Energy of an arbitrary electromagnetic field in a medium with dispersion of the dielectric permittivity and magnetic permeability

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The conservative and dissipative terms in the Poynting equation are distinguished in accordance with their behavior under time reversal. The conservative terms are integrated over the time and a general expression is obtained for the energy of an electromagnetic field with an arbitrary time dependence. As the condition for the existence of the energy, symmetry properties of the parts of the dielectric-permittivity and magnetic-permeability tensors that are even in the frequency are obtained. In the case when the spatial dispersion is neglected an expression is also obtained for the volume density of the electromagnetic-field energy, and a spatially nonuniform medium is considered. In the particular case of a harmonic time dependence of the field, the previously known result is obtained for the time-averaged energy density. In the case when there are no external field sources, normal field coordinates are introduced, the transition to operators is performed, and quantum values of the field energy are obtained. It is shown that, in a quantum-electrodynamical treatment of the interaction with photons of charges introduced into the medium, as compared with the case of a medium with no dispersion, additional frequency dependences appear in the probabilities of the different processes. A generalization of the Planck formula is obtained.

The expression

$$U=\frac{1}{8\pi}(\varepsilon\mathbf{E}^2+\mu\mathbf{H}^2) \tag{1}$$

for the energy density U is inapplicable if ϵ or μ depends on the frequency ω of the field oscillations. For the particular case of a harmonically oscillating field the correct value of the average of U over a period of the oscillations was obtained in [1,2]; it differs from (1). The inapplicability of (1) is also obvious in cases when, in certain ranges of ω, $\epsilon(\omega) < 0$ or $\mu(\omega) < 0$ and it follows from (1) that the greater the field amplitude the lower the energy. Negative values of $\epsilon(\omega)$ are encountered, e.g., in the Lorentz theory of the dispersion of a medium consisting of elastically bound electrons (with small damping), in a plasma, in a free-electron gas, etc. Analogously, cases when $\mu(\omega) < 0$ are encountered.

The purpose of this paper is to obtain a general expression for the energy of an electromagnetic field that varies arbitrarily in time and in space. The treatment is carried out in the framework of phenomenological electrodynamics, without specifying the Hamiltonian of the medium and without using models of the medium.

In the case of a spatially nonuniform medium, only frequency dispersion of ϵ and μ is taken into account. In the case of a spatially uniform medium, spatial dispersion is also taken into account.

We have not attempted to express the field energy in terms of the coefficients of the Fourier-integral expansion of the field as a function of time. The point is that to determine the Fourier coefficients it is necessary to specify the field at all times, from $-\infty$ to $+\infty$. But the energy of the field at a given time t should be determined only by the values of the fields at preceding times. The introduction, in the future, of new field sources, changing the behavior of the field, should not change the field energy at the time t (the causality principle); however, the Fourier coefficients do change.

The situation is different with the Fourier expansion of the field as a function of the coordinates. In the case of spatial dispersion it is necessary to specify the field in all space in order to determine not only the field energy but even the functions **D** and **B** at any space-point **r**. Then the Fourier coefficients are also completely determined. Therefore, in the treatment of spatial dispersion below, we shall use the Fourier expansion of the field as a function of **r**. The above-mentioned feature of the case of spatial dispersion prevents us from introducing the concept of the energy density of the field, and enables us, in this case, to determine only the integral energy over the volume.

1. FIELD-ENERGY DENSITY IN AN ANISOTROPIC SPATIALLY NONUNIFORM MEDIUM WITH FREQUENCY DISPERSION BUT NO SPATIAL DISPERSION

In this case the electric displacement $\mathbf{D}(\mathbf{r}, t)$ and magnetic induction $\mathbf{B}(\mathbf{r}, t)$ are determined by the values of the field intensities $\mathbf{E}(\mathbf{r}, t')$ and $\mathbf{H}(\mathbf{r}, t')$ at the same point **r** at all times $t' \le t$. Assuming that the dependence of $\mathbf{E}(\mathbf{r}, t')$ on t' is analytic for $t' \le t$, we can expand $\mathbf{E}(\mathbf{r}, t')$ in a Taylor series in powers of $t' - t$. Thus, $\mathbf{E}(\mathbf{r}, t')$ is completely determined by giving the infinite set of derivatives $\mathbf{E}^{(n)}(\mathbf{r}, t)$, n = 0, 1, 2, 3, ... at the point t (n is the order of the derivative of $\mathbf{E}(\mathbf{r}, t')$ with respect to t'). In the framework of linear electrodynamics, $\mathbf{D}(\mathbf{r}, t)$ is a linear functional of $\mathbf{E}(\mathbf{r}, t')$, and therefore $\mathbf{D}(\mathbf{r}, t)$ is a linear function of an infinite number of arguments $\mathbf{E}^{(n)}(\mathbf{r}, t)$:

$$\mathbf{D}(\mathbf{r},t)=\sum_{n=0}^{\infty}\varepsilon_n(\mathbf{r})\mathbf{E}^{(n)}(\mathbf{r},t). \tag{2}$$

Here the second-rank tensors ϵ_n are the polarization characteristics of the medium. Since the medium is nonuniform, the ϵ_n depend on **r**. But the ϵ_n do not depend on t, since the properties of the medium are assumed to be constant in time. It is important to emphasize that $\mathbf{E}^{(n)}$ and **D** in (2) are real; therefore all the ϵ_n are real.

The meaning of the tensors ϵ_n becomes clear if we apply formula (2) to a field of the form

$$\mathbf{E}(\mathbf{r}, t') = \mathbf{E}_0(\mathbf{r}) e^{-i\omega t'}. \qquad (3)$$

It is then found that the usual complex dielectric-permittivity tensor $\epsilon(\mathbf{r}, \omega)$ is equal to

$$\varepsilon(\mathbf{r}, \omega) = \sum_{n=0}^{\infty} \varepsilon_n(\mathbf{r}) (-i\omega)^n. \qquad (4)$$

Thus, the ϵ_n are the tensor coefficients of the expansion of $\epsilon(\mathbf{r}, \omega)$ in powers of $(-i\omega)$. It may turn out that in the actual range of frequencies the infinite series (4) diverges. In this case we must retain a very large, but finite, number of terms in the infinite sum (4) and regard $\epsilon_n(\mathbf{r})$ as coefficients of a polynomial (4) that approximates $\epsilon(\mathbf{r}, \omega)$ with arbitrary accuracy in the actual range of ω. If in (2) also we retain only the finite number of corresponding terms, then (2) will approximate the displacement with arbitrary accuracy, provided that $\mathbf{E}(\mathbf{r}, t)$ as a function of t can be expanded as a Fourier integral.

