Simplest dynamo instability

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A model with a prescribed velocity field $\mathbf{v} = \{0; v_\varphi(r); 0\}$ (in cylindrical coordinates) is considered, the electric conductivity being $\sigma(r)$. In this case the restriction on the dynamo is removed, as is illustrated with a jumplike change of the quantities under discussion as an example. By employing the hypothesis that a growing solution exists when v is a random function of the coordinates and is independent of time, the inverse to the Markov processes are introduced. A characteristic of these processes is that $t \gg \lambda/v$ (t is the correlation time, λ is a characteristic scale and v is the rms velocity. Some applications of the model are discussed.

INTRODUCTION

It is well known that the dynamo theory is subject to a number of restrictions, for example, dynamos cannot be axially symmetrical, two-dimensional, centrally symmetrical, and so on. The problem is essentially three-dimensional in any coordinate system. This circumstance is the main difficulty of the theory itself and of its practical applications. In fact, one would wish to think that, for example, the non-rigid-body rotating star or the convection in the earth's core has axial symmetry. However, in view of the indicated restrictions, it is necessary to resort to subtler effects (the Coriolis force, the density inhomogeneity) in order for asymmetrical convection to be produced. On the other hand, attempts are made to find the simplest motions that deviate as little as possible from axial symmetry. Thus, the Herzenberg dynamo [1] is realized by two rotating cylinders, the axes of which are not parallel to each other (see also [2–4]), while the Lortz dynamo is realized by screw motion [5] (see also [6,7]). Nonetheless, the indicated restrictions have a common feature, namely, the symmetry assumption extends both to the motion and to the field, so that it is meaningful to raise the question whether symmetrical motions are capable of generating an asymmetrical field. An affirmative answer to this question is given by the model of Tverskoĭ [8]—a toroidal vortex. The latter has axial symmetry. Nonetheless, the toroidal vortex is a complex motion; we shall follow the path of maximum simplification.

Is a dynamo possible in the case of differential rotation? This motion is the simplest and probably the most frequently encountered in nature. In addition, is a dynamo possible in the case of two-dimensional motion of the type of differential rotation, i.e., in a cylindrical system $v_r = v_z = 0$; $v_\varphi \neq 0$, $v_\varphi = v_\varphi(r)$? At first glance the answer should be negative, since we have a theorem by Zel'dovich [9] that precludes the possibility of a two-dimensional dynamo for an arbitrary (i.e., not necessarily two-dimensional) magnetic field. In our problem, however, the velocity is even one-dimensional, since $\mathbf{v}$ depends only on r. However, the aforementioned theorem is proved for an unbounded conducting medium. If the conducting medium is located at $r < R$, and the medium at $r > R$ is nonconducting (in particular, vacuum), then this restriction can be lifted and a dynamo is possible. We note that there are no unbounded bodies in nature.

Let us explain why the presence of vacuum lifts the restriction on a two-dimensional dynamo. From the induction equation

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}[\mathbf{v}\times\mathbf{H}] + \nu_m \Delta \mathbf{H} \tag{1}$$

follows, in the given geometry, an equation for H_z:

$$\frac{\partial H_z}{\partial t} + (\mathbf{v}\nabla) H_z = \nu_m \Delta H_z. \tag{2}$$

Since the behavior of H_z is described by the heat-conduction equation and $H_z \to 0$ as $r \to \infty$ in an unbounded liquid, we have $H_z \to 0$ as $t \to \infty$, i.e., H_z attenuates. Further, assuming $H_z = 0$, we can easily show that the equation for the vector potential of the remaining components of the field also takes the form of the heat-conduction equation. Consequently, both H_x and H_y attenuate.

