Kinetic equation for a light wave. Intensity fluctuations

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A derivation is given of the kinetic equation for the probability density of the parameters of a light wave traveling in a randomly inhomogeneous medium. This equation is used in the derivation and solution of a closed system of equations for the average value of the reciprocal of the wave intensity. A relationship between the moments of the light-wave intensity at a fixed point and inside a fixed ray tube is obtained.

INTRODUCTION

The determination of the statistical characteristics of the intensity of a light wave traveling in a randomly inhomogeneous medium is of considerable interest. However, numerous difficulties are encountered in studies of strong fluctuations of the light intensity. Even the mean-square value of this intensity obeys an equation whose solution is far from complete (see, for example,[1-3]); this is even more true of the higher moments of the intensity. In general, the determination of the probability characteristics of the intensity of a light wave is equivalent to the solution of an equation expressed in terms of the variational derivatives of the characteristic functional of the wave, which is an extremely difficult problem (see, for example,[1,4]).

The situation simplifies considerably if we ignore the diffraction effects and describe the propagation of a wave in the geometric optics approximation. In this case, we can derive a kinetic equation for the probability density of a light wave which allows us, in principle, to determine any statistical characteristics of a wave traveling in a medium with large-scale random inhomogeneities. This kinetic equation for a light wave is derived in the present paper using the small-angle approximation and the geometric-optics framework. The kinetic equation is then used in a study of the statistical properties of the wave intensity.

It should be noted that the equation obtained below cannot be derived by classical methods employed to obtain equations of the Einstein-Fokker-Planck type because, even in the geometric-optics approximation, the fluctuations which occur in a light wave do not have the necessary properties of finite-dimensional Markov processes (see, for example,[1,2]).

1. KINETIC EQUATION

1. It is known (see, for example,[1,5]) that the intensity $I(x, \rho)$ and fluctuations of the phase $S(x, \rho)$ of a light wave satisfy the following equations in the small-angle geometric-optics approximation:

$$\frac{\partial I}{\partial x}+\frac{1}{k}\nabla_\perp(I\nabla_\perp S)=0,\qquad \frac{\partial S}{\partial x}+\frac{1}{2k}(\nabla_\perp S)^2=\frac{k}{2}\varepsilon(x,\rho), \tag{1.1}$$

where x is the coordinate along which the wave is traveling; $\rho=\{y_1, y_2\}$ are the transverse coordinates; $\epsilon(x, \rho)$ are fluctuations of the permittivity; k is the wave number. We shall assume that $\epsilon(x, \rho)$ is a Gaussian field with a correlation function

$$\langle\varepsilon(x_1,\rho_1)\varepsilon(x_1+x,\rho_1+\rho)\rangle=A[\rho]\delta(x), \tag{1.2}$$

where $\rho=|\rho|$.

If, using Eq. (1.1), we seek an equation for the probability density $W[I, S; x, \rho]$, we find that this equation cannot be obtained in the closed form. However, the equation for the probability density can be closed if the phase and intensity are supplemented by the transverse components of the local wave vector $v=\nabla_\perp S$ and by quantities representing the curvature of the phase front of the wave. This is due to the fact that it is the fluctuations of the phase-front curvature that give rise to fluctuations of the wave intensity.

2. If the permittivity fluctuations are isotropic in a transverse plane, the most natural and convenient characteristics of the phase front are the average curvature $h=-(u_{11}+u_{22})/2k$ and the Gaussian curvature $p=(u_{11}u_{22}-u_{12}^2)/k^2$ of the phase front, both of which are independent of the coordinate system. Here, $u_{\alpha\beta}=\partial^2 S/\partial y_\alpha\partial y_\beta$. Therefore, we shall seek an equation for the six-dimensional single-point probability density $W[I, S, v, h, p; x, \rho]$. We shall do this by supplementing the system (1.1) with equations for v, h, and p. Differentiating the second of the equations in the system (1.1) with respect to the transverse coordinates, we find that simple transformations yield