In fact, formula (2) is also easily obtained in another way, e.g., by assuming that $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$ can be represented by Fourier integrals. Thus, using the approximation (4), we have

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$
$$= \sum_{n=0}^{\infty} \varepsilon_n(\mathbf{r}) \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) \frac{d^n e^{-i\omega t}}{dt^n} d\omega = \sum_{n=0}^{\infty} \varepsilon_n(\mathbf{r}) \mathbf{E}^{(n)}(\mathbf{r}, t). \qquad (5)$$

Here, in view of the possibility of specifying arbitrarily the time behavior of the field after the time t, the Fourier coefficient $\mathbf{E}(\mathbf{r}, \omega)$ of the electric field is not uniquely defined. However, the right-hand side of (5) is unique if $\mathbf{E}(\mathbf{r}, t')$ depends analytically on t' for $t' \leq t$.

Formula (2) is easily inverted by expressing $\mathbf{E}(\mathbf{r}, t)$ linearly in terms of $\mathbf{D}^{(n)}(\mathbf{r}, t)$. For this we can use the inverse dielectric-permittivity tensor $\eta(\mathbf{r}, \omega)$ and a transformation of the type (5).

In this section we shall be concerned only with one fixed point $\mathbf{r}$ of the medium. For brevity, therefore, we shall omit the argument $\mathbf{r}$ in the formulas.

The displacement current

$$\frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} = \frac{\mathbf{E}^{(1)}}{4\pi} + \mathbf{P}^{(1)} \qquad (6)$$

appears in the Maxwell equations only in a sum with the conduction current $\mathbf{J}$. However, in the case of alternating fields it is impossible to distinguish uniquely the term $\mathbf{J}$ and the polarization current $\mathbf{P}^{(1)}$ in the total current. For example, the conduction-electron current induced by the field is a typical conduction current. But at the same time it can also be treated as a polarization current. To avoid ambiguity we shall assume that $\mathbf{J}$ is included in $\mathbf{P}^{(1)}$ and that the total current coincides with the displacement current. In this case, the Poynting equation has the following form:*

$$\frac{dU}{dt} + Q = -\operatorname{div} \mathbf{S} = \frac{1}{4\pi} [\mathbf{E}\mathbf{D}^{(1)} + \mathbf{H}\mathbf{B}^{(1)}], \quad \mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]. \qquad (7)$$

Here $\mathbf{S}$ is the electromagnetic energy flux density, U is the electromagnetic energy density, and Q is the rate of dissipation of electromagnetic energy per unit volume. Included in Q are, e.g., the Joule heat evolved, the dielectric losses, the work performed by the field in transferring electrons to higher energy levels, and so on.

Inasmuch as Eq. (7) determines only the sum of terms dU/dt + Q, to determine each of them we need a clear criterion for distinguishing the conservative and dissipative terms in the right-hand side of (7), for an arbitrary time dependence of the fields.

The separation of the electromagnetic part and the other forms of energy in the energy of a body is not completely trivial. For example, the thermal optical phonons are usually included in the thermal energy. But they are also typical electromagnetic waves and we have an equal right to include them in the electromagnetic energy. Thus, the creation of optical phonons on absorption of a macroscopic electromagnetic wave can be interpreted either as dissipation of electromagnetic energy or as conservative Raman scattering of light. The choice between these two possibilities is determined by the actual form of the constitutive equations. If the latter are linear, e.g., of the type (2), the system of Maxwell equations does not describe combination scattering at all. Therefore, the above-mentioned absorption of a macroscopic wave can only be described as dissipation. But if nonlinear terms are added to the constitutive equations, it will also be possible to describe the transformation of the energy of the macroscopic wave into optical phonons as conservative combination scattering.

Below we shall start from the linear constitutive equation (2) and the analogous equation expressing $\mathbf{B}$ in terms of $\mathbf{H}^{(n)}$. We shall call those terms which violate the invariance of the Maxwell equations under time reversal, dissipative. When these terms are removed the evolution of the field in time becomes reversible. This means that if $\mathbf{E} = \mathbf{F}(\mathbf{r}, t)$, $\mathbf{H} = \mathbf{\Phi}(\mathbf{r}, t)$ is a solution of the Maxwell equations, then $\mathbf{E} = \mathbf{F}(\mathbf{r}, -t)$, $\mathbf{H} = -\mathbf{\Phi}(\mathbf{r}, -t)$ should also be a solution of them.

By writing the system of Maxwell equations it is easy to see that reversibility obtains if we retain only terms with even n in the sum (2). Consequently, terms with odd n are dissipative. Separating in this way the conservative and dissipative terms in (7), we obtain

$$\frac{dU}{dt} = \frac{1}{4\pi} (\mathbf{E}\mathbf{D}_{\text{even}}^{(1)} + \mathbf{H}\mathbf{B}_{\text{even}}^{(1)}), \qquad (8)$$

$$Q = \frac{1}{4\pi} (\mathbf{E}\mathbf{D}_{\text{odd}}^{(1)} + \mathbf{H}\mathbf{B}_{\text{odd}}^{(1)}), \qquad (9)$$

where

$$\mathbf{D}_{\text{even}} = \sum_{p=0}^{\infty} \varepsilon_{2p} \mathbf{E}^{(2p)}, \quad \mathbf{D}_{\text{odd}} = \sum_{p=0}^{\infty} \varepsilon_{2p+1} \mathbf{E}^{(2p+1)}. \qquad (10)$$

Here, as in problems of linear mechanics, on reversal of the motion dU/dt changes sign and Q does not.

Integrating the right-hand side of (8) by parts over t, we obtain a general expression for the electromagnetic energy density

$$U = \frac{1}{8\pi} \sum_{p=0}^{\infty} \sum_{m=0}^{2p} (-1)^m \{\mathbf{E}^{(m)} \varepsilon_{2p} \mathbf{E}^{(2p-m)} + \mathbf{H}^{(m)} \mu_{2p} \mathbf{H}^{(2p-m)}\}. \qquad (11)$$

It is possible to integrate Eq. (8) fully over t and obtain explicitly an energy density $U(\ldots \mathbf{E}^{(n)} \ldots \mathbf{H}^{(l)} \ldots)$ independent of t only if the tensors ϵ_{2p} and μ_{2p} are symmetric:

$$[\varepsilon_{2p}(\mathbf{r})]_{ij} = [\varepsilon_{2p}(\mathbf{r})]_{ji}, \quad [\mu_{2p}(\mathbf{r})]_{ij} = [\mu_{2p}(\mathbf{r})]_{ji}, \quad i, j = 1, 2, 3. \qquad (12)$$

In the case p = 0, i.e., for a static permittivity and permeability, the relations (12) are well-known in electrostatics and magnetostatics.

According to (11), U = 0 if at the given point in the medium $\mathbf{E}$ and $\mathbf{H}$ are equal to zero not only at the given time t, but also at all preceding times t' $(-\infty < t' \leq t)$.