Assume now that we have vacuum at $r > R$. Then, taking the magnetic permeability of the medium to be $\mu = 1$, we have the following boundary conditions: continuity of the field, the field being potential in vacuum, and also

$$\frac{1}{r}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = 0 \tag{3}$$

on the boundary. Condition (3) corresponds to vanishing of the current component normal to the boundary. Now, the heat-conduction equation (2) with boundary conditions (3) need not result at all in an attenuation of the field H_z! In general, all the theorems that state that a dynamo is impossible are proved when one of the field components becomes separated from the other, i.e., behaves independently of them. Here, however, H_z is connected with H_φ by the boundary condition (3). It will be shown below that it is precisely this circumstance that eliminates the restriction on the one-dimensional dynamo.

One might object, presumably, that vacuum boundary conditions on the boundary of a celestial body are not very realistic. The sun for example, is surrounded by a highly conducting corona, which goes over directly into solar wind. Are vacuum conditions realistic in our case? However, the H_z component of the field is not separated from the others even if the electric conductivity does depend on r (so that ν_m is also axially symmetrical). In fact, the equation for H_z takes in this case the form

$$\frac{\partial H_z}{\partial t} + (\mathbf{v}\nabla) H_z = \nu_m \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \varphi^2} \right) - \frac{\partial \nu_m}{\partial r}\frac{\partial H_r}{\partial z} \tag{4}$$

It is seen from (4) that the conclusion that H_z attenuates is again impossible (the author is indebted to D. D. Ryutov for the last remark). Field generation exists also in the case (4). The assumption of an inhomogeneous $\nu_m(r)$ is, naturally, connected with the very fact that the celestial bodies are bounded, and ν_m change jumpwise in the particular case of vacuum, since $\nu_m = \infty$ in vacuum.

1. SOLUTION OF ONE-DIMENSIONAL PROBLEM WITH VACUUM

We specify $v_\varphi(r)$ in the following manner:

$$\text{region I:}\quad v_\varphi=\omega_0 r \quad \text{at} \quad r<r_0,$$
$$\text{region II:}\quad v_\varphi=0 \quad \text{at} \quad r>r_0;\quad r_0<R. \tag{5}$$

the rigid-body motion at $r < r_0$ simplifies the calculations, although it is immaterial in principle whether the angular velocity $\omega(r)$ is discontinuous or continuous. Then the equations for H_r, H_φ, and H_z take the form

$$\begin{gathered}\frac{\partial H_r}{\partial t}=-\omega_0\frac{\partial H_r}{\partial\varphi}+\nu_m\left(\Delta H_r-\frac{1}{r^2}H_r-\frac{2}{r^2}\frac{\partial}{\partial\varphi}H_\varphi\right),\\ \frac{\partial H_\varphi}{\partial t}=-\omega_0\frac{\partial H_\varphi}{\partial\varphi}+rH_r\frac{\partial\omega_0}{\partial r}+\nu_m\left(\Delta H_\varphi-\frac{1}{r^2}H_\varphi+\frac{2}{r^2}\frac{\partial}{\partial\varphi}H_r\right).\\ \frac{\partial H_z}{\partial t}=-\omega_0\frac{\partial H_z}{\partial\varphi}+\nu_m\Delta H_z.\end{gathered} \tag{6}$$

It is natural to seek the solution of the system (6) in the form

$$H_{r,\varphi,z}=f_{r,\varphi,z}(r)\exp[Et+i(m\varphi+kz)]. \tag{7}$$

Introducing, in analogy with [7], the function $f_\pm = f_r \pm if$, we obtain Bessel equations for regions I, II, and the vacuum (region III):

$$\begin{gathered}\frac{\partial^2 f_z}{\partial z^2}+\frac{1}{z}\frac{\partial f_z}{\partial z}-(1+m^2/z^2)f_z=0,\\ \frac{\partial^2 f_\pm}{\partial z^2}+\frac{1}{z}\frac{\partial f_\pm}{\partial z}-[1+(m\pm1)^2/z^2]f_\pm=0,\end{gathered} \tag{8}$$

where in region I we have $z = \beta r$, $\beta = \sqrt{(E + \nu_m k^2 + im\omega_0)/\nu_m}$; in region II we have $z = \kappa r$, $\kappa = \sqrt{(E + \nu_m k^2)/\nu_m}$; in region III we have $z = kr$.