$$\frac{\partial \mathbf{v}}{\partial x}+\frac{1}{k}(\mathbf{v}\nabla_\perp)\mathbf{v}=\frac{k}{2}\nabla_\perp\varepsilon,$$
$$\frac{\partial h}{\partial x}+\frac{1}{k}(\mathbf{v}\nabla_\perp)h-2h^2+p=-\frac{1}{4}\Delta_\perp\varepsilon,$$
$$\frac{\partial p}{\partial x}+\frac{1}{k}(\mathbf{v}\nabla_\perp)p-2hp=-\left(h\Delta_\perp\varepsilon+\frac{1}{2k}u_{\alpha\beta}\frac{\partial^2\varepsilon}{\partial y_\alpha\partial y_\beta}\right). \tag{1.3}$$

Following our earlier investigations,[6,7] we shall differentiate the average value of an arbitrary function of six arguments $\varphi[I, S, \mathbf{v}, h, p]$ with respect to x. Using Eqs. (1.1) and (1.3), we find that

$$\frac{\partial\langle\varphi\rangle}{\partial x}+\frac{1}{k}\operatorname{div}\langle\mathbf{v}\varphi\rangle+2\langle h\varphi\rangle-\left\langle(2h^2-p)\frac{\partial\varphi}{\partial h}\right\rangle-2\left\langle hp\frac{\partial\varphi}{\partial p}\right\rangle$$
$$-2\left\langle h\frac{\partial\varphi}{\partial I}\right\rangle-\frac{1}{2k}\left\langle \mathbf{v}^2\frac{\partial\varphi}{\partial S}\right\rangle=\frac{k}{2}\left\langle\varepsilon\frac{\partial\varphi}{\partial S}\right\rangle+\frac{k}{2}\left\langle\left(\frac{\partial\varphi}{\partial\mathbf{v}}\nabla_\perp\varepsilon\right)\right\rangle$$
$$-\frac{1}{4}\left\langle\Delta_\perp\varepsilon\frac{\partial\varphi}{\partial h}\right\rangle-\left\langle\left(h\Delta_\perp\varepsilon+\frac{1}{2k}u_{\alpha\beta}\frac{\partial^2\varepsilon}{\partial y_\alpha\partial y_\beta}\right)\frac{\partial\varphi}{\partial p}\right\rangle. \tag{1.4}$$

We shall now express the averages explicitly dependent on $\epsilon(x, \rho)$ in Eq. (1.4) in terms of the averages which are governed entirely by the required probability density. This can be done in the case of a Gaussian field ϵ characterized by the correlation function (1.2), using, for example, the Furutsu-Novikov formula (see, for example,[1,2]). Then, bearing in mind the arbitrary

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nature of φ, we can pass from Eq. (1.4) to the required equation for W (details of the procedure used in this case are given in[6,7]):

$$\frac{\partial W}{\partial x}+\frac{1}{k}(\mathbf{v}\nabla_\perp)W+2hW+\frac{\partial}{\partial h}(2h^2-p)W+2h\frac{\partial}{\partial p}pW$$
$$+2h\frac{\partial}{\partial I}IW+\frac{1}{2k}v^2\frac{\partial W}{\partial S}=kD\left[\frac{1}{2}\frac{\partial^2 W}{\partial h\,\partial S}+h\frac{\partial^2 W}{\partial S\,\partial p}\right]$$
$$+\frac{k^2}{4}D\left[\frac{\partial^2 W}{\partial v_1^2}+\frac{\partial^2 W}{\partial v_2^2}\right]+\frac{1}{2}B\frac{\partial^2 W}{\partial h^2}+2B\frac{\partial}{\partial h}h\frac{\partial W}{\partial p}$$
$$+3Bh^2\frac{\partial^2 W}{\partial p^2}-B\frac{\partial^2}{\partial p^2}pW+\frac{k^2}{8}A\frac{\partial^2 W}{\partial S^2}. \qquad (1.5)$$

Here, the following representation $A[\rho]$ is used for small values of ρ:

$$A[\rho]=A-D\rho^2+{}^1/_4B\rho^4-\dots$$

The above kinetic equation for a light wave is obviously valid for any fluctuations in intensity because they are not assumed to be small in the derivation of this equation.