In practice, it is sufficient that **E** and **H** be absent over a period of time considerably longer than the relaxation time of the polarization of the medium.

If in the dielectric permittivity $\epsilon(\mathbf{r}, \omega)$ we separate out the part even in ω, $\epsilon_{\text{even}}(\mathbf{r}, \omega)$, consisting of the even terms in the sum (4), and the odd part $\epsilon_{\text{odd}}(\mathbf{r}, \omega)$ consisting of the odd terms, then

$$\mathbf{D}^{(1)}_{\text{even}}=\frac{\partial}{\partial t}\varepsilon_{\text{even}}\left(\mathbf{r}, i\frac{\partial}{\partial t}\right)\mathbf{E},\quad \mathbf{D}^{(1)}_{\text{odd}}=\frac{\partial}{\partial t}\varepsilon_{\text{odd}}\left(\mathbf{r}, i\frac{\partial}{\partial t}\right)\mathbf{E}. \quad (13)$$

Analogous formulas can be written for $\mathbf{B}^{(1)}_{\text{even}}$ and $\mathbf{B}^{(1)}_{\text{odd}}$. As a result, the expression (9) for Q can be rewritten in the form

$$Q=\frac{1}{4\pi}\left[\mathbf{E}\frac{\partial}{\partial t}\varepsilon_{\text{odd}}\left(\mathbf{r}, i\frac{\partial}{\partial t}\right)\mathbf{E}+\mathbf{H}\frac{\partial}{\partial t}\mu_{\text{odd}}\left(\mathbf{r}, i\frac{\partial}{\partial t}\right)\mathbf{H}\right]. \quad (14)$$

Formulas (9), (13) and (14) are valid for a dielectric medium. But if the medium is a conductor and is characterized by the static conductivity tensor σ, then in the right-hand side of the expression (13) for $\mathbf{D}^{(1)}_{\text{odd}}$ we must add the term $4\pi\sigma\mathbf{E}$, and in the right-hand side of (14) we must add the term $\mathbf{E}\sigma\mathbf{E}$.

For fields of the type of the real part of (3), oscillating harmonically in time, from (14) we obtain

$$Q=-\frac{i\omega}{4\pi}[\mathbf{E}\varepsilon_{\text{odd}}(\mathbf{r},\omega)\mathbf{E}+\mathbf{H}\mu_{\text{odd}}(\mathbf{r},\omega)\mathbf{H}]. \quad (15)$$

If the medium is in thermal equilibrium, the time-average of the quantity (15) is positive, i.e., the principal values of the tensors $-i\omega\epsilon_{\text{odd}}(\mathbf{r}, \omega)$ and $-i\omega\mu_{\text{odd}}(\mathbf{r}, \omega)$ should be positive for all ω. If there is no thermal equilibrium, e.g., there is an inverted population of the energy levels, then it is possible that the averaged $Q < 0$ (amplification of the electromagnetic wave).

If the dependence of **E** and **H** on t is of a more general form, then, even if the medium is in thermal equilibrium, it is possible that $Q < 0$ over particular periods of time. But, provided that $\mathbf{E} \to 0$, $\mathbf{H} \to 0$ as $t \to -\infty$ and $t \to \infty$,

$$\int_{-\infty}^{\infty} Q\,dt \geqslant 0. \quad (16)$$

This inequality is easily obtained by expanding **E** and **H**, as functions of t, in Fourier integrals in (14) and using the positivity of the principal values of $-i\omega\epsilon_{\text{odd}}$ and $-i\omega\mu_{\text{odd}}$.

It is now easy to show that the above separation of the conservative and dissipative terms in the right-hand side of (7) is the only possible separation for constitutive equations of the type (2). Indeed, in deriving the expression (11) from (8) it was proved that each term of the right-hand side of Eq. (8) can be represented as the time derivative of a certain function of $\ldots\mathbf{E}^{(n)}\ldots\mathbf{H}^{(l)}\ldots$ that does not depend explicitly on t and is invariant under time reversal. But not one of the terms in the right-hand side of Eq. (9) possesses such a property, and, therefore, not one can be transferred to the right-hand side of (8). In order to show this, we shall consider an integral, of the type (16), of a general term of the right-hand side of (9), when the field varies arbitrarily in time but tends to zero as $t \to \pm\infty$:

$$\int_{-\infty}^{\infty}\mathbf{E}\varepsilon_{2p+1}\mathbf{E}^{(2p+2)}\,dt=(-1)^{p+1}\int_{-\infty}^{\infty}\mathbf{E}^{(p+1)}\varepsilon_{2p+1}\mathbf{E}^{(p+1)}\,dt. \quad (17)$$

It is obvious that, generally speaking, this integral is nonzero, and consequently, the integrand is not a time derivative of a function of the above-mentioned type.

2. ELECTROMAGNETIC FIELD ENERGY IN A SPATIALLY UNIFORM MEDIUM WITH SPATIAL AND FREQUENCY DISPERSION

In this case the electric displacement $\mathbf{D}(\mathbf{r}, t)$ is determined by the values of the electric field $\mathbf{E}(\mathbf{r}_1, t')$ ($t' \le t$) not only at the same point **r** but also at all other space points $\mathbf{r}_1 \neq \mathbf{r}$. In the linear approximation we now have, in place of (2),

$$\mathbf{D}(\mathbf{r},t)=\sum_{n=0}^{\infty}\int d^3\mathbf{r}_1\,\alpha_n(\mathbf{r}-\mathbf{r}_1)\mathbf{E}^{(n)}(\mathbf{r}_1,t). \quad (18)$$

Here, for simplicity, it is assumed that the medium is spatially uniform. Therefore, the coordinates **r** and $\mathbf{r}_1$ appear in the tensors α_n as the difference. A formula analogous to (18) can also be written for the magnetic induction.

As in the preceding section of the article, the even terms in the sum (18) are conservative and the odd are dissipative. Since all the $\mathbf{E}^{(n)}$ are real, the tensors α_n are also real.

The Poynting equation (7) is valid as before, since it does not depend on the constitutive equations. But it cannot be interpreted as an energy-balance equation if spatial dispersion is important. The point is that the Poynting vector **S** in this case is not the energy flux density[1)]. In fact, we shall consider, e.g., an almost longitudinal polarization wave in a crystal with an ionic lattice. This wave is electromagnetic and creates an energy flux. But for selected propagation directions in the crystal this wave becomes strictly longitudinal and does not produce a magnetic field (since the electric field is irrotational), and, consequently, $\mathbf{S} = 0$. However, the energy flux density in this case is certainly nonzero. A more general example is a light-exciton (polariton) wave[4]: if it is accompanied by a longitudinal electric field, then $\mathbf{H} = 0$ and, consequently, $\mathbf{S} = 0$, but the energy flux density is nonzero.

