Laser heating of a transparent medium containing random absorbing inhomogeneities

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Heating of a transparent medium containing randomly distributed absorbing particles of arbitrary size by laser radiation is considered. The probability distribution of the temperature field produced in the medium is calculated. It is shown that under real conditions the distribution is not Gaussian and is determined mainly by the large-size particles. The shape of the distribution does not depend on the heat-exchange conditions at the boundary of the body.

In this paper we consider the heating, by laser radiation, of a transparent medium containing random inclusions of foreign particles that absorb the radiation. The inclusions produce in the medium a complicated random temperature field, which can lead to a number of physical effects, for example, to melting of the medium in strongly heated regions, and so on. To describe these phenomena it is necessary to know the probability characteristics of the temperature field.

The purpose of this study was to calculate the probability density $T_{x,t}(T)$ of obtaining a temperature T at the point **x** at the instant of time t. A similar problem was considered by I. M. Lifshitz[1]; the method used by us is analogous to that developed in his paper, except for one essential difference that will be indicated later on.

The temperature field $T(\mathbf{x}, t)$ in the medium is determined by the thermal-conductivity equation

$$\frac{\partial T}{\partial t}=\chi\Delta T+j(t)\,q(\mathbf{x})\sum_k \alpha(R_k)\,\delta(\mathbf{x}-\mathbf{x}_k) \qquad (1)$$

with the initial condition

$$T(\mathbf{x},0)=0$$

and a suitable boundary condition, which we shall not spell out here concretely. The functions $q(\mathbf{x})$ and f(t) describe the spatial and temporal dependences of the laser- and radiation flux density ($j(t) = 0$ at $t < 0$); χ is the thermal diffusivity of the medium; $\mathbf{x}_k$ is the random radius vector of the k-th inhomogeneity; R_k is its random dimension. The particle dimensions are assumed to be small in comparison with the distances between them; such particles can be regarded as point sources of heat. The dimensions of the particles determine here the source power, and this is described by the function $\alpha(R)$, which can be obtained by solving the problem of diffraction of light by a particle. Depending on the optical parameters of the inclusion, and also on its shape and dimensions, the absorption can occur either in the entire volume of the particle (in which case the power is proportional to R^3), or else on its surface (power $\propto R^2$), and an intermediate case is also possible. It can be shown[2] that a power-law dependence of α on R holds at any ratio of R to the radiation wavelength. It is therefore natural to assume that in the entire range of dimensions of interest to us we can interpolate $\alpha(R)$ with sufficient accuracy by the expression

$$\alpha(R)=R^l \quad (l=2\div3).$$

The constant coefficient in $\alpha(R)$ is actually of no importance for our purpose, and we include it in $q(\mathbf{x})$.

The spatial distributions of the absorbing particles will be assumed to be independent and homogeneous in the entire considered volume of the sample V. In random realizations of such a system, the number of particles N contained in the volume V is a random quantity with a Poisson distribution and with a mean value nV, where n is the particle-number density.

The normalized probability density for the random radius of the particle will be designated f(R),

$$\int_0^\infty f(R)\,dR=1.$$

As shown in[3], the experimental results on optical breakdown of transparent dielectrics with absorbing inclusions agrees with the assumption that at large R the function f(R) decreases in power-law fashion:

$$f(R)\approx A/R^{m+1} \quad (m\sim4). \qquad (2)$$

On the other hand, the behavior of the function f(R) at small values of R, as will be seen later on, is immaterial for our analysis.

The solution of Eq. (1) can be written in general form with the aid of the Green's function $G(\mathbf{x}, \boldsymbol{\xi}, t-t')$ of the thermal-conductivity equation in the considered region, namely, introducing the following symbol for the temperature field of a unit source:

$$\tau(\mathbf{x},\boldsymbol{\xi},t)=\int_0^t j(t')\,q(\boldsymbol{\xi})\,G(\mathbf{x},\boldsymbol{\xi},t-t')\,dt',$$

we have

$$T(\mathbf{x},t)=\sum_{k=1}^N \alpha_k\tau_k, \qquad \alpha_k=\alpha(R_k),\quad \tau_k=\tau(\mathbf{x},\mathbf{x}_k,t). \qquad (3)$$

Averaging (3) over N, R_k, and $\mathbf{x}_k$, we obtain for the average temperature

$$\bar{T}(\mathbf{x},t)=nV\bar{\alpha}\bar{\tau}(\mathbf{x},t), \qquad \bar{\alpha}=\int_0^\infty f(R)\,\alpha(R)\,dR,\quad \bar{\tau}=\frac{1}{V}\int_V \tau(\mathbf{x},\boldsymbol{\xi},t)\,d^3\xi. \qquad (4)$$

The character of the average distribution of the temperature and its dependence on the time are determined by the boundary conditions in Eq. (1).

Although the temperature, according to (3), is indeed the sum of a large number of independent equally-distributed random quantities, the probability distribution for it is generally speaking not determined by the central limit theorem of probability theory, since an individual term in (3) may not have a variance (see below). However, even in this case the probability distribution of the suitably normalized sum

$$\Theta(\mathbf{x}, t)=[T(\mathbf{x}, t)-\bar{T}(\mathbf{x}, t)](nV)^{-\nu}$$

(the exponent will be defined later on) can converge to a certain stable law, which is obtained in the usual manner [4].

