lambda \approx 0$, where perturbation theory is fully applicable for the calculation of the coefficients in the renormalization-group equations.

The renormalization-group equations used in the present paper make it also possible to answer the following important theoretical question: Jackiw[5] calculated the effective potential for the case of scalar electrodynamics in the single-loop approximation at an arbitrary gauge of the photon field (Coleman and Weinberg[3] carried out the calculations only in the Landau gauge). It has turned out that in this approximation $V_{eff}(\varphi)$ depends on the longitudinal part of the photon propagator α. Jackiw[5] expressed the fear that the very fact of dynamic symmetry breaking and the final formulas for the observed quantities may depend on the assumed gauge α.

It turns out that to answer this question it is likewise impossible to confine oneself to the single-loop approximation. In this paper we perform the calculations in an arbitrary gauge for scalar electrodynamics and for a nonabelian gauge theory, and verify with the aid of the renormalization-group technique that the final expressions are gauge-invariant (in the first nontrivial approximation in the coupling constants).

Finally, we wish to note one technical simplification in comparison with the approach of Coleman and Weinberg.[3] They determined the coefficients in the renormalization-group equations for the effective potential and for certain other analogous quantities from the single-loop approximation. We shall demonstrate, however, that in the lowest-order approximation in the coupling constants the coefficients in these equations coincide with the coefficients in the usual Callan-Symanzik renormalization-group equation, so they can be calculated not by summing an infinite number of diagrams, but using only the first few graphs.

In Sec. 2 we recall briefly the known "effective-action" formalism.[6] In Sec. 3 we consider the quantum electrodynamics of scalar massless particles for the case when the ratio of the scalar-particles interaction constant λ and the charge e^2 is arbitrary. We recall that in the electrodynamics of scalar particles it is always necessary to introduce into the Lagrangian a term of the type $\lambda\varphi^4$, without which (in contrast to electrodynamics of spinor particles) the theory cannot be renormalized (see, e.g. [7]). In Sec. 4 we investigate nonabelian gauge theories.

2. GENERAL FORMALISM

To make the exposition concise, we start with a brief description of the effective-action formalism.[6,3,8] As is well known (see [9], Chap. IV), the generating functional $W(J, J_\mu)$ for connective vertex functions $G^{(n)}$ of gauge theory can be written in terms of the functional integral

$$\exp\{iW(J,J_\mu)\}=\int D\varphi DA_\mu \exp\left\{i\int(L(\varphi,A_\mu)+J\varphi+J_\mu A_\mu)\,dx\right\}F(A_\mu)\Phi(A_\mu)\cdot\left[\int D\varphi DA_\mu \exp\left\{i\int L(\varphi,A_\mu)\,dx\right\}F(A_\mu)\Phi(A_\mu)\right]^{-1}, \quad (3)$$

where $F(A_\mu)$ is a functional that fixes the gauge and $\Phi(A_\mu)$ is the Faddeev-Popov gauge-invariant functional.

We have

$$W(J,J_\mu)=\sum_{n,k}\frac{1}{n!k!}\int dx_1\ldots dx_n\,dy_1\ldots dy_k G^{(n,k)}_{\mu_1\ldots\mu}(x_1,\ldots,y_1,\ldots)\, J(x_1)\ldots J(x_n)\cdot J_{\mu_1}(y_1)\ldots J_{\mu_k}(y_k), \quad (4)$$

where $G^{(n,k)}$ is a connective Green's function of n scalar and k vector fields. Here

$$\delta W/\delta J(x)=\langle\varphi(x)\rangle\equiv\varphi(x),\quad \delta W/\delta J_\mu(y)=\langle A_\mu(y)\rangle\equiv A_\mu(y),$$

where φ and A_μ are the vacuum-averaged or "classical" fields that become established in space under the influence of the field sources J and J_μ.

The effective action Γ is defined by the Legendre transformation

$$\Gamma(\varphi,A_\mu)=W-\int\varphi(x)J(x)\,dx-\int A_\mu(y)J_\mu(y)\,dy. \quad (5)$$

The most important property of the effective action, which justifies its introduction, is that

$$\delta\Gamma/\delta\varphi(x)=-J(x),\quad \delta\Gamma/\delta A_\mu(y)=-J_\mu(y).$$

It follows therefore that the stationary state of the system of fields remaining after the external sources are turned on is determined by the equation

$$\delta\Gamma=0.$$

If this equation has a solution at a nonzero mean value of the field φ, then spontaneous symmetry breaking sets in. We seek a solution of the type $|\varphi| = v = \text{const} \neq 0$, $A_\mu = 0$, which does not violate the Lorentz invariance of the theory. In fact, if the average field φ were to depend on the coordinates or if the average value of the vector A_μ were different from zero, then the Lagrangian resulting from the symmetry breaking would contain an explicit dependence on x or on a certain arbitrary direction.

It is easy to show that Γ is a generating function for vertex functions that are irreducible with respect to the single-particle divisions $\Gamma^{(n,k)}$:

$$i\Gamma=\sum_{n,k}\frac{1}{n!k!}\int dx_1\ldots dx_n\,dy_1\ldots dy_k\Gamma^{(n,k)}_{\mu_1\ldots\mu_k}(x_1,\ldots,x_n,y_1,\ldots,y_k)\cdot\varphi(x_1)\ldots\varphi^*(x_n)A_{\mu_1}(y_1)\ldots A_{\mu_k}(y_k). \quad (6)$$

Since we shall consider henceforth a complex (charged) field, half of the factors $\varphi(x_1)\ldots\varphi^*(x_n)$ in (6) are chosen to be complex-conjugate. It is seen from (6) that the effective action Γ is an invariant of the renormalization; in fact, if we change over to renormalized fields $\varphi\to\varphi Z_\varphi^{-1/2}$, $A_\mu\to A_\mu Z_A^{-1/2}$ and to renormalized vertex functions $\Gamma^{(n,k)}\to Z_\varphi^{n/2}Z_A^{k/2}\Gamma^{(n,k)}$, then the expression for Γ in terms of the renormalized quantities retains the same form. We shall henceforth assume that the renormalization had been carried out.

We note that both W and Γ depend on the choice of the functional $F(A_\mu)$, i.e., on the gauge of the vector field. In Sec. 3 we shall consider scalar electrodynamics, where we choose a functional that fixes the gauge in the form

$$F(A_\mu)=\exp\left\{-\frac{i}{2\alpha}\int dx(\partial_\mu A_\mu)^2\right\}. \quad (7)$$

As is well known, α has the meaning of the longitudinal part of the photon propagator.[9] Then $\Phi(A_\mu)$ can be regarded as equal to unity. In Sec. 4 we shall consider the theory of the interaction of a scalar field with the Yang-Mills field and also choose

$$F(\mathbf{A}_\mu)=\exp\left\{-\frac{i}{2\alpha}\int dx(\partial_\mu\mathbf{A}_\mu)^2\right\}. \quad (8)$$

It will then be convenient to write down $\Phi(\mathbf{A}_\mu)$ by introducing "ghosts" after Faddeev and Popov

$$\Phi(\mathbf{A}_\mu)=\int D\chi D\chi^*\exp\left\{i\int(-\partial_\mu\chi\partial_\mu\chi^*-g(\partial_\mu\chi^*[\mathbf{A}_\mu\chi]))\,dx\right\}, \quad (9)$$

after which we can consider the fictitious particles χ on a par with the particles φ and A_μ, and in particular we can introduce in formula (3) the source J_χ, which "raises the ghosts," and accordingly modify formulas (4)–(6).