Inasmuch as (7) is not an energy-balance equation, it is not possible to determine the electromagnetic energy density from (7), as was done in the preceding section of the article. However, if we integrate (7) over an infinite volume, assuming that the fields fall off sufficiently rapidly at infinity, (or over the volume of the region defined by the cyclic boundary conditions), then **S** vanishes completely and equations are obtained for the volume-integrated field energy W and dissipation rate R:

$$\frac{dW}{dt}=\frac{1}{4\pi}\int(\mathbf{E}\mathbf{D}^{(1)}_{\text{even}}+\mathbf{H}\mathbf{B}^{(1)}_{\text{even}})\,d^3\mathbf{r}, \quad (19)$$

$$R=\frac{1}{4\pi}\int(\mathbf{E}\mathbf{D}^{(1)}_{\text{odd}}+\mathbf{H}\mathbf{B}^{(1)}_{\text{odd}})\,d^3\mathbf{r}. \quad (20)$$

Here $\mathbf{D}_{\text{even}}$ and $\mathbf{D}_{\text{odd}}$ are respectively the sums of the terms with even and odd n in the right-hand side of (18). The quantities $\mathbf{B}_{\text{even}}$ and $\mathbf{B}_{\text{odd}}$ are defined analogously. Substituting these sums into (19) and integrating over t by parts, we obtain the desired general expression for W for fields varying arbitrarily in time:

$$W=\frac{1}{8\pi}\sum_{p=0}^{\infty}\sum_{m=0}^{2p}(-1)^m\int\int d^3\mathbf{r}\,d^3\mathbf{r}_1\{\mathbf{E}^{(m)}(\mathbf{r},t)\alpha_{2p}(\mathbf{r}-\mathbf{r}_1)\mathbf{E}^{(2p-m)}(\mathbf{r}_1,t) + \mathbf{H}^{(m)}(\mathbf{r},t)\alpha_{2p}'(\mathbf{r}-\mathbf{r}_1)\mathbf{H}^{(2p-m)}(\mathbf{r}_1,t)\}. \quad (21)$$

Here the α'_{2p} are the magnetic analogs of the tensors α_{2p}. In order that the integration over t can be performed, i.e., in order that the energy W exist, it is necessary that α_{2p} and α'_{2p} possess symmetry properties of the following type:

$$[\alpha_{2p}(\mathbf{r})]_{ij}=[\alpha_{2p}(-\mathbf{r})]_{ji}, \quad i,j=1,2,3. \tag{22}$$

With no loss of generality, we can expand **E** and **D**, as functions of **r**, in Fourier series by imposing cyclic boundary conditions:

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\mathbf{E}_{\mathbf{k}}(t)e^{i\mathbf{kr}}, \quad \mathbf{D}(\mathbf{r},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\mathbf{D}_{\mathbf{k}}(t)e^{i\mathbf{kr}}. \tag{23}$$

Here V is the volume of the "cyclicity region," $\mathbf{E}_{-\mathbf{k}} = \mathbf{E}_{\mathbf{k}}^*$ and $\mathbf{D}_{-\mathbf{k}} = \mathbf{D}_{\mathbf{k}}^*$. Substituting (23) into (18) leads to the relations

$$\mathbf{D}_{\mathbf{k}}(t)=\sum_{n=0}^{\infty}\varepsilon_n(\mathbf{k})\mathbf{E}_{\mathbf{k}}^{(n)}(t), \quad \varepsilon_n(\mathbf{k})=\int\alpha_n(\mathbf{r})e^{-i\mathbf{kr}}d^3\mathbf{r}. \tag{24}$$

Application of formula (24) to a field of the type (3), harmonically oscillating in time, leads to the relations:

$$\mathbf{D}_{\mathbf{k}}(t)=\varepsilon(\omega,\mathbf{k})\mathbf{E}_{\mathbf{k}}(t), \quad \varepsilon(\omega,\mathbf{k})=\sum_{n=0}^{\infty}\varepsilon_n(\mathbf{k})(-i\omega)^n. \tag{25}$$

We note that in these formulas ω and **k** are independent, i.e., are not related by any dispersion law. Moreover, one ω can be associated with an infinite set of the vectors **k** necessary for the Fourier expansion of the field (3). $\epsilon(\omega, \mathbf{k})$ is the usual dielectric permittivity in a medium with spatial and frequency dispersion, and $\epsilon_n(\mathbf{k})$ are the tensor coefficients of its expansion in powers of $(-i\omega)$.

Since the α_n are real, it follows from (24) that

$$\varepsilon_n(-\mathbf{k})=\varepsilon_n^*(\mathbf{k}), \tag{26}$$

and from (25) we then obtain the well-known relation

$$\varepsilon(-\omega,-\mathbf{k})=\varepsilon^*(\omega,\mathbf{k}). \tag{27}$$

From (22) and (24) we obtain

$$[\varepsilon_{2p}(\mathbf{k})]_{ij}=[\varepsilon_{2p}(-\mathbf{k})]_{ji}, \ [\varepsilon_{\text{even}}(\omega,\mathbf{k})]_{ij}=[\varepsilon_{\text{even}}(\omega,-\mathbf{k})]_{ji}, \quad i,j=1,2,3, \tag{28}$$

where $\epsilon_{\text{even}}(\omega, \mathbf{k})$ is the sum of the terms with even n in the expansion (25). Correspondingly, $\epsilon_{\text{odd}}(\omega, \mathbf{k})$ below denotes the sum of the terms with odd n.

If we substitute the expansion (23) into (21), W can be represented in the form

$$W=\frac{1}{8\pi}\sum_{\mathbf{k}}\sum_{p=0}^{\infty}\sum_{m=0}^{2p}(-1)^m\{\mathbf{E}_{-\mathbf{k}}^{(m)}\varepsilon_{2p}(\mathbf{k})\mathbf{E}_{\mathbf{k}}^{(2p-m)}+\mathbf{H}_{-\mathbf{k}}^{(m)}\mu_{2p}(\mathbf{k})\mathbf{H}_{\mathbf{k}}^{(2p-m)}\}, \tag{29}$$

where $\mu_{2p}(\mathbf{k})$ is the magnetic analog of $\epsilon_{2p}(\mathbf{k})$.