2. PROBABILITY DENSITY FOR A PLANE WAVE

1. Let us assume that a plane wave falls normally on a half-space $x > 0$ filled with a randomly inhomogeneous medium. Since we are only interested in fluctuations of the wave intensity and arrival angles $\theta = |\mathbf{v}|/k$, we shall go over from Eq. (1.5) to an equation for the five-dimensional probability density $w[I, \mathbf{v}, h, p; x, \rho]$, by integrating Eq. (1.5) with respect to S between infinite limits. Bearing in mind that, in our case, the wave is statistically homogeneous in any transverse plane, so that the probability density is independent of ρ, we obtain

$$\frac{\partial w}{\partial x}+2hw+\frac{\partial}{\partial h}[(2h^2-p)w]+2h\frac{\partial}{\partial p}pw+2h\frac{\partial}{\partial I}Iw$$
$$=\frac{k^2}{4}D\left[\frac{\partial^2 w}{\partial v_1^2}+\frac{\partial^2 w}{\partial v_2^2}\right]+\frac{B}{2}\frac{\partial^2 w}{\partial h^2}+2B\frac{\partial}{\partial h}h\frac{\partial w}{\partial p}$$
$$+3Bh^2\frac{\partial^2 w}{\partial p^2}-B\frac{\partial^2}{\partial p^2}pw. \qquad (2.1)$$

The boundary condition corresponding to a plane wave is

$$w[I,\mathbf{v},h,p;0]=\delta(I-1)\delta(\mathbf{v})\delta(h)\delta(p). \qquad (2.2)$$

2. We shall seek the solution of Eq. (2.1) in the form

$$w=V[\mathbf{v};x]H[I,h,p;x]. \qquad (2.3)$$

Substituting Eq. (2.3) into (Eq. (2.1), we can easily show that the latter equation separates into two independent expressions:

$$\frac{\partial V}{\partial x}=\frac{k^2}{4}D\left[\frac{\partial^2 V}{\partial v_1^2}+\frac{\partial^2 V}{\partial v_2^2}\right], \qquad (2.4)$$

$$\frac{\partial H}{\partial x}+2hH+\frac{\partial}{\partial h}[(2h^2-p)H]+2h\frac{\partial}{\partial p}pH+2h\frac{\partial}{\partial I}IH$$
$$=\frac{B}{2}\frac{\partial^2 H}{\partial h^2}+2B\frac{\partial}{\partial h}h\frac{\partial H}{\partial p}+3Bh^2\frac{\partial^2 H}{\partial p^2}-B\frac{\partial^2}{\partial p^2}pH. \qquad (2.5)$$

The boundary conditions for the above equations follow in an obvious manner from Eq. (2.2).

Equations (2.4), together with Eqs. (2.3) and (2.5), demonstrate that fluctuations of the arrival angles and of the intensity of a plane wave are statistically independent. On the other hand, it follows from Eq. (2.5) that—as expected—there is a strong statistical relationship between the intensity fluctuations and the curvature of the wave phase front.

Equation (2.4), subject to Eq. (2.2), can be reduced to the Rayleigh distribution of the wave arrival angles $\theta = |\mathbf{v}|/k$:

$$W_\theta[\theta;x]=\frac{2\theta}{Dx}\exp\left\{-\frac{\theta^2}{Dx}\right\}.$$

It is clear from the above expression that $\langle\theta^2\rangle = Dx$, which is identical with the results obtained earlier in the geometric-optics approximation for small fluctuations of the wave amplitude,[5] and which generalizes the former expression to the case of arbitrary fluctuations of the amplitude.