It is obvious that any functional Γ can be represented in the form of an expansion in the fields and derivatives of various orders of the fields. Since we are interested in large but almost constant scalar fields and small vector fields, we confine ourselves to the first terms of such an expansion.[3] For scalar electrodynamics, for example, we can then write

$$\Gamma=\int dx\Big(-V(|\varphi|)+|i\partial_\mu\varphi+eA_\mu\varphi|^2Z(|\varphi|)-\frac{1}{4}F_{\mu\nu}{}^2T(|\varphi|)+\frac{1}{2\alpha}(\partial_\mu A_\mu)^2R(|\varphi|)+\ldots\Big), \quad (10)$$

where $|\varphi| = (\varphi\varphi^*)^{1/2}$, $V(|\varphi|)$, $Z(|\varphi|)$, etc. are certain unknown functions of the field ($V(|\varphi|)$ is called the "effective potential"). The fact that the common coefficient $Z(\varphi)$ can be used with an entire combination of fields will be proved below. (In nonabelian gauge theory this, generally speaking, cannot be done—see Sec. 4.)

It is easily seen that in the "tree approximation" the functional $W(J, J_\mu)$ takes the form

$$W_{tree}=\int L\,dx+\int J\varphi\,dx+\int J_\mu A_\mu\,dx,$$

so that the function Γ simply coincides with the usual action:

$$\Gamma_{tree}=\int L\,dx$$

(in these formulas it is necessary to include in L also the terms that fix the gauge). Therefore in the tree approximation expression (10) does not contain any terms except those written out above, with[2)] $Z(\varphi) = T(\varphi) = R(\varphi) = 1$, and $V(\varphi) = 1/6\lambda(\varphi\varphi^*)^2$. After calculating (10) with allowance for the higher approximations of perturbation theory, the expression under the integral sign can be taken to mean the "effective Lagrangian," which contains even in the lowest perturbation-theory approximation processes that take into account an arbitrary number of φ particles (in analogy with the Heisenberg-Euler Lagrangian for photon interaction).

If the equation $\delta\Gamma = 0$, which defines the stationary vacuum, has a solution at $|\varphi| = v \neq 0$, then the really observed particles will be the quanta of the field $\eta = |\varphi| - v$. The effective Lagrangian can be then rewritten in terms of the new field η, for which it remains, naturally, the effective Lagrangian as before, i.e., it includes in the lowest order processes in which an arbitrary number of η particles take part. The term proportional to η^2 will be the mass term of the new field.

Equating to zero the variation of the effective action (10) with respect to the fields (it is convenient to parametrize $\varphi(x) = 2^{-1/2}\rho(x)e^{i\theta(x)}$ beforehand), and recognizing that no constant vectors can be used in the construction of the solution, we arrive at the conclusion that the solution is $A_\mu = \partial_\mu\theta = 0$, while v is defined by the equation

$$V'|_{\rho=v}=0. \tag{11}$$

Finally, changing over to the momentum representation and comparing expressions (6) and (10), we obtain

$$V(\varphi)=-\sum_n \Gamma^{(n,0)}(p_i=0)\frac{\varphi^{n/2}(\varphi^*)^{n/2}}{n!}; \tag{12}$$

$$Z(\varphi)=-\sum_n \Gamma^{(n,0)\prime\prime}(p_i=0)\frac{\varphi^{(n-2)/2}(\varphi^*)^{(n-2)/2}}{(n-2)!},$$

$$\Gamma^{(n,0)\prime\prime}g_{\mu\nu}=\frac{\partial^2\Gamma^{(n,0)}(p_i)}{\partial p_{i\mu}\,\partial p_{j\nu}}\Big|_{i\neq j,\,p_i=0} \tag{13}$$

where $\Gamma^{(n,0)}(p_i)$ is the vertex function corresponding to the aggregate of the single-particle irreducible diagrams with n scalar-particle external lines; it is understood here that $(2\pi)^4\delta(\sum p_i)$ has been separated from $\Gamma^{(n,0)}(p_i)$, and that all the momenta are set equal to zero.

3. SCALAR ELECTRODYNAMICS

In this section we find the effective Lagrangian (10) for massless scalar electrodynamics, without fixing the gauge parameter α (7), and verify, first, that in this theory spontaneous breaking of local gauge invariance must take place, i.e., that the "photon" acquires a mass, and second, that the observed quantities are independent here of the choice of the number α.

It is appropriate to recall here the already mentioned fact (see, e.g. [7]) that pure scalar electrodynamics is not a closed theory. Speaking more concretely, in addition to the usual primitive divergences (of the mass, charge, scattering of light by light), which can be eliminated by renormalizing the mass and the charge and also by the gauge-invariance requirement, in scalar electrodynamics there is also one logarithmic divergence, namely, scalar-particle scattering due to exchange of two photons. This divergence can be eliminated only by introducing the counterterm $\lambda\varphi^4$, i.e., the selfaction of the scalar particles, and thus, the scattering amplitude of scalar charge particles is not determined uniquely via the renormalized mass and charge even in second order, but also via one arbitrary parameter. We note that analogous counterterms must be used also in the nonabelian theory. Thus, it is meaningless to consider "pure" scalar electrodynamics; self-action of the type $-\lambda|\varphi|^4/6$ must additionally be introduced into the Lagrangian.

We turn to the calculation of the effective potential in such a theory. As seen from the definition (12), the potential is determined by diagrams with different numbers of external scalar lines. These diagrams are infrared-divergent. As shown in [3], after summing over n, the infrared divergence vanishes. Instead of the explicit summation we wish to use now the equations of the renormalization group. It is quite natural to consider first, instead of diagrams with zero external momenta, diagrams in which the external momenta have certain nonzero values (space-like). For example, all the momenta can be equal in magnitude and form a regular closed n-sided polygon, so that $\sum p_i = 0$. It is convenient to introduce

$$\tilde{V}=-\sum_n \Gamma^{(n,0)}(p_i)\frac{|\varphi|^n}{n!} \tag{14}$$

and also, in addition to (13), the quantity

$$\tilde{Z}=-\sum_n \Gamma^{(n,0)\prime\prime}(p_i)\frac{|\varphi|^{n-2}}{(n-2)!}\quad \text{etc.} \tag{15}$$

In the quantities $\Gamma^{(n,0)}(p_i)$, $\Gamma^{(n,0)\prime\prime}(p_i)$, etc. it is impossible to take the limit as $p_i \to 0$, owing to the infrared divergences. This can be done, however, after summing over n for the quantities $\tilde{V}$ and $\tilde{Z}$; in this case $\tilde{V} \to V$ and $\tilde{Z} \to Z$ ($p_i \to 0$).