For fields of the form of the real part of (3) we have

$$\mathbf{E}_{\mathbf{k}}(t)={}^1/_2[\mathbf{E}_{0\mathbf{k}}e^{-i\omega t}+(\mathbf{E}_0^*)_{\mathbf{k}}e^{i\omega t}]. \tag{30}$$

In this case we obtain from (29)

$$W=\frac{1}{8\pi}\sum_{\mathbf{k}}\Big[\mathbf{E}_{-\mathbf{k}}(t)\varepsilon_{\text{even}}(\omega,\mathbf{k})\mathbf{E}_{\mathbf{k}}(t)+H_{-\mathbf{k}}(t)\mu_{\text{even}}(\omega,\mathbf{k})\mathbf{H}_{\mathbf{k}}(t)$$
$$+\frac{1}{2}E_{0,-\mathbf{k}}\omega\frac{\partial\varepsilon_{\text{even}}(\omega,\mathbf{k})}{\partial\omega}(\mathbf{E}_0^*)_{\mathbf{k}}+\frac{1}{2}\mathbf{H}_{0,-\mathbf{k}}\omega\frac{\partial\mu_{\text{even}}(\omega,\mathbf{k})}{\partial\omega}(\mathbf{H}_0^*)_{\mathbf{k}}\Big]. \tag{31}$$

Here the derivative of $\epsilon_{\text{even}}(\omega, \mathbf{k})$ and $\mu_{\text{even}}(\omega, \mathbf{k})$ with respect to ω must be taken at fixed **k**.

From (18), (20) and (23) we obtain the following expression for the rate of dissipation:

$$R=\frac{1}{4\pi}\sum_{\mathbf{k}}\sum_{p=0}^{\infty}[\mathbf{E}_{-\mathbf{k}}(t)\varepsilon_{2p+1}(\mathbf{k})\mathbf{E}_{\mathbf{k}}^{(2p+2)}(t)+\mathbf{H}_{-\mathbf{k}}(t)\mu_{2p+1}(\mathbf{k})\mathbf{H}_{\mathbf{k}}^{(2p+2)}(t)]. \tag{32}$$

If the field is represented by the real part of (3), it follows from (32), when (25) and (30) are taken into account, that

$$R=-\frac{i\omega}{4\pi}\sum_{\mathbf{k}}[\mathbf{E}_{-\mathbf{k}}(t)\varepsilon_{\text{odd}}(\omega,\mathbf{k})\mathbf{E}_{\mathbf{k}}(t)+\mathbf{H}_{-\mathbf{k}}(t)\mu_{\text{odd}}(\omega,\mathbf{k})\mathbf{H}_{\mathbf{k}}(t)]. \tag{33}$$

On the basis of (27), this expression is real in the absence of any symmetry of the tensor ϵ_{odd}, μ_{odd} in the indices i and j.

If the medium is in thermal equilibrium, then, in the case of a plane traveling monochromatic wave, the quantity (33) must be positive. Hence it follows that the Hermitian tensor

$$-i\omega[\varepsilon_{\text{odd}}(\omega,\mathbf{k})-\tilde{\varepsilon}_{\text{odd}}^*(\omega,\mathbf{k})] \tag{34}$$

should have positive principal values. Here ~ denotes the transpose.

As in the preceding section of the article, it can be shown that neither the quantity (32) nor individual terms of the right-hand side of (32) are time derivatives of some function of ... $\mathbf{E}_{\mathbf{k}}^{(n)}(t)$ ... $\mathbf{H}_{\mathbf{k}}^{(l)}(t)$ ... that does not depend explicitly on t. Therefore, the terms mentioned cannot be included in the conservative part of the energy-balance equation.

If the medium possesses a static electrical-conductivity tensor $\sigma(\mathbf{k})$, we must add the term

$$\sum_{\mathbf{k}}\mathbf{E}_{-\mathbf{k}}(t)\sigma(\mathbf{k})E_{\mathbf{k}}(t).$$

3. EXAMPLES

In a spatially nonuniform medium without spatial dispersion the energy density of the electromagnetic field is determined by formula (11).

We shall consider the case when the time dependence of the field is determined by the real part of (3) (an analogous expression is assumed for the magnetic field). Then, taking (4) into account, we obtain from (11)

$$U=\frac{1}{8\pi}\Big\{\mathbf{E}(\mathbf{r},t)\varepsilon_{\text{even}}(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},t)+\mathbf{H}(\mathbf{r},t)\mu_{\text{even}}(\mathbf{r},\omega)\mathbf{H}(\mathbf{r},t)$$
$$+\frac{1}{2}\mathbf{E}_0^*(\mathbf{r})\omega\frac{\partial\varepsilon_{\text{even}}(\mathbf{r},\omega)}{\partial\omega}\mathbf{E}_0(\mathbf{r})+\frac{1}{2}\mathbf{H}_0^*(\mathbf{r})\omega\frac{\partial\mu_{\text{even}}(\mathbf{r},\omega)}{\partial\omega}\mathbf{H}_0(\mathbf{r})\Big\} \tag{35}$$

If this expression is averaged over the period of the oscillations of the field, we obtain for the average energy density $\bar{U}$:

$$\bar{U}=\frac{1}{16\pi}\Big\{\mathbf{E}_0^*(\mathbf{r})\frac{\partial}{\partial\omega}(\omega\varepsilon_{\text{even}})\mathbf{E}_0(\mathbf{r})+\mathbf{H}_0^*(\mathbf{r})\frac{\partial}{\partial\omega}(\omega\mu_{\text{even}})\mathbf{H}_0(\mathbf{r})\Big\}, \tag{36}$$

which, in the isotropic case, coincides with the result of Landau and Lifshitz [1].

As a second example we shall consider a time dependence of the following form for the fields:

$$\mathbf{E}=\mathbf{A}(\mathbf{r})e^{at}, \quad \mathbf{H}=\mathbf{A}'(\mathbf{r})e^{a't}. \tag{37}$$

In this case we obtain from (11) and (4)

$$U=\frac{1}{8\pi}[\mathbf{E}\varepsilon_{\text{even}}(\mathbf{r},ia)\mathbf{E}+\mathbf{H}\mu_{\text{even}}(\mathbf{r},ia')\mathbf{H}]. \tag{38}$$

The superficial similarity of (38) and (1) is curious. In fact, these expressions coincide only if ϵ_{even} and μ_{even} do not depend on ω.

As a third example we shall consider the fields $\mathbf{E} = \mathbf{A}(\mathbf{r})t^2$, $\mathbf{H} = 0$. From (11) we obtain

$$U=\frac{1}{8\pi}[\mathbf{E}\varepsilon_0(\mathbf{r})\mathbf{E}+\mathbf{E}^{(2)}\varepsilon_4(\mathbf{r})\mathbf{E}^{(2)}]. \tag{39}$$

We note that the fields (37), and also those proportional to t^2, cannot be Fourier-expanded as functions of t. Therefore, attempts to express U in terms of Fourier coefficients are doomed to failure. But even if the increase of the fields as $t \to \infty$ is slowed down to the extent that the Fourier coefficients become finite, but very large, it is not possible to express U in terms of them at, e.g., the time t = 0, when U is not large and does not depend on the behavior of the fields as $t \to \infty$.