To this end we calculate the mean value ($\langle\ldots\rangle$) of the quantity $e^{-i\omega\Theta}$ (the characteristic function). The averaging over N is carried out immediately. Using (3) and (4), we obtain

$$\langle\exp\{-i\omega\Theta\}\rangle=\exp\left\{i\omega\frac{\bar{T}}{(nV)^{\nu}}\right\}\left\langle\prod_{k=1}^{N}\exp\left\{-i\omega\frac{\alpha_k\tau_k}{(nV)^{\nu}}\right\}\right\rangle$$

$$=\exp\left\{i\omega\frac{\bar{T}}{(nV)^{\nu}}\right\}\sum_{N}\frac{(nV)^{N}}{N!}\exp\{-nV\}\left(\left\langle\exp\left\{-i\omega\frac{\alpha\tau}{(nV)^{\nu}}\right\}\right\rangle\right)^{N}$$

$$=\exp\{\varphi(\omega)\},$$

where

$$\varphi(\omega)=n\int_0^\infty\int_V\left[i\omega\frac{\alpha(R)\tau(\mathbf{x},\xi,t)}{(nV)^{\nu}}+\exp\left\{-i\omega\frac{\alpha(R)\tau(\mathbf{x},\xi,t)}{(nV)^{\nu}}\right\}-1\right]f(R)\,dR\,d^3\xi. \quad (5)$$

On the other hand, by definition,

$$\langle\exp\{-i\omega\Theta\}\rangle=\int_{-\infty}^{+\infty}P_{\mathbf{x},t}(\Theta)\exp\{-i\omega\Theta\}d\Theta. \quad (6)$$

The integration in (6) has been formally continued to $-\infty$, since $P_{x,t}(\Theta)=0$ at $\Theta<-(nV)^{-\nu}\bar{T}$. Using the inverse Fourier transform, we obtain finally

$$P_{\mathbf{x},t}(\Theta)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[i\omega\Theta+\varphi(\omega)]d\omega. \quad (7)$$

Let us consider the case of greatest interest, when the average number of particles included in the volume V is large ($nV\gg1$). Then we can[1)] calculate the integral (7) by using not the exact expression for the function $\varphi(\omega)$ but its asymptotic form as $\omega\to0$. The exponent ν must be chosen such that the principal term of the asymptotic expression does not contain nV, and then all the subsequent terms are small relative to this parameter.

As can be easily seen from (5), $\varphi(0)=\varphi'(0)=0$. If the second derivative $\varphi''(0)=-(nV)^{-2\nu+1}\overline{\alpha^2}\,\overline{\tau^2}$ is finite, then $\varphi(\omega)\approx\frac{1}{2}\varphi''(0)\omega^2$, and we obtain from (7) a Gaussian distribution (which trivially generalizes the central limit theorem to the case when the number of terms has a Poisson distribution). It is necessary here to put $\nu=\frac{1}{2}$.

It is easy to show that the integral that determines $\overline{\tau^2}$ converges. As to $\overline{\alpha^2}$, the corresponding integral converges only if $m>2l$. Therefore at $m\le2l$ the sought distribution differs from a Gaussian one. We note that the divergence of $\overline{\alpha^2}$ means that the random terms that enter in (3) have no variance[2)], but their mean value, and consequently also the average temperature, remain finite to the extent that the inequality $m>l$ is satisfied (see (4)).

Let us consider separately the case $l<m<2l$. It is clear from the foregoing that now the function $\varphi(\omega)$ has at small ω a power-law asymptotic form with an exponent ranging from 1 to 2. To obtain this asymptotic form, we note that the integral (5) remains convergent if we replace in it f(R) by expression (2) all the way to R = 0. Since, however, the sought asymptotic form is due to the behavior of the integrand at large R, such a replacement gives an asymptotically correct result (in terms of the parameter nV). This result takes the form

$$\varphi(\omega)|_{\omega\to0}\approx F(\mathbf{x},t)\begin{cases}\omega^{m/l}(-\cos p\pi+i\sin p\pi), & \omega\geqslant0\\ (-\omega)^{m/l}(-\cos p\pi-i\sin p\pi); & \omega\leqslant0\end{cases}$$

$$p\equiv1-(m/2l),$$

$$F(\mathbf{x},t)\equiv\overline{\tau^{(m/l)}(\mathbf{x},t)}\frac{Al}{m(m-l)}\Gamma\left(2-\frac{m}{l}\right),$$

and ν is chosen here to be equal to l/m. Consequently, in this case

$$P_{\mathbf{x},t}(\Theta)=\frac{1}{\pi}\int_0^\infty\exp\{-F\omega^{m/l}\cos p\pi\}\cos[\omega\Theta+F\omega^{m/l}\sin p\pi]d\omega.$$

The sought temperature probability density $P_{x,t}(T)$ is obtained from this by a trivial change of variables. It must be remembered, that the error of the probability density $P_{x,t}(\Theta)$ is uniformly small in the Fourier-transform space, something that cannot be said concerning $P_{x,t}(T)$ [4].

