Evolution of a nonlinear Langmuir wave in a weakly inhomogeneous plasma

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We obtain an expression for the amplitude and the phase of a nonlinear electrostatic wave in an inhomogeneous plasma; it follows from this that even a small gradient in the concentration changes appreciably the way the amplitude and phase depends on the distance from the source—as compared to the homogeneous plasma. Moreover, the additional phase shift, caused by the inhomogeneous plasma, increases, and in some cases determines the the modulation instability growth rate. An experimental check of the results is possible.

1. Recently considerable success has been achieved in experimental studies of nonlinear effects caused by the resonance interaction of monochromatic Langmuir waves with plasma particles (see, e.g.,[1,2]). The general setup of such experiments consists in the following. One supplies a potential on a grid (the point x = 0 on the figure, see below) with a frequency ω close to the plasma frequency ω_p which generates Langmuir oscillations propagating along the axis of the plasma column. Measurement of the field amplitude at different distances from the source in a stable plasma show initially linear damping with the Landau damping rate γ_L ($\gamma_L < 0$) changing at distances of the order of the nonlinear scale length

$$l_N=2\pi\omega\tau/k, \quad \tau=(m/ekE)^{1/2}, \tag{1}$$

(E(x) is the wave amplitude, k(x) the wave number) to oscillations with period l_N in agreement with the nonlinear theory.[3-5] After that the amplitude in a homogeneous plasma must tend to a constant value due to the establishing of an ergodic state in the resonance region of phase space.[3,5]

We consider in the present paper the influence of effects of the inhomogeneity of the plasma on the behavior of the amplitude and phase of the wave. We shall see below that even a small concentration gradient along the x axis suffices for the wave to be appreciably damped over distances of the order of 5 to 10 times l_N from the source. Moreover, the inhomogeneity in the plasma leads to an additional nonlinear shift in the frequency which under well-defined conditions may cause a modulation instability. The expressions obtained in this paper for the changes in the amplitude and the frequency shift can directly be compared with experiments which, in our opinion, are within the present-day possible limits.

2. We write the wave equation in the form

$$\mathscr{E}=E(x,t)\sin\left(\int_0^x k(x')\,dx'-\omega t+\varphi\right), \tag{2}$$

where E(x, t), k(x), and φ(x, t) are slowly varying functions, while k(x) is determined from the usual dispersion law, in which the plasma frequency ω_p depends on x as a parameter. E(x, t) and φ(x, t) satisfy the equations

$$\begin{aligned}\frac{\partial U}{\partial t}+\frac{\partial}{\partial x}(v_gU)+v_g'\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}U\right)&=-\overline{j\mathscr{E}}\\ &=\frac{1}{2\pi}\frac{e\omega^2}{k}E\int_{t-\pi/\omega}^{t+\pi/\omega}dt'\int_{-\infty}^{\infty}dv\,(f-f_L)\sin\left(\int_0^x k(x')\,dx'-\omega t'+\varphi\right),\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial\varphi}{\partial t}+v_g\frac{\partial\varphi}{\partial x}&=-\delta\omega\\ &=\frac{4e\omega^2}{k}\left(\frac{\partial(\varepsilon\omega)}{\partial\omega}\right)^{-1}\frac{1}{E}\int_{t-\pi/\omega}^{t+\pi/\omega}dt'\int_{-\infty}dv\,(f-f_L)\cos\left(\int_0 k(x')\,dx'-\omega t'+\varphi\right).\end{aligned} \tag{4}$$

Here $U = (E^2/16\pi)\partial(\varepsilon\omega)/\partial\omega$ is the energy density in the wave, $v_g = d\omega/dk$ is the group velocity, $v_g' = dv_g/dk$, f is the distribution function of the resonance electrons, and f_L their distribution function in the linear approximation.