As a last example we shall consider the field

$$\mathbf{E}=\mathbf{A}(\mathbf{r})e^{-\beta t^2}, \quad \mathbf{H}=0. \tag{40}$$

For simplicity we shall confine ourselves to determining U at the time t = 0. From (11) we obtain

$$U=\frac{1}{8\pi}\sum_p s_p(-\beta)^p\mathbf{A}\varepsilon_{2p}(\mathbf{r})\mathbf{A}, \quad s_p=\sum_{q=0}^{p}\frac{(2q)!\,(2p-2q)!}{q!\,(p-q)!} \tag{41}$$

$$s_0=1, \quad s_1=4, \quad s_2=28, \quad s_3=288, \quad s_4=3984.$$

If in formula (41) the summation over p is taken from 0 to ∞, this sum diverges because of the rapid increase of the coefficients s_p. But for sufficiently small $\sqrt{\beta}$ (by comparison with the "eigenfrequencies" of the medium), the terms in the sum decrease at first. Thus, the series is semi-convergent. In this case formula (41) gives a reasonable result if the series is cut off at the terms of smallest magnitude.

4. CANONICAL FORMALISM. QUANTIZATION OF THE FIELD

Below we discard all the dissipative terms and, for simplicity, put $\mu = 1$. The quantity ϵ_{even} is assumed to be isotropic and can possess spatial and frequency dispersion. Because of (28), the terms with indices **k** and $-\mathbf{k}$ in (29) coincide. Therefore, in (29) we can double the coefficient in the right-hand side and perform the summation over the half-space $k_x \geq 0$. Thus, the energy of the system is decomposed into a sum of the energies of conservative noninteracting subsystems. Below we consider only one of these subsystems, with energy $W_{\mathbf{k}\nu}$, where the index ν labels the two mutually orthogonal polarizations of the field (ν = 1, 2). We have

$$W_{\mathbf{k}\nu}=\frac{1}{4\pi}\Big[\sum_{p=0}^{\infty}\sum_{m=0}^{2p}(-1)^m\varepsilon_{2p}(\mathbf{k})\mathbf{E}_{-\mathbf{k}\nu}^{(m)}\mathbf{E}_{\mathbf{k}\nu}^{(2p-m)}+\mathbf{H}_{-\mathbf{k}\nu}\mathbf{H}_{\mathbf{k}\nu}\Big], \tag{42}$$

$$W=\sum_{\nu,k_x\geq 0}W_{\mathbf{k}\nu}.$$

If we exclude from consideration the static part of the field and the longitudinal electric waves, then, introducing the vector potential $\mathbf{A}(\mathbf{r}, t)$, we have

$$\mathbf{H}=\operatorname{rot}\mathbf{A}, \quad \operatorname{div}\mathbf{A}=0, \quad \mathbf{E}=-\frac{1}{c}\mathbf{A}^{(1)}, \quad \mathbf{A}(\mathbf{r},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\mathbf{A}_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{r}}. \tag{43}$$

The system of Maxwell equations reduces to the equation

$$k^2\mathbf{A}_{\mathbf{k}\nu}+\frac{1}{c^2}\varepsilon_{\text{even}}\Big(i\frac{\partial}{\partial t},\mathbf{k}\Big)\mathbf{A}_{\mathbf{k}\nu}^{(2)}=0, \quad \mathbf{A}_{\mathbf{k}\nu}\perp\mathbf{k}. \tag{44}$$

It might seem that (44) is a differential equation of infinite order in t, inasmuch as the expansion of $\epsilon_{\text{even}}(i\partial/\partial t,\mathbf{k})$ in powers of the operator argument $i\partial/\partial t$ can be cut off at a term with a derivative of arbitrarily high order. Such arguments, however, are incorrect. For example, let

$$\varepsilon_{\text{even}}(\omega,\mathbf{k})=a+\sum_{j=1}^{s}\frac{b_j}{\omega_j^2-\omega^2}, \tag{45}$$

where the constants a, b_j and ω_j depend on **k**. Then, acting on (44) with the operator

$$\prod_{j=1}^{s}\Big(\omega_j^2+\frac{\partial^2}{\partial t^2}\Big),$$

we obtain for $A_{\mathbf{k}\nu}(t)$ a scalar differential equation of order 2(s + 1), i.e., of finite order. It is equivalent to the initial equation (44) for all harmonics of the field apart from those whose frequencies coincide with ω_j. But such harmonics are not considered below, since they possess an infinite rate of dissipation.

Returning from the example to the general treatment, we note that (44) is a linear homogeneous equation with constant coefficients, containing derivatives of even order only. Therefore, it coincides formally with the classical-mechanical equation of motion for a set of elastically coupled points performing small harmonic oscillations. Such an equation is obtained in mechanics if we go over from the numerous second-order Newtonian equations of motion to one equation of high order.

To apply the traditional methods of quantization developed in mechanics, we find it convenient, conversely, to go over from the above-mentioned high-order equation equivalent to (44) (let its order be 2(s + 1)) to a system of second-order equations of the Newtonian type. For this, in the equation of order 2(s + 1) for $A_{\mathbf{k}\nu}$ we make the substitution

$$A_{\mathbf{k}\nu}^{(2p)}=\sum_{l=0}^{s}M_{pl}q_l, \quad p=0,1,2,\ldots,s, \tag{46}$$

where $q_l(t)$ are the new unknown functions, from which, for simplicity, we have dropped the indices **k** and ν; M_{pl} is any square matrix that is independent of t and has an inverse. For q_l a system of second-order equations, containing no first derivatives, is obtained. We shall call these the equations of motion of the field and q_l the generalized coordinates of the field.

Differentiating the last of Eqs. (46) with respect to t any even number of times and expressing the $q_l^{(2)}$ that arise each time in the right-hand side linearly in terms of $\ldots q_{l'} \ldots$ from the equations of motion, we can express any even derivative of $A_{\mathbf{k}\nu}$ linearly in terms of q_l:

$$A_{\mathbf{k}\nu}^{(2p)}=\sum_{l=0}^{s}M_{pl}q_l, \quad p=s+1,s+2,\ldots \tag{47}$$

Here M_{pl} is a non-square augmentation of the matrix introduced earlier.