3. INTENSITY FLUCTUATIONS AT A PLANE WAVE

1. We shall use Eq. (2.5) to study the statistical properties of the intensity of a plane wave traveling in a randomly inhomogeneous medium. It should be noted that the exact solution of the above equation is unknown. However, some of its consequences, supplemented by physically reasonable assumptions, allow us to find detailed statistical properties of the intensity of such a wave.

2. Equation (2.5) can be transformed into a finite system of closed equations in the case of some averages of physical interest. For example, multiplying Eq. (2.5) by I and integrating with respect to I, h, and p, we obtain the well-known equation

$$d\langle I\rangle/dx=0, \qquad (3.1)$$

which means that the average intensity of a plane wave remains constant in a randomly inhomogeneous medium.

It is clear from the form of Eq. (2.5) that a closed finite system of equations is also satisfied by the averages $\langle I^{-n}\rangle$, where $n = 1, 2, \dots$. Thus, the averages $\langle I^{-1}\rangle$ are defined by the following system of six equations:

$$\frac{d\langle I^{-1}\rangle}{dx}+4\langle hI^{-1}\rangle=0, \quad \frac{d\langle hI^{-1}\rangle}{dx}+2\langle h^2I^{-1}\rangle+\langle pI^{-1}\rangle=0,$$
$$\frac{d\langle h^2I^{-1}\rangle}{dx}+2\langle hpI^{-1}\rangle=B\langle I^{-1}\rangle, \quad \frac{d\langle pI^{-1}\rangle}{dx}+2\langle hpI^{-1}\rangle=0,$$
$$\frac{d\langle hpI^{-1}\rangle}{dx}+\langle p^2I^{-1}\rangle=2B\langle hI^{-1}\rangle, \quad \frac{d\langle p^2I^{-1}\rangle}{dx}=6B\langle h^2I^{-1}\rangle-2B\langle pI^{-1}\rangle. \qquad (3.2)$$

We shall now find the dependence of this average on the longitudinal coordinate. To do this, we note that, in the case of $\langle I^{-1}\rangle$, the system (3.2) reduces to one equation:

$$\frac{d^6\langle I^{-1}\rangle}{dx^6}-28B\frac{d^3\langle I^{-1}\rangle}{dx^3}-80B^2\langle I^{-1}\rangle=0. \qquad (3.3)$$

The boundary conditions corresponding to a plane wave of unit intensity, incident on a half-space $x > 0$, are found from Eqs. (2.2) and (3.2):

$$\langle I^{-1}\rangle_0=1, \quad \left.\frac{d\langle I^{-1}\rangle}{dx}\right|_{x=0}=\left.\frac{d^2\langle I^{-1}\rangle}{dx^2}\right|_{x=0}=0,$$
$$\left.\frac{d^3\langle I^{-1}\rangle}{dx^3}\right|_{x=0}=8B, \quad \left.\frac{d^4\langle I^{-1}\rangle}{dx^4}\right|_{x=0}=\left.\frac{d^5\langle I^{-1}\rangle}{dx^5}\right|_{x=0}=0. \qquad (3.4)$$

The solution of Eq. (3.3) subject to Eq. (3.4) is

$$\langle I^{-1}(x,\rho)\rangle=\left(\frac{1}{2}-\frac{3}{\sqrt{376}}\right)\left[\frac{1}{3}\exp\{\varkappa_1 x\}\right.$$
$$\left.+\frac{2}{3}\exp\left\{-\frac{\varkappa_1 x}{2}\right\}\cos\frac{\sqrt{3}}{2}\varkappa_1 x\right]+\left(\frac{1}{2}+\frac{3}{\sqrt{376}}\right)\cdot$$
$$\cdot\left[\frac{1}{3}\exp\{-\varkappa_2 x\}+\frac{2}{3}\exp\left\{\frac{\varkappa_2 x}{2}\right\}\cos\frac{\sqrt{3}}{2}\varkappa_2 x\right];$$
$$\varkappa_1=[B(\sqrt{376}+14)]^{1/3}, \quad \varkappa_2=[B(\sqrt{376}-14)]^{1/3}. \qquad (3.5)$$