Omitting the corresponding calculations, we present the results for the temperature probability density $P_{x,t}(T)$ in various limiting cases.

As $T-\bar{T}\to-\infty$ we have

$$P_{\mathbf{x},t}(T)\sim\exp\left\{-\frac{m-l}{m}\left(\frac{l}{nVmF}\right)^{l/(m-l)}|T-\bar{T}|^{m/(m-l)}\right\}. \quad (8)$$

From physical considerations, however, it is obvious that the temperature cannot drop anywhere below the initial value; consequently, the exact formula (7) yields a zero value for the probability density at negative T. Therefore the criterion for the applicability of the proposed approximation consists in the fact that the integral of the obtained density over the region of negative temperatures be small in comparison with unity. According to (8), this yields

$$(nV\bar{\tau})^{m/l}\gg nV\overline{\tau^{m/l}}. \quad (9)$$

In other words, the temperature fields of different particles should overlap strongly. It is clear that this condition is particularly stringent during the initial stage of the heating. It is easy to show that for short times ($\chi t\ll D^2$ and $\chi t\ll L^2$, where D is the distance to the nearest boundary and L is the characteristic dimension over which the radiation flux density changes), when the Green's function "does not feel" the boundary conditions and is a rapidly varying function in comparison with q(x), the inequality (9) reduces to

$$n(\chi t)^{3/2}\gg1 \quad (10)$$

In the case of long times, on the other hand, condition (9) is automatically satisfied because of the macroscopically large factor nV. Thus, the results become valid anytime after the start of the heating, and this time is shorter the larger the density n (10).

At $T-\bar{T}\to0$ we have

$$P_{\mathbf{x},t}(T)\approx a_0-a_1(T-\bar{T}),$$

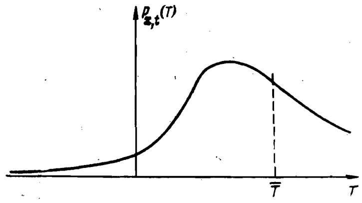
$$a_0\equiv\frac{1}{\pi}\cdot\frac{l}{m}(nVF)^{-l/m}\Gamma\left(\frac{l}{m}\right)\cos\frac{\pi pl}{m},$$

$$a_1\equiv\frac{1}{\pi}\frac{l}{m}(nVF)^{-2l/m}\Gamma\left(\frac{2l}{m}\right)\sin\frac{2\pi pl}{m}.$$

We see that the obtained distribution is not symmetrical—the mean value of the temperature does not coincide with the modal value.

As $T-\bar{T}\to\infty$ we have

$$P_{\mathbf{x},t}\approx\frac{1}{\pi}nVF\Gamma\left(1+\frac{m}{l}\right)\sin\left(\frac{m-l}{l}\pi\right)(T-\bar{T})^{-(m+l)/l}. \quad (11)$$



Such a slow (power-law) decrease of the probability density has a clear-cut physical meaning. It means that the temperature at some point may turn out to be large not only because of random accumulation of a large number of particles with medium dimensions (the probability of such an event is exponentially small), but because a large strongly heating particle may find its way accidentally near the observation point. Thus, at $nV \gg 1$ the form of the functional dependence of $P_{x,t}(T)$ on the temperature is determined only by the behavior of $f(R)$ as $R \to \infty$ and depends neither on the shape of the sample, nor on the conditions of heat exchange on its boundary, nor on the character of the space-time distribution of the laser-pulse radiation. These factors determine only the dependence of $\overline{T}$ and of the function F (or T^2 in the case of a Gaussian distribution) on the coordinates and on the time. At short times we have $\overline{T} \approx \text{const} \cdot q(\mathbf{x})t$, while $F(\mathbf{x}, t) \approx \text{const} \cdot q^{m/l}(\mathbf{x})t^{(3l-m)/2l}$. At long times, on the other hand, the average temperature and $F(\mathbf{x}, t)$ approach exponentially the stationary values with a decrement determined by the smallest eigenvalue of the corresponding boundary-value problem, provided only the boundary is not thermally insulated. An approximate plot of the distribution $P_{x,t}(T)$ at fixed x and t is shown in the figure.

We present one typical application of the obtained distribution. Assume that the medium melts at a certain relative high temperature T*. Then, neglecting the latent heat of the melting and the presence of the thermophysical properties of the solid and liquid phases, we can easily estimate the mean value of the volume v(t) of matter that melts by the instant of time t. We have

$$v(t) \approx \int_V \int_{T^*}^{\infty} P_{x,t}(T)\, d^3x\, dT.$$

According to (11), for short times and $\overline{T} \ll T^*$ we obtain

$$v(t) \sim AnV\overline{q^{m/l}}(\chi t)^{(3l-m)/2l}(\chi T^*)^{-m/l},$$

where the bar denotes averaging over the volume V.

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1) For details of the proof of this statement see, e.g., the monograph [4].
2) Our problem differs from that analyzed in [1] in the character of the divergence of the variances of the random terms.

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