Stability of the isotropy of a random electromagnetic-wave field during stimulated scattering

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The small angular perturbations that arise in a spherically symmetric photon distribution during stimulated scattering by free electrons evolve in time. The perturbations can propagate upward along the frequency axis, whereas the main photon flux moves downward along the same axis.

The classical random electromagnetic-wave field can be described as an ensemble of photons. The fact that photons are bosons is important in the investigation of the interaction of a photon gas with electrons: the probability expression contains the characteristic factor $n + 1$, which means that the photons in a given cell effectively "attract" other photons into the cell.

The question arises whether an anisotropy does not spontaneously arise in the photon distribution in phase space in processes in which the direction of motion of the photons can change, i.e., during their scattering. Will a photon excess in a certain solid angle not attract into this solid angle other photons, thereby enhancing the anisotropy? In other words, it is necessary to consider the stability of an isotropic (spherically symmetric in phase space) photon distribution against angle-dependent perturbations, i.e., perturbations which destroy the isotropy.

Let us recall that, depending on the form of the initial spectrum, stimulated scattering can lead to the contraction or divergence of a directed radiation beam[1] with a certain angular aperture, whereas, evidently, spontaneous scattering rapidly equalizes any anisotropy. It is expedient to consider the problem at once in the limiting case when stimulated scattering predominates. As is well known, in this case the quantum language is only a convenient (but not an indispensable) means of describing the classical electromagnetic field and its scattering in the random-phase approximation. Stimulated scattering leads to effects that depend on small frequency shifts $\Delta\nu/\nu \approx h\nu/mc^2$ during the scattering; therefore, the condition for the predominance of stimulated scattering turns out to be not $n \gg 1$, but the stronger condition $(\partial n/\partial\nu)\Delta\nu \gg 1$, which, for $\partial n/\partial\nu \sim n/\nu$, leads to $n > mc^2/h\nu$ and $kT_b = nh\nu > mc^2$, where T_b is the brightness temperature.

In the isotropic problem, the equation describing the evolution of the spectrum was derived by the late A. S. Kompaneets in his outstanding paper[2]. In the limiting case when stimulated scattering predominates and in the nonstationary problem without sources and without photon leakage (an unbounded, homogeneous, coordinate space) an equation in the characteristics is obtained which can lead to the generation in phase space of a shock wave[3] with a peculiar structure[4].

In such a nonstationary formulation of the problem, the initial nonspherical perturbations also move along the spectrum, but with a velocity and an amplitude-variation law that are different from the velocity of and amplitude-variation law for the main unperturbed spherically symmetric (isotropic) photon distribution. Consequently, the ratio of the perturbation to the unperturbed background does not remain constant. Below we give formulas that describe this phenomenon.

However, the treatment of the steady-state (i.e., time-independent) unperturbed situation corresponds more to the usual terminology in stability problems. For this purpose, it is necessary to include in the equation, besides scattering, the production of photons and their absorption—true of effectively describing the departure of photons contained in a spatially bounded system.

According to general theorems, perturbations depend on time exponentially, i.e., vary as $e^{\lambda t}$, if they develop in a background of a steady-state, unperturbed solution. The sign of λ determines the stability. Here, however, it becomes evident that the photons flow monotonically and "hydrodynamically" downward along the frequency axis, and do not diffuse upward and downward at least in the regime in which stimulated scattering predominates. For this reason, a nonspherical perturbation in a source with $\nu = \nu_0$ does not intensify at this frequency, although, as will be shown below, it does give rise to nonsphericity downstream or upstream. We find that $\lambda \equiv 0$ and that the instability effectively manifests itself in the growth of a specific type of nonsphericity as ν is decreased or increased.

In future, the stability analysis should be extended to perturbations of spatial homogeneity and polarization, considering the consistent dependence of the perturbations on angle, coordinates, and polarization. The analysis carried out below may seem inexact, owing to the correlation between the scattering angle and the polarization of the scattered wave, but the corrections are, probably, small. The questions touched upon in the present note apply almost in their entirety to other types of plasma and acoustic oscillations.

As long as the random-phase approximation is valid, the introduction of quasiparticles—photons, phonons, etc.—and the construction of kinetic equations for them did and still do constitute a powerful method of describing nature.