In order that the q_l be normal coordinates, a special choice of the matrix M_{pl} is necessary; for this choice the particular integral of the problem has the form

$$q_{l'}=\delta_{l'l}e^{-i\omega_l t}, \quad l'=0,1,2,\ldots,s. \tag{48}$$

Substituting (48) into (46) and (47), we obtain

$$M_{pl}=(-\omega_l^2)^p. \tag{49}$$

Substituting (46)—(49) into (44), we obtain the dispersion equation

$$k^2=\frac{\omega_l^2}{c^2}\varepsilon_{\text{even}}(\omega_l,\mathbf{k}), \tag{50}$$

in which **k** is fixed. The roots $\pm\omega_0, \pm\omega_1, \ldots, \pm\omega_s$ of this equation are the discrete eigen-frequencies of the conservative subsystem k, ν under consideration, and the number of these roots defines the true order of Eq. (44). Thus, if ϵ_{even} has the form (45), then, as was proved above, (44) is equivalent to an equation of order 2(s + 1).

The number of roots of the dispersion equation in this case is also $2(s + 1)$.

Expanding ϵ_{even} in (50) in powers of $-i\omega_l$, we obtain an equation which, when we add to it the infinite set of identities

$$\omega_l^2 \sum_{p=0}^{\infty} \varepsilon_{2(p+p')}(\mathbf{k}) (-\omega_l^2)^p (-\omega_{l'}^2)^{p'} \equiv \omega_l^2 \sum_{p=0}^{\infty} \varepsilon_{2(p+p')}(\mathbf{k}) (-\omega_l^2)^p (-\omega_{l'}^2)^{p'}, \quad p'=1,2,3,\dots \tag{51}$$

leads to a certain relation. If in the latter we interchange the indices $l \rightleftarrows l'$, $p \rightleftarrows p'$ and subtract the result from the original relation, we obtain the useful "orthogonality condition"

$$\sum_{p,p'=0}^{\infty} \varepsilon_{2(p+p')}(\mathbf{k}) (-\omega_l^2)^p (-\omega_{l'}^2)^{p'} = 0, \quad \text{if} \quad \omega_l^2 \neq \omega_{l'}^2. \tag{52}$$

If we first divide the relation mentioned by ω_l^2, and then make the interchange of indices and the subtraction, we obtain another "orthogonality condition," used below:

$$c^2\mathbf{k}^2 - \sum_{p,p'=0}^{\infty} \varepsilon_{2(p+p'+1)}(\mathbf{k}) (-\omega_l^2)^{p+1} (-\omega_{l'}^2)^{p'+1} = 0, \quad \text{if} \quad \omega_l^2 \neq \omega_{l'}^2. \tag{53}$$

We shall return now to that stage in the treatment in which the q_l were not yet specific functions of t, of the type (48), but were the unknown functions—the generalized coordinates of the field. In the expression (42) for the energy we substitute (43), (46), (47), (49) and (50), taking into account that $A_{-\mathbf{k}\nu} = A^*_{\mathbf{k}\nu}$, and introduce the notation

$$m(\omega_l) \equiv m_l = \frac{\omega_l^2}{2\pi c^2} \frac{\partial}{\partial(\omega_l^2)} \varepsilon_{\text{even}}(\omega_l, \mathbf{k}) + \frac{\mathbf{k}^2}{2\pi\omega_l^2} \equiv \frac{1}{2\pi c^2} \frac{\partial}{\partial(\omega_l^2)} [\omega_l^2 \varepsilon_{\text{even}}(\omega_l, \mathbf{k})]. \tag{54}$$

As a result, the energy is written in the form

$$W_{\mathbf{k}\nu} = T + V, \quad T = \frac{1}{2} \sum_{l=0}^{s} m_l q_l^{(1)*} q_l^{(1)}, \quad V = \frac{1}{2} \sum_{l=0}^{s} m_l \omega_l^2 q_l^* q_l. \tag{55}$$

Here T and V must be interpreted as the kinetic and potential energies, respectively. The real and imaginary parts of q_l are the real normal coordinates of the system. Because of the condition (52), mixed products $q_l^{(1)*} q_{l'}^{(1)}$ with $l' \neq l$ have not appeared in T, and, because of (53), mixed products $q_l^* q_{l'}$ have not appeared in V. The system of equations of motion that can be obtained from the Lagrangian $L = T - V$ is equivalent to the Maxwell equation (44).

To quantize the field it is necessary to introduce the canonically conjugate momenta, transform to operators, and write down the well-known commutation relations between the coordinates and momenta. From these follow commutation relations for q_l, q_l^+, $q_l^{(1)}$ and $q_l^{(1)+}$. For brevity, we shall not write out either of these sets of relations.

Having disposed of the treatment of an individual conservative subsystem with fixed $\mathbf{k}$ and ν and energy $W_{\mathbf{k}\nu}$, we turn to consider the whole assembly of such subsystems with all possible $\mathbf{k}$ and ν. It is now necessary to attach the indices $\mathbf{k}$, ν to the quantities q_l and $q_l^{(1)}$, and the index $\mathbf{k}$ to the quantities s, ω_l and m_l. We recall that for all these quantities the index $\mathbf{k}$ was positioned in the half-space $k_x \geq 0$ (see (42) and the preceding text). The two operators $q_{l\mathbf{k}\nu}$ and $q_{l\mathbf{k}\nu}^{(1)}$, defined in the above-mentioned half-space of $\mathbf{k}$, can be expressed in terms of the two operators $a_{l,\mathbf{k}\nu}$ and $a_{l,-\mathbf{k}\nu}^+$ defined in complementary half-spaces:

$$q_{l\mathbf{k}\nu} = \sqrt{\frac{\hbar}{m_{l\mathbf{k}}\omega_{l\mathbf{k}}}} (a_{l,\mathbf{k}\nu} + a_{l,-\mathbf{k}\nu}^+), \quad q_{l\mathbf{k}\nu}^{(1)} = -i\sqrt{\frac{\hbar\omega_{l\mathbf{k}}}{m_{l\mathbf{k}}}} (a_{l,\mathbf{k}\nu} - a_{l,-\mathbf{k}\nu}^+), \quad \omega_{l\mathbf{k}} > 0. \tag{56}$$

From the commutation relations for $q_{l\mathbf{k}\nu}$, $q_{l\mathbf{k}\nu}^+$ and $q_{l\mathbf{k}\nu}^{(1)}$, $q_{l\mathbf{k}\nu}^{(1)+}$ we obtain

$$[a_{l\mathbf{k}\nu}, a_{l'\mathbf{k}'\nu'}^+] = \delta_{ll'}\delta_{kk'}\delta_{\nu\nu'}, \quad a_{l\mathbf{k}\nu}^+ a_{l\mathbf{k}\nu} = n_{l\mathbf{k}\nu} = 0, 1, 2, \dots \tag{57}$$

Thus, $a_{l\mathbf{k}\nu}^+$ and $a_{l\mathbf{k}\nu}$ are ordinary Bose creation and annihilation operators. Substituting Eqs. (55) and (56) into (42), we obtain

$$W = \sum_{\mathbf{k}} \sum_{l=0}^{s_{\mathbf{k}}} \sum_{\nu=1}^{2} \hbar\omega_{l\mathbf{k}} a_{l\mathbf{k}\nu}^+ a_{l\mathbf{k}\nu}, \tag{58}$$

where the summation over $\mathbf{k}$ must be performed not over the half-space but over the full space.