It is understood that (14) and (15) contain renormalized vertex functions $\Gamma^{(n,k)}$ that depend explicitly on the normalization point and also on the renormalized λ, e^2, α ($d\Gamma^{(2,0)}(p^2)/dp^2 = 1$, $\Gamma^{(4,0)}(p^2) = \lambda$ etc. at $p^2 = -M^2$). Then, as already discussed in the introduction, the renormalization of the mass is also assumed to be satisfied and the renormalized mass is set equal to zero. The quantities $\Gamma^{(n,k)}$ satisfy the "massless" Callan-Symanzik equation:

$$\left(M\frac{\partial}{\partial M}+\hat{D}+n\gamma_\varphi+k\gamma_A\right)\Gamma^{(n,k)}(p_i,M,\lambda,e^2,\alpha)=0; \tag{16}$$

$$\hat{D}=\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_{e^2}\frac{\partial}{\partial e^2}+\delta\frac{\partial}{\partial\alpha},\quad \beta_\lambda=M\frac{\partial\lambda}{\partial M},\quad \beta_{e^2}=M\frac{\partial e^2}{\partial M},$$

$$\delta=M\frac{\partial\alpha}{\partial M},\quad \gamma_\varphi=-\frac{1}{2}M\frac{\partial\ln Z_\varphi}{\partial M},\quad \gamma_A=-\frac{1}{2}M\frac{\partial\ln Z_A}{\partial M}. \tag{17}$$

The latter quantities are obtained from the equations that connect the constants specified for the different values of the normalization mass:

$$\lambda=Z_\varphi^2Z_\lambda^{-1}\lambda_0,\quad e=Z_A^{1/2}Z_\varphi Z_e^{-1}e_0,\quad \alpha=Z_A^{-1}\alpha_0, \tag{18}$$

where Z_φ and Z_A are the renormalization constants of the wave function of the field φ and the field A_μ; Z_λ^{-1} and Z_e are the renormalization constants of the corresponding vertices. From these equations we obtain two relations, of which the first

$$\delta=\frac{\partial\alpha}{\partial\ln M}=\alpha_0\frac{\partial Z_A^{-1}}{\partial\ln M}=-(2\alpha_0Z_A^{-1})\frac{1}{2}\frac{\partial\ln Z_A}{\partial\ln M}=2\alpha\gamma_A \tag{19}$$

holds also in nonabelian theory, and the second is typical only of electrodynamics, since it makes use of the Ward identity $Z_\varphi = Z_e$, whence

$$\beta_{e^2}=e_0^2\frac{\partial Z_A}{\partial\ln M}=2(e_0^2Z_A)\frac{1}{2}\frac{\partial\ln Z_A}{\partial\ln M}=-2e^2\gamma_A. \tag{20}$$

We note that of all the quantities in (17), only γ_φ is not gauge-invariant. Equation (16) (with k = 0) is satisfied also by the function $\Gamma^{(n,0)\prime\prime}$ which enter in (15).

It follows from the Eqs. (16) jointly with the definitions (14) and (15) that $\widetilde{V}$ and $\widetilde{Z}$ satisfy the peculiar renormalization-group equations:

$$\left(M\frac{\partial}{\partial M}+\hat{D}+\gamma_\varphi|\varphi|\frac{\partial}{\partial|\varphi|}\right)\widetilde{V}=0,$$

$$\left(M\frac{\partial}{\partial M}+\hat{D}+\gamma_\varphi|\varphi|\frac{\partial}{\partial|\varphi|}+2\gamma_\varphi\right)\widetilde{Z}=0. \tag{21}$$

As already mentioned, $\widetilde{V}\to V$ and $\widetilde{Z}\to Z$ as $p_i\to 0$. It is easy to verify that the real condition under which $\widetilde{V}\approx V$ and $\widetilde{Z}\approx Z$ is $p_i^2\ll\lambda|\varphi|^2, e^2|\varphi|^2$. This can be verified directly, say in the single-loop approximation, by carrying out the summation in (12), (13), and in (14), (15). There is, however, a more general argument. The point is that summation over a number of external lines (to each of which there corresponds a coupling constant multiplied by the value of the field φ) is equivalent to replacing the internal propagators in a certain skeleton diagram by an exact propagator in the "external" field φ.[3)] The propagator $a/(-k^2)$ of the field φ must then be replaced by $1/(-k^2+\lambda\varphi^2/2)$, and the quantity $e^2|\varphi|^2$ must be added in the denominator of the transverse part of the photon propagator. It is therefore clear that if the external momenta $p_i^2\ll\lambda|\varphi|^2, e^2|\varphi|^2$, then the answer turns out to be the same as if they were equal to zero. Consequently, V and Z satisfy the same equations (21) as $\widetilde{V}$ and $\widetilde{Z}$.

To determine V and Z from these equations, it is necessary to specify the initial conditions for a certain value of φ, say at $\varphi = M$. But at φ close to M we can find V and Z from the single-loop approximation, inasmuch as we assume the coupling constants to be small. We shall prove below, incidentally, that to find V at Z in the lowest order of perturbation theory it is not even necessary to sum the single-loop diagrams, but for the time being we shall carry out a detailed analysis for the sake of greater clarity.

Direct calculation of V and Z in the single-loop approximation yields

$$V(|\varphi|)=\frac{|\varphi|^4}{6}\left[\lambda+\frac{\lambda^2}{16\pi^2}\frac{10}{3}\ln\left(\frac{|\varphi|}{M}\xi_1\right)\right. \tag{22}$$

$$\left.-\frac{\lambda e^2\alpha}{16\pi^2}4\ln\left(\frac{|\varphi|}{M}\xi_2\right)+\frac{e^4}{16\pi^2}36\ln\left(\frac{|\varphi|}{M}\xi_3\right)\right],$$

$$Z(|\varphi|)=1+\frac{e^2}{16\pi^2}6\ln\left(\frac{|\varphi|}{M}\eta_1\right)-\frac{e^2\alpha}{16\pi^2}2\ln\left(\frac{|\varphi|}{M}\eta_2\right). \tag{23}$$

The first terms in both expressions are the contributions of the tree approximation. $\xi_i, \eta_i = O(\sqrt{\lambda}, e)$. Putting $|\varphi| = M$ in (22) and (23) we get

$$\left.\frac{6V}{|\varphi|^4}\right|_{\varphi=M}=\lambda+O(\lambda^2, e^4, \lambda e^2\alpha),$$

$$Z|_{\varphi=M}=1+O(e^2, e^2\alpha).$$

Thus, in the lowest order of perturbation theory we can use the following initial conditions (it is convenient to introduce in place of V the dimensionless quantity $U = 6V/\lambda\varphi^4$, and regard U and Z as functions of the dimensionless quantity $t = \ln(|\varphi|/M)$):

$$U|_{t=0}=1,\quad Z|_{t=0}=1. \tag{24}$$

The equations for these quantities take the form

$$\left(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+4\gamma_\varphi+\frac{\beta_\lambda}{\lambda}\right)U=0, \tag{25}$$

$$\left(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\varphi\right)Z=0. \tag{26}$$

We add to them the equations for the functions T and R (see (10))

$$\left(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_A\right)T=0, \tag{27}$$

$$\left(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}\right)R=0 \tag{28}$$

with the initial conditions

$$T|_{t=0}=1,\quad R|_{t=0}=1. \tag{29}$$

Equations (25)–(27) (in the Landau gauge: $\alpha = 0$) were written out in [3]; we wish only to emphasize here that in the lowest order of perturbation theory there is no need to determine the coefficients of the equations by summing diagrams with different numbers of external lines—these coefficients can be obtained in accordance with the usual prescriptions (17) and (18).

What happens in the next higher orders of perturbation theory? We can reason here in two ways. If we leave unchanged Eqs. (15)–(28), in which the functions β and γ are defined in terms of the ordinary "momentum" renormalization constants (18), then it becomes necessary to modify somewhat the initial conditions for the sought functions, namely, instead of (24) and (29) it is necessary to find expressions for these functions in the form (22) and (23), but with greater accuracy, and to put in them $|\varphi| = M$. We can, however, proceed differently. The initial conditions (24) and (29), which are taken to be exact, define certain "field" coupling constants that do not coincide exactly with the usual λ and e^2. The field and ordinary coupling constants are expanded in series in terms of one another. Naturally, the field functions β and γ, i.e., the coefficients in Eqs. (25)–(28), expressed in terms of the field constants, will likewise, generally speaking, not coincide with the "momentum" functions β and γ (as we have shown, only the first terms of the expansion coincide).