1. THE BASIC EQUATION

Induced Compton scattering of radiation by free electrons in an unbounded homogeneous medium is described by the integro-differential equation[4-6]

$$\frac{\partial n(\nu,\theta,\varphi)}{\partial t} = n(\nu,\theta,\varphi)\int A(\nu,\nu',\alpha)\,n(\nu',\theta',\varphi')\,d\nu'\,d\cos\theta'\,d\varphi' \qquad (1)$$

with the antisymmetric kernel

$$A(\nu,\nu',\alpha)=\frac{3}{8\pi}\frac{\sigma_T N_e h}{mc}\frac{1}{\sqrt{2\pi}}\left(\frac{mc^2}{kT_e}\right)^{1/2}\frac{(1+\cos^2\alpha)}{(1-\cos\alpha)^{1/2}}$$
$$\times\frac{\nu'^2(\nu-\nu')}{\nu^3}\exp\left\{-\frac{mc^2(\nu-\nu')^2}{4kT_e\nu^2(1-\cos\alpha)}\right\}.$$

In (1), $n = c^2J_\nu/8\pi h\nu^3$ is the photon occupation number in phase space and the scattering angle α is determined by the relation

$$\cos\alpha=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi').$$

The kernel $A(\nu, \nu', \alpha)$ is the derivative of the Gaussian function describing the change in frequency during the spontaneous scattering of a monochromatic line by Maxwellian electrons.

In the case of an isotropic field of radiation with a broad spectrum $\Delta\nu \gg \Delta\nu_D = (2kT_e/mc^2)^{1/2}\nu$, Eq. (1) can be reduced to the following differential equation, first derived by A. S. Kompaneets[2]:

$$\frac{\partial f}{\partial t}=af\frac{\partial f}{\partial \nu};$$
$$f=n\nu^2,\quad a=\frac{2\sigma_T N_e h}{mc},\quad \sigma_T=\frac{8\pi}{3}r_0^2. \tag{2}$$

In the anisotropic case[1)], (1) also gets simplified when the spectrum is broad:

$$\frac{\partial f(\nu,\theta,\varphi,t)}{\partial t}$$
$$=bf(\nu,\theta,\varphi,t)\int\frac{\partial f(\nu,\theta',\varphi',t)}{\partial\nu}(1-\cos\alpha)(1+\cos^2\alpha)\,d\cos\theta'\,d\varphi',$$
$$b=\frac{3}{8\pi}\frac{\sigma_T N_e h}{mc}=\frac{3}{16\pi}a. \tag{3}$$

2. STABILITY OF THE SPHERICALLY SYMMETRIC SOLUTION

Let f_0 be the spherically symmetric solution to the Eqs. (2) and (3), and let δ be a small perturbation of this solution. Then Eq. (3) assumes the form ($f = f_0(\nu, t) + \delta(\nu, \theta, \varphi, t)$):

$$\frac{\partial\delta}{\partial t}=\delta\frac{\partial\ln f_0}{\partial t}+bf_0\int\frac{\partial\delta}{\partial\nu}(1-\cos\alpha)(1+\cos^2\alpha)\,d\cos\theta'\,d\varphi'. \tag{4}$$

Let us expand the perturbation δ in a series in terms of the spherical functions:

$$\delta=\sum_{n=0}^{\infty}\sum_{m=-n}^{m=n}u_{nm}(\nu,t)Y_{nm}(\theta,\varphi). \tag{5}$$

The transformation angle function in (4) can be expanded in terms of the Legendre polynomials P_n:

$$(1-\cos\alpha)(1+\cos^2\alpha)={}^4/_3P_0-{}^8/_5P_1(\cos\alpha)+{}^2/_3P_2(\cos\alpha)-{}^2/_5P_3(\cos\alpha). \tag{6}$$

On account of the addition theorem for spherical harmonics,

$$P_n(\cos\alpha)=\frac{4\pi}{2n+1}\sum_{m=-n}^{m=n}Y_{nm}(\theta,\varphi)Y_{nm}^*(\theta',\varphi'), \tag{7}$$

where Y^*_{nm} is the complex conjugate of Y_{nm}.