3. Experiment and a qualitative analysis (see, for example,[5]) show that the amplitude of a plane wave $\chi = \frac{1}{2} \ln I$ has a single-point Gaussian distribution. If this is correct, the knowledge of the averages $\langle I \rangle$ and $\langle I^{-1} \rangle$ is sufficient for the determination of all the probability parameters of this amplitude. In this case, it follows from Eq. (3.1)[5] that $\sigma_\chi^2 = -\langle \chi \rangle$, where σ_χ^2 is the variance of the amplitude fluctuations and $\langle I^{-1} \rangle = \exp\{4\sigma_\chi^2\}$ determines, together with Eq. (3.5), the dependence of σ_χ^2 and, at the same time, of $\langle \chi \rangle$ on x. As expected, in the range of small fluctuations in amplitude (where $Bx^3 \ll 1$), the expression for σ_χ^2 obtained in this way leads to the well-known cubic dependence of the variance of the amplitude fluctuations on the path traveled by the wave, whereas, in the range of large fluctuations ($Bx^3 \sim 1$), it leads to the experimentally observed slower rise.

4. It has been suggested in[8,9] that the statistical characteristics of the intensity of a light wave can be determined by investigating fluctuations of the cross-sectional area of ray tubes. In particular, this approach is used in[8,9] to show that, in the case of a plane wave traveling in a randomly inhomogeneous medium, the average of the reciprocal of the intensity inside a fixed ray tube does not vary with the distance traveled by the wave. This result is not in conflict with the expression (3.5) for $\langle I^{-1} \rangle$, which is the average of the reciprocal of the wave intensity at a fixed point because the averages found in[8,9] and calculated above apply to different ensembles. We shall demonstrate this below. We shall consider the specific case when rays do not intersect.

In the case of a plane wave of unit intensity incident on a half-space filled with a randomly inhomogeneous medium, the reciprocal of the intensity inside the fixed ray tube emerging from a point ρ_0 in the plane x = 0 is

$$\frac{1}{I_r(x,\rho_0)} = \frac{1}{I(x,\rho)} = \frac{d\rho(\rho)}{d\rho_0(\rho_0)} = J. \qquad (3.6)$$

Here, $I_r(x, \rho_0)$ is the intensity inside a given ray tube; $d\rho_0(\rho_0)$ is the fiinfinitesimally small area of the cross section of this tube in the x = 0 plane; $d\rho(\rho)$ is the corresponding area in a plane x; ρ is the coordinate of the ray tube in the same plane; J is the Jacobian used to transform ρ into ray coordinates ρ_0.

For convenience, we shall assume that fluctuations of the intensity of a light wave in a fixed ray tube and at a fixed ray point are ergodic in the transverse plane. We can then easily show that the following formula applies:

$$\left\langle \frac{1}{I_r(x,\rho_0)} \right\rangle = \langle J \rangle = \lim_{L\to\infty} \frac{1}{4L^2} \iint_{-L}^{L} J\, d\rho_0. \qquad (3.7)$$

Here and later, $\langle J \rangle$ and similar averages are understood to be the averages of the Jacobian of the transformation of J (or of its function) corresponding to a ray tube emerging from the point $(0, \rho_0)$.