Let us turn to first-order perturbation theory in scalar electrodynamics. The calculation of the logarithmically-divergent part of certain simple diagrams, shown in Fig. 2, leads to the following expressions:

$$Z_A=1-\frac{e^2}{16\pi^2}\frac{2}{3}\ln M,\quad Z_\varphi=1+\frac{e^2}{16\pi^2}(6-2\alpha)\ln M,$$

$$Z_\lambda^{-1}=1-\left(\frac{\lambda}{16\pi^2}\cdot\frac{10}{3}+\frac{e^4/\lambda}{16\pi^2}\cdot 36-\frac{e^2\alpha}{16\pi^2}\cdot 4\right)\ln M, \tag{30}$$

whence, using (17)–(20), we obtain

$$\gamma_A=-\frac{e^2}{16\pi^2}\frac{1}{3},\quad \beta_{e^2}=\frac{e^4}{16\pi^2}\frac{2}{3},\quad \gamma_\varphi=\frac{e^2}{16\pi^2}(3-\alpha),$$

$$\delta=-\frac{e^2\alpha}{16\pi^2}\frac{2}{3},\quad \beta_\lambda=\frac{1}{16\pi^2}\left(\frac{10}{3}\lambda^2-12e^2\lambda+36e^4\right). \tag{31}$$

We call attention to the fact that β_λ, as expected, does not depend on the gauge α.

The general solution of (25)–(28) with arbitrary initial conditions can be represented in the form

$$\begin{aligned} U&=F_U(\lambda(t), e^2(t), \alpha(t))\,\zeta^2(t)\,\varkappa(t),\\ Z&=F_Z(\lambda(t), e^2(t), \alpha(t))\,\zeta(t),\\ T&=F_T(\lambda(t), e^2(t), \alpha(t))\,\xi(t)\\ R&=F_R(\lambda(t), e^2(t), \alpha(t)), \end{aligned} \tag{32}$$

where $\lambda(t)$, $e^2(t)$, and $\alpha(t)$ are the effective charges and

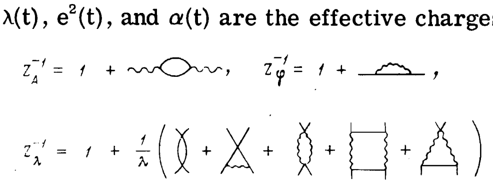

FIG. 2

the gauge, defined by equations of the Gell Mann–Low type:

$$\frac{d\lambda(t)}{dt}=\bar\beta_\lambda(\lambda(t),e^2(t),\alpha(t)),$$

$$\frac{de^2(t)}{dt}=\bar\beta_{e^2}(\lambda(t),e^2(t),\alpha(t)),\quad \frac{d\alpha(t)}{dt}=\delta(\lambda(t),e^2(t),\alpha(t));$$

$$\bar\beta_\lambda=\frac{\beta_\lambda}{1-\gamma_\varphi},\quad \bar\beta_{e^2}=\frac{\beta_{e^2}}{1-\gamma_\varphi},\quad \bar\gamma_\varphi=\frac{\gamma_\varphi}{1-\gamma_\varphi},$$

$$\bar\gamma_A=\frac{\gamma_A}{1-\gamma_\varphi},\quad \bar\delta=\frac{\delta}{1-\gamma_\varphi}\tag{33}$$

with the initial conditions

$$\lambda(0)=\lambda,\quad e^2(0)=e^2,\quad \alpha(0)=\alpha.\tag{34}$$

The remaining quantities in (32) will be written out in the form

$$\zeta(t)=\exp\left\{2\int_0^t\bar\gamma_\varphi(\lambda(t'),e^2(t'),\alpha(t'))dt'\right\},$$

$$\varkappa(t)=\exp\left\{\int_0^t\frac{\bar\beta_\lambda(\lambda(t'),e^2(t'),\alpha(t'))}{\lambda(t')}dt'\right\}=\exp\left\{\int_0^t\frac{d\ln\lambda(t')}{dt'}dt'\right\}=\frac{\lambda(t)}{\lambda},$$

$$\xi(t)=\exp\left\{2\int_0^t\bar\gamma_A(\lambda(t'),e^2(t'),\alpha(t'))dt'\right\}=\frac{e^2}{e^2(t)}.\tag{35}$$

$F_U,\ldots$ are arbitrary functions of their arguments, and can be determined in principle from the initial conditions. Taking the latter in the form (24) and (29), we see that all these functions should be identically equal to unity. We note that formulas (32)–(35) are exact and are not connected with perturbation theory.

Substituting (32) in (10) we obtain the following effective Lagrangian:

$$L_{eff}=-\frac{\lambda(t)}{6}|\varphi|^4\zeta^2(t)+|i\partial_\mu\varphi+eA_\mu\varphi|^2\zeta(t)-\frac{1}{4}F_{\mu\nu}^2\xi(t)-\frac{1}{2\alpha}(\partial_\mu A_\mu)^2$$

or, writing φ in the form $\varphi(x)=2^{-1/2}\rho(x)e^{i\theta(x)}$, we have

$$L_{eff}=-\frac{\lambda(t)\zeta^2(t)\rho^4}{4!}+\frac{\zeta(t)}{2}(\partial_\mu\rho)^2+\frac{\rho^2}{2}\zeta(t)\,(eA_\mu+\partial_\mu\theta)^2$$
$$-\frac{1}{4}\xi(t)F_{\mu\nu}^2-\frac{1}{2\alpha}(\partial_\mu A_\mu)^2,\tag{36}$$

where $t=\ln(\rho/\Lambda\sqrt2$. Equation (11) for finding the vacuum mean field ρ now takes the form

$$\frac{\partial V}{\partial\rho}=\frac{\zeta^2(t)\rho^3}{3!}\left[\lambda(t)+\frac{1}{4}\left(\lambda'(t)+\frac{2\lambda(t)\zeta'(t)}{\zeta(t)}\right)\right]$$
$$=\frac{\zeta^2\rho^3}{3!}\left(\lambda+\frac{1}{4}\bar\beta_\lambda+\lambda\bar\gamma_\varphi\right)=\frac{\zeta^2\rho^3}{3!}\frac{1}{1-\gamma_\varphi}\left(\lambda+\frac{1}{4}\beta_\lambda\right)=0.$$

Thus, the condition for spontaneous symmetry breaking takes the following simple form:

$$\lambda(t)+{}^1/_4\beta_\lambda(\lambda(t),e^2(t))=0.\tag{37}$$

In order that the stationary point be stable, it is necessary that $\partial^2V/\partial\rho^2$ at $\rho=v$ be a positive quantity. We have:

$$\left.\frac{\partial^2V}{\partial\rho^2}\right|_{\rho=v}=\left.\frac{\partial^2V}{\partial\rho^2}\right|_{\lambda+\beta_\lambda/4=0}$$
$$=\frac{\zeta^2v^2}{6}\left[\beta_\lambda\left(1+\frac{1}{4}\frac{\partial\beta_\lambda}{\partial\lambda}\right)+\frac{1}{4}\left(\frac{\partial\beta_\lambda}{\partial e^2}\right)\beta_{e^2}\right]\left.\frac{1}{(1-\gamma_\varphi)^2}\right|_{\lambda+\beta_\lambda/4=0}>0.\tag{38}$$

For small coupling constants, (37) and (38) reduce to the conditions

$$\lambda(t)\approx0,\quad \beta_\lambda(\lambda=0,e^2)>0.\tag{39}$$

The conditions (39) are general in character. Specifically, for scalar electrodynamics, the solution of (33) with initial conditions (34) yields (see Fig. 3)

$$e^2(t)=e^2\Big/\left(1-\frac{e^2}{16\pi^2}\frac{2}{3}t\right),$$

$$\lambda(t)=\frac{e^2}{10(1-{}^2/_3(e^2/16\pi^2)t)}\Bigg\{19$$
$$+\sqrt{719}\,\mathrm{tg}\Bigg[-\frac{\sqrt{719}}{2}\ln\left(1-\frac{e^2}{16\pi^2}\frac{2}{3}t\right)$$
$$+\mathrm{arctg}\frac{10\lambda/e^2-19}{\sqrt{719}}\Bigg]\Bigg\}.\tag{40}$$

It is clear from Fig. 3 that for any choice of the initial λ and e^2 there exists a point t_V at which Eq. (37) is satisfied ($t_V=\ln(v/M\sqrt2)$). At $\lambda=-\beta_\lambda/4\approx0$ we have $\beta_\lambda=9e^4/4\pi^2>0$, i.e., the stationary solution is stable. Thus, in this scheme scalar electrodynamic exhibits a property such that for any choice of the interaction constants λ and e^2 (provided they are much smaller than unity) spontaneous violation of local gauge invariance must take place (as a result of which the photon acquires mass).