We seek a solution of the system (8) in the form

region I: $f_z=AI_m(z)$, $f_\pm=B_\pm I_\pm(z)$ (bounded at zero);
region II: $f_z=CI_m(z)+DK_m(z)$, $f_\pm=L_\pm I_\pm(z)+M_\pm K_\pm(z)$;

region III: $f_z=FK_m(z)$, $f_\pm=iFK_\pm(z)$ (vanishing at infinity),

where $I_\pm = I_{m\pm1}$ and $K_\pm = K_{m\pm1}$. In region III we have used the condition curl **H** = 0.

Matching the solutions in the three regions yields a system of algebraic equations for the coefficients. The matching condition is the following: continuity of all the solutions and

$$\left.\frac{\partial f_z}{\partial r}\right|_{\rm I}=\left.\frac{\partial f_z}{\partial r}\right|_{\rm II},\quad \left.\left(\frac{\partial f_\pm}{\partial r}\pm\frac{ir_0\omega_0}{2\nu_m}(f_+ + f_-)\right)\right|_{\rm I}=\left.\frac{\partial f_\pm}{\partial r}\right|_{\rm II}. \tag{9}$$

Altogether we obtain 9 equations and 10 coefficients. We obtain the tenth equation by using the condition div **H** = 0 (condition (3) follows from those written out above). To derive this equation, we take the divergence of (6), and obtain

$$E\gamma-im\omega_0\gamma=\nu_m\Delta\gamma,\qquad \gamma=\operatorname{div}\mathbf{H}. \tag{10}$$

Further, in order to have $\gamma \equiv 0$, it suffices to stipulate that γ vanish on the boundary with the vacuum (this follows from the uniqueness of the solution (10) at the given boundary conditions). Writing out div **H** = 0 on the boundary of regions II and III

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_r)+\frac{1}{r}\frac{\partial}{\partial\varphi}H_\varphi+\frac{\partial}{\partial z}H_z=0$$

and using the fact that the field passes continuously through the boundary, we can easily see that $\partial H_\varphi/\partial\varphi$ and $\partial H_z/\partial z$ are also continuous on the boundary; hence

$$\left.\frac{\partial H_r}{\partial r}\right|_{\rm II}=\left.\frac{\partial H_r}{\partial r}\right|_{\rm III}. \tag{11}$$

Condition (11) is indeed the sought tenth equation. The determinant of tenth order is represented in the form of a product of two factors, one of which does not yield a dynamo.

We assume that βr_0, κr_0, $kR \gg 1$. We shall see below that this situation corresponds to a large magnetic Reynolds number $R_M = \omega_0 r_0^2/\nu_m \gg 1$. Now $z \gg 1$ on the boundaries, and we can use the asymptotic expressions

$$I_m(z)=(2\pi z)^{-1/2}[e^z+e^{-z-(m+1/2)\pi i}],\qquad K_m(z)=(\pi/2z)^{1/2}e^{-z}. \tag{12}$$

The second of the factors of the determinant is simplified if account is taken of the fact that $\tanh\varphi \approx \pm1$, $\varphi = \beta r_0 + \frac{1}{2}i\pi(m + \frac{1}{2})$, a relation satisfied with exponential accuracy. This factor then breaks up in turn into two factors, one of which yields the equation

$$\operatorname{th}\varkappa\Delta r=\mp\varkappa/\beta,\qquad \Delta r=R-r_0, \tag{13}$$

which does not result in a dynamo. The second corresponds to the equation

$$\operatorname{th}\varkappa\Delta r=\varkappa\frac{k\mp\beta}{\varkappa^2\mp\beta k} \tag{14}$$