From the dispersion equation (50) there follows an expression for the density of vibrational states. Thus, in the simplest case of no spatial dispersion, the number of values of the vector $\mathbf{k}$ in the frequency interval $d\omega$ and within the solid angle $d\Omega$ is equal to

$$\rho(\omega)\, d\omega\, d\Omega = V \frac{\sqrt{\varepsilon_{\text{even}}(\omega)}}{4\pi^2 c} m(\omega)\, \omega^2\, d\omega\, d\Omega, \tag{59}$$

where V is the volume of the basic region of cyclicity. In an unbounded medium there exist waves with real $\mathbf{k}$ only, for which $\epsilon_{\text{even}}(\omega) \geq 0$. Consequently, $\rho(\omega) = 0$ in the frequency ranges in which $\epsilon_{\text{even}}(\omega) < 0$.

In the case of thermal equilibrium, for the spectral density of the energy of the electromagnetic waves we obtain the generalized Planck formula

$$dW = \frac{2}{\pi c} \varepsilon_{\text{even}}^{1/2}(\omega)\, m(\omega) \frac{\hbar\omega^3\, d\omega}{e^{\hbar\omega/kT} - 1}, \quad \text{if} \quad \varepsilon_{\text{even}}(\omega) > 0; \qquad dW = 0, \quad \text{if} \quad \varepsilon_{\text{even}}(\omega) \leq 0. \tag{60}$$

When the quantity (60) is integrated over ω, the Stefan-Boltzmann formula, generalized to the case of a dispersive medium, is obtained. If $\epsilon_{\text{even}}(\omega)$ has the form (45), then as $\omega \to \omega_j$ spatial dispersion becomes important and formulas (59) and (60) are inapplicable.

If we introduce additional charges (electrons, impurity molecules) into a dispersive medium and consider their interaction with photons on the basis of quantum electrodynamics, the probabilities of the different processes are expressed in terms of matrix elements of the quantities (56) and in terms of the density $\rho(\omega)$. The factors $m(\omega)$ and $\epsilon_{\text{even}}^{1/2}$ appearing in these quantities introduce extra frequency dependences into the above-mentioned probabilities, as compared with the case of a medium without dispersion. The probability of one-photon processes is proportional to

$$\rho(\omega)\, |\langle n_{l\mathbf{k}\nu}+1 | q_{l\mathbf{k}\nu} | n_{l\mathbf{k}\nu}\rangle|^2.$$

In this product the factor $m(\omega)$ cancels out, but the factor $\epsilon_{\text{even}}^{1/2}(\omega)$ remains; because of this, in the probability there appears the singularity $\sim(\omega_j - \omega)^{-1/2}$ when $\omega_j - \omega \to +0$, if $\epsilon_{\text{even}}(\omega)$ has the form (45). In the region

$\epsilon_{even}(\omega) \leq 0$ the probability of the process is equal to zero.

5. IMPROVEMENT OF THE CONVERGENCE OF THE EXPANSIONS USED

The final results (15), (31), (33), (35), (38), (39), (47)—(50) and (54)—(60) do not contain infinite sums over the index n or p, although such series were used for their derivation. These series diverge for many forms of the electromagnetic field—in particular, for many normal vibrations of the field. We shall show that the final results are applicable far outside the limits of the class of fields for which the above-mentioned series converge. For this we shall obtain the same results using other expansions which are somewhat more complicated but always convergent.

The fact that $\mathbf{D}(\mathbf{r}, t)$ is a linear functional of $\mathbf{E}(\mathbf{r}, t')$ ($t' \leq t$) can be written, in the general case, in the form

$$\mathbf{D}(\mathbf{r},t)=\int_0^\infty f(\tau)\mathbf{E}(\mathbf{r},t-\tau)d\tau=\int_0^\infty f(\tau)d\tau\, e^{-\tau\partial/\partial t}\mathbf{E}(\mathbf{r},t), \qquad (61)$$

where the tensor $f(\tau) \to 0$ as $\tau \to \infty$. If we expand the exponential in (61) in powers of $\partial/\partial t$, an expression of the type (2) is obtained, with

$$\varepsilon_n=\int_0^\infty f(\tau)d\tau(-\tau)^n/n! \qquad (62)$$

(the argument $\mathbf{r}$ of ϵ_n and f is omitted for brevity). So long as the series (2) converges, formulas (2) and (61) are equivalent. But (61) also remains meaningful when (2) diverges. Thus, if $\mathbf{E} \sim e^{-i\omega t}$, from formula (62) we obtain

$$\mathbf{D}(\mathbf{r},t)=\varepsilon(\omega)\mathbf{E}(\mathbf{r},t), \quad \varepsilon(\omega)=\int_0^\infty f(\tau)e^{i\omega\tau}d\tau. \qquad (63)$$

This integral is finite even for very large ω, which cannot be said of the right-hand side of (4). Thus, formula (61) gives a convergent result if the field is expressed by an arbitrary linear combination of exponential harmonics.

All the results of this paper are easily followed through by applying formula (61), in which we must first sum the series and then integrate, in place of formula (2). Ultimately, expressions are obtained that differ from those obtained earlier in that, in place of the tensors ϵ_n, we have the tensors $f(\tau)(-\tau)^n/n!$ (which fall off much faster with increasing n), and after summing over n the integration is performed over τ from 0 to ∞.

Completely analogously, in the case when spatial dispersion is taken into account we must use, in place of formula (24), the formula

$$\mathbf{D}_\mathbf{k}(t)=\int_0^\infty f(\tau,\mathbf{k})d\tau\, e^{-\tau\partial/\partial t}\mathbf{E}_\mathbf{k}(t), \qquad \varepsilon_n(\mathbf{k})=\int_0^\infty f(\tau,\mathbf{k})d\tau\frac{(-\tau)^n}{n!}. \qquad (64)$$

In those cases in which it was possible, when using the original expansions, to contract them and represent the result without the sum over n, when the expansions (61) or (64) are used it is possible, in addition, to perform the integration over τ. Thus, all the final results listed at the beginning of this Section of the article conserve their form exactly.

1) This fact was not taken into account in [3], in which, for a plane monochromatic wave with frequency ω and wave-vector $\mathbf{k}$ in the presence of spatial dispersion, it was proposed that the electromagnetic energy density be determined from the formula proved for the case of frequency dispersion only, by first substituting into the dielectric permittivity $\epsilon(\omega, \mathbf{k})$ the explicit ω-dependence of $\mathbf{k}$ obtained in solving Maxwell's equations.

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