Thus, let Eq. (37) have a solution at certain t_V and let the condition (38) be satisfied. As already indicated, we must now introduce a field $\eta(x)=\rho(x)-v$, after which the Lagrangian (36) becomes the effective Lagrangian for the field $\eta(x)$. We see, however, that the terms corresponding to the kinetic energy are written out in this Lagrangian not in canonical form; to arrive at the usual form of the terms it is necessary to renormalize the wave functions of the fields $\eta(x)$ and $A_\mu(x)$. (In this case the quantities $\zeta(t)$ and $\xi(t)$ can be replaced by $\zeta(t_V)$ and $\xi(t_V)$ with accuracy up to corrections of order $\sim e^2\ln(1+\eta/v)\ll1$ or $\sim\lambda\ln(1+\eta/v)\ll1$). Thus, we introduce

$$\eta'=\eta\sqrt{\zeta(t_v)},\quad v'=v\sqrt{\zeta(t_v)},\quad A_\mu'=A_\mu\sqrt{\xi(t_v)}.\tag{41}$$

after which we obtain the following Lagrangian:[4)]

$$L'=-\frac{v'^2}{6}\beta_\lambda(\lambda=0)\frac{\eta'^2}{2!}-\frac{5v'}{6}\beta_\lambda(\lambda=0)\frac{\eta'^3}{3!}-\frac{11}{6}\beta_\lambda(\lambda=0)\frac{\eta'^4}{4!}-\ldots$$
$$\ldots+\frac{1}{2}(\partial_\mu\eta')^2+\frac{1}{2}(e(t_v)A_\mu'+\partial_\mu\theta)^2(v'+\eta')^2-\frac{1}{4}F_{\mu\nu}'^2-\frac{e^2(t_v)}{2\alpha e^2}(\partial_\mu A_\mu')^2.\tag{42}$$

We introduce a new field $A_\mu''=A_\mu'+e^{-1}(t_V)\partial_\mu\theta$ and use the fact that by virtue of (33) and (34) we have $\alpha(t)=\alpha e^2/e^2(t)$; the last terms of (42) then are rewritten in the form

$$\frac{1}{2}e^2(t_v)A_\mu''^2(v'+\eta')^2-\frac{1}{4}(F_{\mu\nu}'')^2-\frac{1}{2\alpha(t_v)}\left(\partial_\mu A_\mu''-\frac{1}{e(t_v)}\partial^2\theta\right)^2.$$

The first two terms contain here the free Lagrangian of the massive vector meson with mass $m_V=e(t_V)v'$ and the

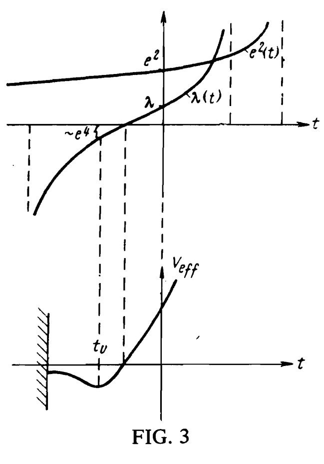

FIG. 3

interaction of this meson with the field η'. It is easily seen that the last term does not lead in general to any observable effect whatever, and can be omitted. In fact, the functional integral that defines any Green's function contains integrations with respect to $DA''_\mu(x)$ and $D^\theta(x)$. Carrying out first the integration with respect to $D^\theta(x)$ at constant $A''_\mu(x)$, we can change variables by putting

$$\theta(x)=\theta'(x)+\theta_0(x),$$

where $\theta_0(x)$ is an arbitrary solution of the equation

$$\Box\theta_0(x)=e(t_v)\,\partial_\mu A_\mu''(x).$$

The last term in (42) is rewritten in the form

$$-(\partial^2\theta'(x))^2/2\alpha(t_v)e^2(t_v),$$

after which it is clear that the integration with respect to $D\theta'$ (the Jacobian of the transition from $\theta(x)$ to $\theta'(x)$ is equal to unity) is separated from the integration with respect to DA''_μ and yields a constant factor that is cancelled out upon normalization. The field θ' itself does not interact with anything, so that the corresponding degrees of freedom are fictitious, as they should be. We note that the foregoing proof of the possibility of leaving out the last terms in the Lagrangian (42) applies also to the case of the usual Higgs mechanism of spontaneous symmetry breaking.

We rewrite once more, taking the foregoing remark into account, the Lagrangian (12), omitting the primes in the symbols for the fields η and A_μ:

$$L'=-\frac{v'^2}{6}\beta_\lambda(0)\frac{\eta^2}{2!}-\frac{5v'}{6}\beta_\lambda(0)\frac{\eta^3}{3!}-\frac{11}{6}\beta_\lambda(0)\frac{\eta^4}{4!}-\ldots+\frac{1}{2}(\partial_\mu\eta)^2$$
$$+\frac{1}{2}e^2(t_v)v'^2A_\mu^2+\frac{1}{2}e^2(t_v)A_\mu^2(2\eta v'+\eta^2)-\frac{1}{4}F_{\mu\nu}^2. \qquad (43)$$

We see that the Lagrangian is determined by two dimensionless gauge-invariant quantities and one dimensional quantity v'. It follows therefore that all the dimensionless parameters of the new theory do not depend on the gauge in which the calculations are performed. The ratio of the squares of the masses of the scalar and vector particles is

$$\frac{m_s^2}{m_v^2}=\frac{\beta_\lambda(\lambda(t_v)\approx 0)}{6e^2(t_v)}=\frac{6e^2(t_v)}{16\pi^2}.$$

Substituting here the value $e^2(t_V)$, which can be obtained from (40) ($\lambda(t_V\approx 0)$, we obtain

$$\frac{m_s^2}{m_v^2}=\frac{6e^2}{16\pi^2}\exp\left\{-\frac{2}{\sqrt{719}}\left(\operatorname{arctg}\frac{19}{\sqrt{719}}+\operatorname{arctg}\frac{10\lambda/e^2-19}{\sqrt{719}}\right)\right\}. \qquad (44)$$

Expression (44) generalizes the result of Coleman and Weinberg,[3] where the mass ratio of the scalar and vector particles was written out under the assumption $\lambda\sim e^4$ (with $m_S^2/m_V^2=6e^2/16\pi^2$).