(the upper sign corresponds to $\tanh\varphi = \pm1$) or, in dimensionless form ($z_1 = \kappa\Delta r$),

$$\operatorname{th}z_1=z_1\frac{D-(z_1^2+iC)^{1/2}}{z_1^2-D(z_1^2+iC)^{1/2}},\quad \operatorname{Re}(z_1^2+iC)^{1/2}>0,\quad D=k\Delta r,\quad C=m\omega_0(\Delta r)^2/\nu_m. \tag{15}$$

If we put $\omega_0 = 0$, i.e., $C = 0$ in (15) (absence of rotation, trivial case), then (15) has no roots and it is necessary to use other cofactors of the determinant, which result in an attenuating field. If $m = 0$ (purely axially symmetrical case, i.e., both the velocity and **H** are axially symmetrical), then $C = 0$, which is equivalent to $\omega_0 = 0$, and the dynamo is impossible (the theorem of S. I. Braginskiĭ [10]). If $k = 0$, i.e., $D = 0$ (purely two-dimensional case), then the equation is analogous to (13), but the right-hand side contains β/κ instead of κ/β. This equation also has no growing solutions (a theorem by Ya. B. Zel'dovich [9]). Letting $\Delta r \to 0$, we obtain a rigid cylinder rotating in a vacuum. Naturally, the dynamo is impossible ($D = C = 0$). We see therefore that the field will be essentially three-dimensional and will contain all three components.

It is easily seen that for dynamo solution the root should lie in the region $\operatorname{Re}(z_1^2 - k^2) > 0$. It is convenient to seek the solution (15) in the $C - D$ plane, specifying z_1. Thus, putting $z_1 = 1.00 - i0.50$, we obtain graphically $C = 0.11$ and $D = -0.53$. It is easy to verify that this root corresponds to the assumed approximations and yields a dynamo. In fact, $\kappa\Delta r \approx \beta\Delta r \approx k\Delta r \approx 1$ does not contradict the use of the asymptotic expressions (12), if $r_0 \gg \Delta r$; the latter condition corresponds to $R_M = Cr_0^2/m(\Delta r)^2 \gg 1$, i.e., to a large Reynolds number. Furthermore

$$E=(\varkappa^2-k^2)\nu_m=\frac{\nu_m}{(\Delta r)^2}(0.47-i1.00). \tag{16}$$

Using the fact that $C \approx 1$, we obtain $\operatorname{Re}E \sim m\omega_0$, which is perfectly natural.

2. "EMPIRICAL" RULES FOR THE DETERMINATION OF THE EXISTENCE OF A DYNAMO. PROCESSES INVERSE TO THE MARKOV PROCESS

The considered example confirms the following reasoning: a dynamo exists in all those cases when it is impossible to prove the opposite by using certain standard rules. Probably, this statement cannot be proved, but nonetheless it is practically always helpful.

Thus, to verify whether a dynamo exists in a definite situation, it is necessary to proceed as follows:

1. Write down the induction equation in a natural system of curvilinear coordinates.

2. Examine whether one of the field components is separated; if such a component exists, this leads to attenuation and the dynamo is impossible, since the remaining components must also attenuate (this is proved by transforming to the equation for the vector potential).

It is precisely in this manner that the theorems stating that the dynamo is impossible have been proved. Thus, it can be proved that if the geometry is completely "straightened out," i.e., if we consider the analog of the problem of Sec. 1 for Cartesian coordinates, ($\mathbf{v} = \{0; v_y(x); 0\}$), then there are no growing solutions. To illustrate the method, let us consider an example analyzed by Lerche [11] wherein $\mathbf{v}(\mathbf{x})$ does not depend on y, z, or t (ν_m is homogeneous). For H_x, the equation takes the form of the heat-conduction equation

$$\frac{\partial H_x}{\partial t} = -(\mathbf{v}\nabla)H_x + \nu_m \Delta H_x.$$

Consequently, $H_x \to 0$ as $t \to \infty$. Putting $H_x = 0$, we find that the equations for H_y and H_z also take the form of the heat-conduction equation, thus contradicting the nonrigorous conclusions made by Lerche [11]. It is equally obvious that the solution of Lerche's second example [1], in which v does not depend on the coordinates and depends only on the time, is also in error (we note, incidentally, that numerous papers by the same author, which "block" the gyrotropic generation considered by Steenbeck and co-workers (see, e.g., [12]), are just as nonrigorous.