It follows from the orthogonality of the spherical functions and from the formulas (4)–(7) that only the angular perturbations containing the first spherical functions with $n \le 3$ will behave nontrivially. The higher harmonics will neither intensify nor attenuate in stimulated scattering, since they cause the integral on the right-hand side of (4) to vanish. The substitution of (6) and (7) into (5) and (4) allows us to find the equations for u_{nm}:

$$\frac{\partial u_{nm}}{\partial t}=u_{nm}\frac{\partial\ln f_0}{\partial t}+c_nbf_0\frac{\partial u_{nm}}{\partial\nu}, \tag{8}$$

where

$$c_0=16\pi/3,\quad c_1=-32\pi/15,\quad c_2=8\pi/15,\quad c_3=-8\pi/35;$$
$$c_n=0 \text{ for } n>3.$$

Therefore, for example,

$$\frac{\partial u_{5m}}{\partial t}=u_{5m}\frac{\partial\ln f_0}{\partial t},\quad \frac{u_{5m}}{f_0}=\text{const}(t).$$

It can be seen from (8) that the perturbation propagates along the characteristic

$$d\nu/dt=-c_nbf_0 \tag{9}$$

with a velocity proportional to f_0. But the velocity of their motion along the frequency axis differs from the velocity of the downward motion of the main (spherically symmetric) photon flux

$$d\nu/dt=-af_0=-{}^{16}/_3\pi bf_0,$$

which is determined by Eq. (2). Since the coefficients c_n have different signs, it follows from (9) that the perturbations in which only the functions Y_{2m} have been substituted propagate downward along the frequency axis with a velocity equal to one tenth the velocity of the isotropic solution. The perturbations corresponding to Y_{1m} and Y_{3m} move upward along the frequency axis with velocities that are respectively 2.5 and ${}^{70}/_3$ times less than in the fundamental solution[2)]. The spherically symmetric perturbation (corresponding to Y_{00}) naturally moves downward along the frequency axis with the same velocity as the fundamental solution.

The second distinguishing feature of the equation (8) for the harmonics is the presence of the inhomogeneous term, so that the quantity u_{nm} for $1 \le n \le 3$ is not conserved along the characteristic (9). In order to graphically present the result, let us find the quantity that is conserved along this characteristic. Let us write, dropping part of the indices, the system of equations (8) and (2) in the form

$$\frac{\partial u}{\partial t}=u\frac{\partial\ln f}{\partial t}+d_nf\frac{\partial u}{\partial\nu},\quad \frac{\partial f}{\partial t}=af\frac{\partial f}{\partial\nu},$$

where $d_n = c_nb$, $f = f_0$, and $u = u_{nm}$. It is not difficult to find such $r = a/(d_n - a)$ that

$$\partial uf^r/\partial t=d_nf\partial uf^r/\partial\nu$$

and uf^r = const along the characteristic $d\nu/dt = -d_nf$, which clearly coincides with (9).

Hence it is easy to also determine the law of variation of the relative amplitude of the perturbation:

$$u_{nm}/f_0=\text{const}\cdot f_0^{-1-r}=\text{const}\cdot f_0^{d_n/(a-d_n)}. \tag{10}$$

For all $1 \le n \le 3$, the difference $a - d_n$ is positive, i.e., the sign of the power $-(r + 1) = d_n/(a - d_n)$ depends only on the sign of $d_n = c_nb$. This power is positive for $n = 2$ and is equal to $-(r + 1) = 1/9$ and negative for $n = 1$ and $n = 3$, it being respectively equal to $-(r + 1) = -2/7$ and $-(r + 1) = -3/73$.

To solve the problem completely, we must first solve the unperturbed problem, and then, knowing $f_0(\nu, t)$, construct the characteristics u_{nm} of the perturbations. Owing to their different velocity, they do not coincide with the characteristics of f_0, so that $f_0 \ne$ const along the u_{nm} characteristics. Consequently, the rate $d\nu/dt$ is also not a constant along the u_{nm} characteristics;

they are curves in the $\nu - t$ plane. Since f_0 is not a constant, the relative perturbation is, according to (10), also not a constant, but the sign of the answer for a given harmonic (the growth or decrease of the relative perturbation) depends on whether we are dealing with an increasing or a decreasing unperturbed solution.

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[1)]If the radiation is effectively concentrated in a cone of width θ_0, i.e., if $f \approx 0$ for $\theta > \theta_0$, then the restriction on the spectral width becomes less rigid: $\Delta\nu \gg \Delta\nu_D(1 - \cos\theta_0)^{1/2}$.

[2)]The motion of the characteristic upward along the frequency axis is surprising. The equation is valid for any electron temperature—in particular, at zero temperature, in which case the photon frequency can only decrease in a scattering process. However, we are here dealing with the propagation of the phase density, and not of the photons themselves. Because of the Bose factors, the scattering of photons of a given frequency ν depends on the density $n(\nu')$ of those photons (with $\nu' < \nu$) that are produced after the scattering: $n(\nu')$ has an influence on $\partial n(\nu)/\partial t$ when $\nu > \nu'$.

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