As to the absolute values of the quantities with dimension of mass, all are proportional to $v'=v\sqrt{\zeta(t_V)}$ and seem to be non-gauge-invariant (since $\zeta(t_V)$ depends on the gauge α of the photon propagator). In fact, however, we cannot state that the average field v itself should be gauge-invariant, since one observes not v but the masses of the scalar and vector particles, which are proportional to v'. If we thus assume v' to be gauge-invariant and v to be non-invariant, this is equivalent to stating that the arbitrary normalization point $M = v\exp(-t_V)$ is also gauge-dependent. The possibility of the dependence of the average field on the gauge was pointed out by Tyutin and Fradkin[10] and by Applequist et al. as cited in [8] (Sec. 21).

Thus, we see that in the first nontrivial order of perturbation theory spontaneous symmetry breaking in Abelian theory does not depend on the chosen gauge of the photon propagator.

The situation with the gauge invariance in the higher orders of perturbation theory does not seem clear to us. On the one hand, the exact stationarity equation (37) and the stability condition (38) have a gauge-invariant form (the noninvariant factor $1-\gamma_\varphi$ is factored out). However, as already discussed, in the higher orders the functions β_λ and β_{e^2} do not coincide with the usual Callan-Symanzik functions and can, generally speaking, depend on the gauge α. It is possible, incidentally, to use also the usual functions β_λ and β_{e^2}, but then the initial conditions in the higher orders will be more complicated and can also depend on α. It must be remembered, however, that there is always a leeway in the choice of the normalization point M; by making it explicitly dependent on α, we can attempt to exclude α from the final answers.

4. SPONTANEOUS VIOLATION OF GAUGE INVARIANCE IN NONABELIAN THEORIES

We consider the Lagrangian of scalar fields that transform in accordance with a complex vector representation of the group SU_N. The effective Lagrangian in the tree approximation can be written in the form

$$L=-\frac{1}{4}(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+g[\mathbf{A}_\mu\times\mathbf{A}_\nu])^2+|i\partial_\mu\boldsymbol{\varphi}+g(\mathbf{A}_\mu\mathbf{t})\boldsymbol{\varphi}|^2-\frac{\lambda}{2}(\boldsymbol{\varphi\varphi}^*)^2$$
$$-[(\partial_\mu\boldsymbol{\chi},\partial_\mu\boldsymbol{\chi}^*)+g(\partial_\mu\boldsymbol{\chi}^*,[\mathbf{A}_\mu\times\boldsymbol{\chi}])]-\frac{1}{2\alpha}(\partial_\mu\mathbf{A}_\mu)^2, \qquad (45)$$

where χ are ghosts.

The effective action is, on the one hand, the generating functional for the irreducible vertex functions:

$$\Gamma=\sum_{n,m,l}\frac{1}{n!m!l!}\int\Gamma^{(n,m,l)}_{\mu_1\ldots\mu_m;\,a_1\ldots a_n,\,b_1\ldots b_m,\,c_1\ldots c_l}(x_1,\ldots;y_1,\ldots,z_1,\ldots)\cdot\varphi^{a_1}(x_1)\ldots$$
$$\ldots\varphi^{a_n}(x_n)A^{b_1}_{\mu_1}(y_1)\ldots A^{b_m}_{\mu_m}(y_m)\chi^{c_1}(z_1)\ldots\chi^{c_l}(z_l)(dx)(dy)(dz), \qquad (46)$$

where $\Gamma^{(n,m,l)}$ is an irreducible vertex with n external scalar particles, m vector particles, and l ghosts particles (summation is carried out with respect to $\mu_1,\ldots,\mu_m, a_1,\ldots,a_n, b_1,\ldots, c_1\ldots$; $2\le a, c\le 2N$, $1\le b\le N^2-1$, with n and l running through only even values and m through arbitrary values). On the other hand, as discussed in Sec. 2, the effective action can be represented as an integral of the effective Lagrangian taken in the form (45), where each term has a factor that depends on $|\varphi| = \sqrt{\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}^*}$. We omit here terms of higher order either in the vector field $\mathbf{A}_\mu$, or in the number of differentiations of the field $\boldsymbol{\varphi}$.

Thus,

$$L=-{}^1/_4(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu)^2T(\varphi)-{}^1/_2gC_{abc}(\partial_\mu A_\nu^a-\partial_\nu A_\mu^a)A_\mu^bA_\nu^cT_1(\varphi)$$
$$-{}^1/_4g^2C_{abc}C_{adc}A_\mu^bA_\nu^cA_\mu^dA_\nu^eT_2(\varphi)+(\partial_\mu\boldsymbol{\varphi},\partial_\mu\boldsymbol{\varphi}^*)Z(\varphi)-ig[(\hat{A}_\mu\boldsymbol{\varphi},\partial_\mu\boldsymbol{\varphi}^*)$$
$$-(\partial_\mu\boldsymbol{\varphi},\hat{A}_\mu\boldsymbol{\varphi}^*)]Z_1(\varphi)+g^2(\hat{A}_\mu\boldsymbol{\varphi},\hat{A}_\mu\boldsymbol{\varphi}^*)Z_2(\varphi)-{}^1/_2\lambda(\boldsymbol{\varphi\varphi}^*)^2U(\varphi)$$
$$-(\partial_\mu\boldsymbol{\chi},\partial_\mu\boldsymbol{\chi}^*)X(\varphi)-g(\partial_\mu\chi^a)A_\mu^b\chi^cC_{abc}X_1(\varphi)-(\partial_\mu\mathbf{A}_\mu)^2R(\varphi)/2\alpha, \qquad (47)$$

$$\hat{A}_\mu=\mathbf{A}_\mu\mathbf{t}.$$

The quantities T, Z, U, . . . satisfy the renormalization-group equations, which are simplest to derive by noting that by virtue of the Callan-Symanzik equations for the irreducible vertices

$$\left[M\frac{\partial}{\partial M}+\hat{D}+n\gamma_\varphi+m\gamma_A+l\gamma_\chi\right]\Gamma^{(n,m,l)}=0,$$

$$\hat{D}=\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_g\frac{\partial}{\partial g}+\delta\frac{\partial}{\partial\alpha},$$

the effective action (46) satisfies the equation

$$\Big(M\frac{\partial}{\partial M}+\hat{D}+\gamma_\varphi\int dx\,\varphi^a(x)\frac{\delta}{\delta\varphi^a(x)}+\gamma_A\int dy\,A_\mu^b(y)\frac{\delta}{\delta A_\mu^b(y)}+\gamma_\chi\int dz\,\chi^c(z)\frac{\delta}{\delta\chi^c(z)}\Big]\Gamma=0. \quad (48)$$

Substituting in (48) the function Γ taken in the form of an integral of the effective Lagrangian (47) over all of full space, and equating separately to zero the coefficients of the different structures in (47), we obtain

$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+4\gamma_\varphi+\frac{\beta_\lambda}{\lambda}\Big)U=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\varphi\Big)Z=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\varphi+\gamma_A+\frac{\beta_g}{g}\Big)Z_1=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\varphi+2\gamma_A+\frac{2\beta_g}{g}\Big)Z_2=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_A\Big)T=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+3\gamma_A+\frac{\beta_g}{g}\Big)T_1=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+4\gamma_A+\frac{2\beta_g}{g}\Big)T_2=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\chi\Big)X=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_\chi+\gamma_A+\frac{\beta_g}{g}\Big)X_1=0,$$
$$\Big(-(1-\gamma_\varphi)\frac{\partial}{\partial t}+\hat{D}+2\gamma_A-\frac{1}{\alpha}\delta\Big)R=0, \quad (49)$$

where the quantities $\beta_\lambda, \ldots, \gamma_A$ are defined in (17), and in addition $\gamma_\chi = -\frac{1}{2}M\partial\ln Z_\chi/\partial M$, $t = \ln(|\varphi|/M)$. The initial conditions obtained from the tree approximation (45) take the form

$$U=Z=Z_1=Z_2=T=T_1=T_2=X=X_1=R=1, \qquad t=0. \quad (50)$$

We confine ourselves below to the first order of perturbation theory. The discussion of the higher orders does not differ in practice from the case of quantum electrodynamics.