Applying Eq. (3.6), we can transform the integrand in Eq. (3.7) to

$$J\, d\rho_0 = J \frac{d\rho_0}{d\rho} d\rho = d\rho.$$

Equation (3.7) then becomes:

$$\left\langle \frac{1}{I_r} \right\rangle = \lim_{L\to\infty} \frac{1}{4L^2} \iint_C d\rho = \lim_{L\to\infty} \frac{S_C}{4L^2}. \qquad (3.8)$$

The integration is carried out over a region bounded by a random contour C, which is the geometric locus of the points of intersection of a plane x by rays emerging from the sides of a square ($y_1 = \pm L$, $y_2 = \pm L$) in the x = 0 plane; S_C is the area of the figure bounded by the contour C. Since the coordinates of the rays at finite distances x change only by a finite amount, it follows that, in the limit $L \to \infty$, we have $4L^2 - S_C \sim L$, i.e., in accordance with Eq. (3.8), we obtain

$$\langle 1/I_r \rangle = \lim_{L\to\infty} S_C/4L^2 = 1,$$

which has been shown, in particular, in[8,9]. For this reason, in the limit $L \to \infty$, we can modify Eq. (3.8) and similar equations by replacing the integration over a region bounded by the contour C with the integration over the interior of the square mentioned above.

We shall now analyze the average of the reciprocal of the intensity at a fixed point (x, ρ). Bearing in mind the ergodicity of $I(x, \rho)$, we obtain

$$\left\langle \frac{1}{I(x,\rho)} \right\rangle = \lim_{L\to\infty} \frac{1}{4L^2} \iint_{-L}^{L} J\, d\rho.$$

However, it follows from (3.6) that

$$J\, d\rho = J \frac{d\rho}{d\rho_0} d\rho_0 = J^2\, d\rho_0,$$

$$\left\langle \frac{1}{I(x,\rho)} \right\rangle = \lim_{L\to\infty} \frac{1}{4L^2} \iint_{C_0} J^2\, d\rho_0, \qquad (3.9)$$

where C_0 is a random contour in the x = 0 plane formed by rays passing in a plane x through the sides of a square ($y_1 = \pm L$, $y_2 = \pm L$). The same procedure as before allows us to ignore in Eq. (3.9) the difference between the integration domains bounded by the contour C_0 and by the sides of a square. Thus, Eq. (3.9) can be rewritten in the form

$$\left\langle \frac{1}{I(x,\rho)} \right\rangle = \lim_{L\to\infty} \frac{1}{4L^2} \iint_{-L}^{L} J^2\, d\rho_0 = \langle J^2 \rangle.$$

It follows that the average of the reciprocal intensity of a plane wave of unit intensity incident on a half-space x > 0 filled with a randomly inhomogeneous medium is equal to the mean-square value of the Jacobian J inside a fixed ray tube.

Similarly, we can derive the following general relationship between the moments of the reciprocal intensity of a plane wave in a fixed ray tube at a fixed point in space and the moments of the Jacobian:

$$\langle I_\pi^{-n-1} \rangle = \langle I^{-n} \rangle = \langle J^{n+1} \rangle, \quad n = 0, \ \pm 1, \ \pm 2 \ldots \qquad (3.10)$$

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1 V. I. Klyatskin and V. I. Tatarskiĭ, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 15, 1433 (1972).
2 V. I. Klyatskin and V. I. Tatarskiĭ, Usp. Fiz. Nauk 110, 499 (1973) [Sov. Phys.-Usp. 16, 494 (1974)].
3 I. M. Dagkesamanskaya and V. I. Shishov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 13, 16 (1970).
4 V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. 56, 2106 (1969) [Sov. Phys.-JETP 29, 1133 (1969)].
5 V. I. Tararskiĭ, Rasprostranenie voln v turbulentnoĭ

atmosfere, Nauka, M., 1967 (Wave Propagation in a Turbulent Medium, Dover, New York, 1968).
[6]A. N. Malakhov and A. I. Saichev, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 17, 699 (1974).
[7]A. N. Malakhov and A. I. Saichev, Zh. Eksp. Teor. Fiz. 67, 940 (1974) [Sov. Phys.-JETP 40, 467 (1975)].
[8]V. I. Klyatskin and V. I. Tatarskiĭ, Teor. Veroyatn. Primen. 14, 357 (1969).
[9]N. V. Zubareva, Izv. Vyssh. Uchebn. Zaved. Radiofiz. 16, 310 (1973).

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