The striking simplicity and symmetry of the considered model give grounds for hoping that the restriction on the dynamo will be lifted for practically all symmetrical models. But then the more complicated asymmetrical motions will be all the more dynamo-unstable. Thus, we can advance the hypothesis that all motions in nature are unstable to magnetic fluctuations. The dynamo problem can by the same token be inverted, namely, seek in real bounded bodies motions that do not result in generation.

Let now $\mathbf{v}$ be a random function of the coordinate but independent of the time. Of course, the "empirical" rules do not make it possible to prove the absence of growing solutions, so that it is natural to assume that such solutions exist. It was shown in Sec. 1 that the growth rate is $\sim\omega_0$ (characteristic frequency), and consequently for a random $\mathbf{v}$ the growth rate is $\sim v/l$ (v is the characteristic velocity and l is the characteristic length). This estimate is valid for any scale of l (if there is a scale spectrum), since the field adjusts itself to any scale and the characteristic scale of the field for a given l will be l/R_M.

Yet the turbulent-dynamo problem is still more complicated: the velocity itself varies with the same frequency v/l. It is therefore reasonable to consider the limiting case $\tau \gg l/v$ (τ is the characteristic time of variation of the field $\mathbf{v}$), which is opposite to the already considered $\tau \ll l/v$ (δ-correlation in time (see [13])). If $\tau = 0$ (δ-correlation), then we deal with stochastic differential equations corresponding to a Markov process; a small "broadening" of the δ-function—a super-Markov process—is obtained in the functional formulation (see [14,15] as applied to the dynamo [16]).

It is now reasonable to approach the real process $\tau = l/v$ from the opposite direction. In the first approximation $\tau = \infty$ we assume that growing solutions exist and the general solutions of the form

$$\mathbf{H}(\mathbf{r}, t) = \sum_n \mathbf{H}_n(\mathbf{r}) \exp(E_n t), \tag{17}$$

where $\mathrm{Re}E_0 > 0$. In the next approximation, the velocity changes slowly

$$\mathbf{H}(\mathbf{r}, t) = \sum_n \mathbf{H}_n(\mathbf{r}, t) \exp\left(\int E_n(t_1)\,dt_1\right),$$

$\mathbf{H}_n(\mathbf{r}, t)$ and $E_n(t)$ are slowly varying ($\tau \gg l/v$) functions of the time. In the asymptotic regime $t \to \infty$, which is indeed the only one of interest in the theory of the turbulent dynamo,

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0(\mathbf{r}, t) \exp\left(\int_0^t E_0(t_1)\,dt_1\right) + \text{c.c.} \tag{18}$$

We obtain the correlation tensor by first averaging over space and then over time (the characteristic length of variation of H_0 is $\sim l$)

$$\begin{aligned} T_{ij}(\mathbf{r}, \mathbf{r}', t, t') &= \langle\langle H_i(\mathbf{r}, t) H_j(\mathbf{r}', t')\rangle\rangle = \Big\langle \langle H_i^0(\mathbf{r}, t) H_j^0(\mathbf{r}', t')\rangle \\ &\times \exp\left(\int_0^t E_0(t_1)\,dt_1 + \int_0^{t'} E_0(t_2)\,dt_2\right)\Big\rangle + \Big\langle \langle H_i^0(\mathbf{r}, t) \overset{*}{H}{}_j^0(\mathbf{r}', t')\rangle \\ &\times \exp\left(\int_0^t E_0(t_1)\,dt_1 + \int_0^{t'} E_0^*(t_2)\,dt_2\right)\Big\rangle + \text{c.c.} \end{aligned} \tag{19}$$