We recall that in Abelian theory we had two relations for the coefficients in the renormalization-group equations, namely (19) and (20). The first is common to all gauge theories; its result, in particular, is that R = 1 for all t. The second is characteristic only of Abelian theory, where there is a simple Ward identity $Z_\varphi = Z_e$; in electrodynamics it has resulted in the fact that three generally speaking different coefficients Z, Z_1, and Z_2 have satisfied the same renormalization-group equations with identical initial conditions, and have therefore coincided. But now $\gamma_A \neq -\beta_g/g$, and therefore $Z \neq Z_1 \neq Z_2$. For the same reason we have $T \neq T_1 \neq T_2$ and $X \neq X_1$.

The solution of (49) with the initial conditions (50), in accordance with the prescription (32)–(35), is given by

$$\begin{gathered} U=\zeta^2(t)\lambda(t)/\lambda, \quad Z=\zeta(t), \\ Z_1=\zeta(t)\xi^{1/2}(t)g(t)/g, \quad Z_2=\zeta(t)\xi(t)g^2(t)/g^2, \\ T=\xi(t), \quad T_1=\xi^{3/2}(t)g(t)/g, \\ T_2=\xi^2(t)g^2(t)/g^2, \quad X=\varkappa(t), \\ X_1=\varkappa(t)\xi^{1/2}(t)g(t)/g, \quad R=1, \end{gathered} \quad (51)$$

where $\lambda(t)$ and $g(t)$ are solutions of equations of the Gell-Mann–Low type

$$\begin{gathered} d\lambda(t)/dt=\beta_\lambda, \quad \lambda(0)=\lambda, \quad dg(t)/dt=\beta_g, \\ g(0)=g, \quad d\alpha(t)/dt=\delta, \quad \alpha(0)=\alpha; \end{gathered} \quad (52)$$

$$\begin{gathered} \zeta(t)=\exp\Big\{2\int_0^t\gamma_\varphi(g(t'),\lambda(t'),\alpha(t'))dt'\Big\}, \\ \xi(t)=\exp\Big\{2\int_0^t\gamma_A(g(t'),\lambda(t'),\alpha(t'))dt'\Big\}, \\ \varkappa(t)=\exp\Big\{2\int_0^t\gamma_\chi(g(t'),\lambda(t'),\alpha(t'))dt'\Big\}. \end{gathered} \quad (53)$$

The equation for "breaking" takes as before the form (37): $\lambda(t_V) + \frac{1}{4}\beta_\lambda(\lambda(t_V), g(t_V)) = 0$, and the stability condition takes the form (38), which reduces to $\beta_\lambda(\lambda = 0) > 0$ in the lowest order of perturbation theory.

We consider next for simplicity the case of the SU_2 group, and then the field $\varphi(x)$ can be parametrized in the following manner:

$$\varphi(x)=2^{1/2}\rho(x)\exp\{i\theta_a t_a\}\begin{pmatrix}0\\1\end{pmatrix}.$$

We introduce, just as in the case of electrodynamics, the shifted field $\eta(x) = \rho(x) - v$ and renormalize the field:

$$\eta'=\eta\sqrt{\zeta(t_v)}, \quad v'=v\sqrt{\zeta(t_v)}, \quad \mathbf{A}_\mu'=\mathbf{A}_\mu\sqrt{\xi(t_v)}, \quad \chi'=\chi\sqrt{\varkappa(t_v)}.$$

Finally, we introduce in place of $\mathbf{A}'_\mu$

$$\mathbf{A}_\mu''=\mathbf{A}_\mu'-g^{-1}(t_v)\partial_\mu\boldsymbol{\theta}+[\boldsymbol{\theta}\times\mathbf{A}_\mu'],$$

and omit the primes on the field symbols. As a result the new Lagrangian takes the form

$$\begin{aligned} L'=&-\frac{1}{4}(\partial_\mu\mathbf{A}_\nu-\partial_\nu\mathbf{A}_\mu+g(t_v)[\mathbf{A}_\mu\times\mathbf{A}_\nu])^2+\frac{1}{2}(\partial_\mu\eta)^2+\frac{1}{8}g^2(t_v)(\eta+v')^2\mathbf{A}_\mu^2 \\ &-v'^2\frac{1}{2}\beta_\lambda(\lambda=0)\frac{\eta^2}{2!}-v'\frac{5}{2}\beta_\lambda(\lambda=0)\frac{\eta^3}{3!}-\frac{11}{2}\beta_\lambda(\lambda=0)\frac{\eta^4}{4!}-\ldots+L(\mathbf{A}_\mu,\chi,\boldsymbol{\theta}). \end{aligned} \quad (54)$$

The last term results from the substitution of $\mathbf{A}''_\mu$ in the remaining part of the effective Lagrangian (47). As a result there appears, at first glance, a system of fields A, χ, and θ which interact in a complicated manner (in contrast to electrodynamics, where A_μ and the "extra" field θ entered only linearly and quadratically, and therefore it was possible to integrate with respect to θ in explicit form). Here, however, just as in electrodynamics, it is possible to integrate formally with respect to all fields $\boldsymbol{\chi}$, $\boldsymbol{\chi}'$ and $\boldsymbol{\theta}$, and to verify that this part of the effective Lagrangian does not result in any observable effects, and can therefore be omitted.

Indeed, let us consider the generating functional of the system (54) (see Sec. 2), which we write in the form

$$\exp\{iW(J,J_\mu)\}=\int D\mathbf{A}_\mu D\eta\, D\boldsymbol{\theta}\, D\chi\, \partial\chi^* \times\exp\Big\{i\int dx(L(A)+L(\eta,A)+L(\eta)+L(\chi,A^\theta)+\mathbf{J}_\mu\mathbf{A}_\mu+J\eta)\Big\}F(A^\theta),$$

where

$$\begin{gathered} \mathbf{A}_\mu^\theta=\mathbf{A}_\mu-\frac{1}{g(t_v)}\partial_\mu\boldsymbol{\theta}+[\boldsymbol{\theta}\times\mathbf{A}_\mu], \\ L(A)=-\frac{1}{4}\mathbf{F}_{\mu\nu}^2, \quad L(\eta)=-V(\eta)+\frac{1}{2}(\partial_\mu\eta)^2, \\ L(A,\eta)=\frac{1}{8}g^2(t_v)(\eta+v')^2\mathbf{A}_\mu^2, \\ L(\chi,A^\theta)=-(\partial_\mu\chi\partial_\mu\chi^*)-g(t_v)(\partial_\mu\chi^*[\mathbf{A}_\mu\times^\theta\chi]), \\ F(A)=\exp\Big\{-\frac{i}{2\alpha}\int dx(\partial_\mu\mathbf{A}_\mu)^2\Big\}. \end{gathered}$$

We recall that

$$\int D\chi D\chi^*\exp\Big\{i\int L(\chi,A)dx\Big\}=\Phi(A)=\Big(\int F(A^\theta)D\theta\Big)^{-1}$$

is the Faddeev-Popov gauge-invariant functional. Hence

$$\int D\theta D\chi\,\partial\chi^{*}\exp\left\{i\int L(A^{\theta},\chi)\,dx\right\}F(A^{\theta})=\int D\theta\,F(A^{\theta})\Big/\int D\theta'\,F(A^{\theta\theta'})=1$$

by virtue of the known group property of the measure: $D(\theta\theta') = D(\theta')$. Thus, the sought generating functional of the theory, following the spontaneous breaking of the local symmetry is simply

$$\exp\{iW(J,J_\mu)\}=\int DA_\mu D\eta\exp\left\{i\int dx\,(L(A)+L(\eta)+L(A,\eta)+J_\mu A_\mu+J\eta)\right\},$$

which coincides with the results that will be obtained from (54) if the last term were omitted.