The lower eigenfunction H^0 by itself, as well as the value of E_0, should be homogeneous in time in the statistical sense, just as the field $\mathbf{v}$ itself, i.e., $\langle H^0(t) H^0(t')\rangle$ should depend only on $t - t'$, therefore fixing the time interval $t - t'$ and averaging over one of the times, t or t′, we obtain

$$\begin{aligned} T_{ij}(\mathbf{r}, \mathbf{r}', t, t') &= T_{ij}^1(\mathbf{r}-\mathbf{r}', t-t') \Big\langle \exp\left(\int_0^t E_0(t_1)\,dt_1 + \int_0^{t'} E_0(t_2)\,dt_2\right)\Big\rangle \\ &+ T_{ij}^2(\mathbf{r}-\mathbf{r}', t-t') \Big\langle \exp\left(\int_0^t E_0(t_1)\,dt_1 + \int_0^{t'} E_0^*(t_2)\,dt_2\right)\Big\rangle + \text{c.c.}; \end{aligned} \tag{20}$$

$$T_{ij}^1(\mathbf{r}-\mathbf{r}', t-t') = \langle H_i^0(\mathbf{r}, t) H_j^0(\mathbf{r}', t')\rangle, \quad T_{ij}^2 = \langle H_i^0 \overset{*}{H}{}_j^0\rangle.$$

We note further that $E_0(t) = \langle E_0\rangle + \delta E_0(t)$, and by assumption we have $\langle E_0\rangle \neq 0$ and $\mathrm{Re}\langle E_0\rangle > 0$. It is meaningful to consider expression (20) at very large t and t′ (the asymptotic regime), or more accurately $t, t' \gg \tau$. Recognizing that $\int_0^t E_0(t_1)\,dt_1$ increases with t at best like $t^{1/2}$ (inasmuch as

$$\int_0^t\int_0^t \langle \delta E_0(t_1)\delta E_0(t_2)\rangle\,dt_1\,dt_2 = \int_0^t (t-s) f(s)\,ds \sim t$$

at large t), we find that all the average exponentials can be expressed simply in terms of $\langle E_0\rangle$:

$$\Big\langle \exp\left(\int_0^t E_0\,dt_1 + \int_0^{t'} E_0\,dt_2\right)\Big\rangle = \exp[\langle E_0\rangle (t+t')]$$

etc. The field energy increases like $\exp(\langle E_0\rangle 2t)$. Thus, all the characteristics of the magnetic field depend on the lower eigenfunction in a rather simple manner. The eigenfunction can be sought by the WKB method, inasmuch as the highest order derivative $\nabla^2\mathbf{H}$ is preceeded by a small quantity (the large parameter necessary for the WKB method is R_M). To be sure, in view of the complexity (and not the hermiticity) of the operator (1) and the three-dimensional character of the vector field, the question of the behavior of the solution in the vicinity of the singular points is very complicated, and can be solved for the time being only in the simplest cases [17].

The question of processes that are inverse to Markov processes can be formulated in the same manner for the temperature field in a turbulent liquid. An important role is played here by the harmonic that attenuates most slowly.

3. CONCLUSION

Inasmuch as the dynamo solution exhibits instability to fluctuations of the magnetic field, the latter must inevitably be excited by the thermodynamic fluctuations, regardless of whether this agrees with the observation data or not. One can there ask why the period of the solar cycle is ≈ 22 years when $\omega_0 \sim 10^{-6}$ sec^{-1}, i.e., the field growth time due to the presence of differential rotation is less than 1 month; why do not all celestial bodies have magnetic fields, etc. On the other hand, the model considered here is close to the rotating-galaxy model; the growth rate is $r(\partial\omega/\partial r) \sim \omega$, i.e., the field increases during the period of revolution of the galaxy, i.e., quite rapidly. Finally, both the appearance and the enhancement of the field on the sun's surface (for example, sunspots), as well as the vanishing of the field, can be easily attributed to differential rotation by resorting only to either the simplest dynamo or the simplest "antidynamo" [17].

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