All the dimensionless ratios in the Lagrangian (54) are gauge-invariant quantities. As to the dimensional quantities, the reasoning at the end of the preceding section applies to them.

We have thus arrived at a simple rule for finding the effective Lagrangians. The procedure is to take the initial Lagrangian (without functionals that fix the gauge) and simply rewrite it by substituting for the interaction constants effective charges that depend on the values of the field and are determined by equations of the Gell-Mann–Low type. (We recall that the effective Lagrangian acquires this form after suitable renormalization of the wave functions of the fields.) The condition for "breaking" is determined from (37) and (38).

Let us apply the developed formalism to some concrete example. We consider N complex scalar fields that transform in accordance with the vector representation of the group SN_N and interact with $N^2 - 1$ Yang-Mills vector fields. In addition, let the Yang-Mills fields interact with a definite number of fermions, and let there be no interaction of the Yukawa type. The quantities β_λ and β_g are obtained by using the calculations of Gross and Wilczek[1]

$$\beta_g=-{}^1\!/_2 b_0 g^3,\quad \beta_\lambda=A\lambda^2+B'\lambda g^2+Cg^4, \tag{55}$$

where $b_0 = 1/8\pi^2(^{11}\!/_3 N - {}^1\!/_6$ are the contributions from the fermions),

$$A=\frac{1}{8\pi^2}(N+4),$$
$$B'=-\frac{1}{8\pi^2}\frac{3(N^2-1)}{N},$$
$$C=\frac{1}{8\pi^2}\frac{3(N-1)(N^2+2N-2)}{4N^2} \tag{56}$$

The renormalization-group equations take here the form

$$dg^2(t)/dt=-b_0g^4(t),\quad g^2(0)=g^2,$$
$$d\lambda(t)/dt=A\lambda^2(t)+ +B'\lambda(t)g^2(t)+Cg^4(t),\quad \lambda(0)=\lambda. \tag{57}$$

The first of these equations can be solved directly

$$g^2(t)=g^2/(1+b_0g^2t), \tag{58}$$

and corresponds to the antiscreening character of the Yang-Mills fields (for positive b_0). We solve the second equation by introducing the variable $\kappa(t) = \lambda(t)/g^2(t)$. The equation for $\kappa(t)$ is

$$-\frac{d\kappa(t)}{d\ln g^2(t)}=a\kappa^2(t)+b\kappa(t)+c, \tag{59}$$

where $a = A/b_0$, $b = (B' + b_0)/b_0$, $c = C/b_0$.

Assume that we have introduced a sufficiently large number of fermion multiplets, so that their condition

$$b<-\sqrt{4ac} \tag{60}$$

is satisfied. Then the first part of (59) has two positive real roots

$$\kappa_{1,2}=(-b\mp\sqrt{b^2-4ac})/2a,\quad 0<\kappa_1<\kappa_2. \tag{61}$$

Integrating (59) and using (58), we obtain $\lambda(t)$:

$$\lambda(t)=\frac{g^2}{1+b_0g^2t}\,\frac{\kappa_2(\kappa-\kappa_1)-\kappa_1(\kappa-\kappa_2)(1+b_0g^2t)^\Delta}{(\kappa-\kappa_1)-(\kappa-\kappa_2)(1+b_0g^2t)^\Delta}, \tag{62}$$

where $\Delta = a(\kappa_2 - \kappa_1) > 0$, $\kappa = \lambda(0)/g^2(0) = \lambda/g^2$. Depending on which of the two regions is chosen to contain the initial value of κ, we obtain three possible behaviors of $\lambda(t)$—see Fig. 4, where curve 1 corresponds to $\kappa > \kappa_2$, curve 2 to $\kappa_1 < \kappa < \kappa_2$, and curve 3 to $\kappa < \kappa_1$.

The first possibility corresponds to an ultraviolet pole of $\lambda(t)$; the second and third correspond to asymptotically free behavior of both effective charges, but with poles in the infrared region. However, only the third possibility leads to spontaneous symmetry breaking, for only in this case does the equation $\lambda(t_V) \approx 0$ have a solution. It is easy to show here that the resultant masses of the scalar and vector particles turn out to be larger than those values of the momenta at which $\lambda(\ln(P/M))$ or $g^2(\ln(P/M))$ become infinite. The reason is that the masses of the particles are equal, generally speaking, to $\sim M\exp(t_V)$, whereas the pole value is ~~xxxxx~~ $p_\infty \sim M\exp(t_1)$ and $t_V - t_1 \sim \mathrm{const}/g^2 > 0$. Thus, if (60) is satisfied and if at some normalization point λ and g^2 are such that $\lambda/g^2 < \kappa_1$, then, first, asymptotic freedom with respect to both constants will obtain in the theory, and second, spontaneous breaking of local gauge invariance will take place, and the infrared poles will turn out to be cancelled out.[5)]

We note that if the group and number of fermion multiplets were chosen such that (16) were not satisfied, then at all values of λ and g^2 the function $\lambda(t)$ would pass through zero, just as in electrodynamics. But then there would be no asymptotic freedom in terms of the interaction constant $\lambda(t)$.

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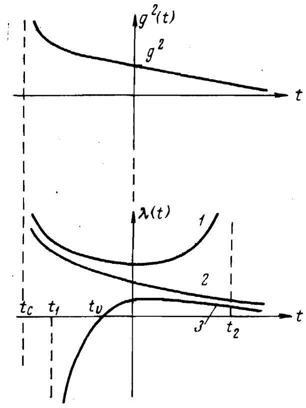

FIG. 4. The functions $g^2(t)$ and $\lambda(t)$; here

$$t_1=t_c\left[1-\left(\frac{\kappa_1-\kappa}{\kappa_2-\kappa}\right)^{1/\Delta}\right]<0,$$
$$t_v\approx t_c\left[1-\left(\frac{\kappa_2}{\kappa_1}\frac{\kappa_1-\kappa}{\kappa_2-\kappa}\right)^{1/\Delta}\right]>t_1,$$
$$t_2=-t_c\left[\left(\frac{\kappa-\kappa_1}{\kappa-\kappa_2}\right)^{1/\Delta}-1\right]>0$$

[1]We note incidentally that, just as in the case when the spontaneous symmetry breaking is of the Higgs type, it is very difficult to choose a variant in which all of the vector fields acquire mass (with the possible exception of the electromagnetic one). The number of vector fields that remain massless is determined not by any concrete dynamic mechanism but from general group-theoretical considerations.

[2]The factor 1/6 in the coupling constant is chosen in accordance with [3].

[3]This statement is not quite correct for the single-loop approximation, which is, by virtue of the cyclic symmetry, a degenerate case calling for greater accuracy. Qualitatively, however, this reasoning is of course true also for this approximation.

[4]The exact expression for the effective potential, written in terms of the new field η', can be represented in the form

$$V(\eta')=\frac{1}{3!}\beta_{\cdot}(\lambda=0)v'^4\int\limits_0^{\eta'/v'}(1+x)^3\ln(1+x)\,dx.$$

[5]In the considered concrete case, only $2N-1$ vector particles acquire mass, and the remaining N^2-2N remain massless. Asymptotic freedom with respect to the constant λ is possible here only at $N\geqslant 3$, and only if there is a sufficiently large number of fermion multiplets [1].

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