We have shown earlier[6] that under certain conditions which will be discussed below (see (9)) the right-hand side of (3) takes the form[1)]. It will be clear from the expression to be obtained below for $\delta\omega$ that the term with $\partial\varphi/\partial x$ in Eq. (3) is small compared to (5). Putting further the time-derivative in (3) equal to zero, we get the following equation for the field in the stationary regime:

$$\overline{j\mathscr{E}}=-\frac{2}{\pi^3}\frac{\partial(\varepsilon\omega)}{\partial\omega}\frac{\omega^3\gamma_L}{k_0}\left(\frac{m}{ek}\right)^{3/2}E^{1/2}\frac{1}{k}\frac{dk}{dx}\ln\frac{k}{k_0}. \tag{5}$$

$$\frac{\partial}{\partial x}(v_gU)=-\overline{j\mathscr{E}}. \tag{3a}$$

We shall now assume that the wave parameters satisfy the conditions

$$(\omega\tau_0)^{-3}\ll|\gamma_L/(kv_g)|\ll(\omega\tau_0)^{-1} \tag{6}$$

(here and henceforth the index 0 will indicate values of the various quantities at x = 0). The last of inequalities (6) means that the wave must be sufficiently nonlinear, and the first one (as will become clear from what follows) guarantees the fastest damping of the wave for very small density gradients. It will, in particular, become clear in what follows that, thanks to the first of conditions (6), the amplitude is already considerably damped for small $\Delta k = k - k_0$. Bearing this in mind we can write the solution of Eqs. (3) and (5) in the form

$$\left(\frac{k}{k_0}\right)^{3/4}\left(\frac{E}{E_0}\right)^{3/2}=1+\frac{12}{\pi^2}\frac{\gamma_L(\omega\tau_0)^3}{kv_g}\ln^2\frac{k}{k_0}. \tag{7}$$

It follows from (7) that $E \ll E_0$ when

$$\frac{\Delta k}{k}\approx\pi\left|\frac{kv_g}{12\gamma_L(\omega\tau_0)^3}\right|^{1/2}. \tag{8}$$

In that case $\Delta k \ll k$ by virtue of (6).[2)]

Equation (5) is valid when[6]

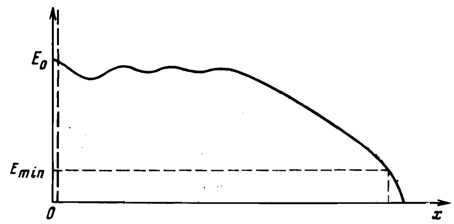
$$\frac{4}{\pi k}\frac{d}{dx}\left(\frac{1}{\omega\tau}\right)<\frac{1}{k^2}\frac{dk}{dx}\ll\frac{1}{(\omega\tau)^2}. \tag{9}$$

Substituting (7) into (9) we find that the first of inequalities (9) is satisfied when $E > E_1$, where

$$E_1=E_0\frac{16}{\pi^2}\left|\frac{\gamma_L\omega\tau_0}{3kv_g}\right|^{2/3}. \tag{10}$$

It is clear from (10) and (6) that $E_1 \ll E_0$. The second of inequalities (9) is satisfied when $E \gg E_2$ where

$$E_2=E_0\frac{(\omega\tau_0)^2}{k^2}\frac{dk}{dx}. \tag{11}$$



Damping of a Langmuir wave excited by a source (x = 0), in an inhomogeneous plasma column.

Hence it follows that $E_2 \ll E_0$ when[3]

$$\frac{1}{k^2}\frac{dk}{dx} \ll (\omega\tau_0)^{-2}. \tag{12}$$

The solution (7) describes thus the decrease of the field from E_0 to E_{min} where $E_{min} = \mathrm{Max}(E_1, E_2)$. One can show that when $E < E_{min}$ the decrease of the field will be faster than is described by Eq. (7) (see figure). Under the conditions (6) and (12) one may thus assume that the field decreases to values $E \ll E_0$ for Δk given by Eq. (8). Estimating the corresponding distance x from the source from the relation $x = \Delta k/(dk/dx)$ we get

$$x = \frac{\pi}{2k}\left(\frac{1}{k^2}\frac{dk}{dx}\right)^{-1}\left|\frac{kv_g}{3\gamma_L(\omega\tau_0)^3}\right|^{1/3}. \tag{13}$$

On checks easily, taking inequalities (12) and (6) into account, that $x \gg l_N$. The relative change in the density over such distances can be determined from the relation

$$\Delta n/n = -6(kv_T/\omega)^2 \Delta k/k$$

(provided the radius of the plasma column is appreciably larger than the Debye radius).

3. Let us turn to an evaluation of the correction to the phase $\varphi(x)$ of the wave. This quantity is determined by Eq. (4) where the physical meaning of $\delta\omega$ is that of the frequency shift in a reference system moving with the group velocity of the wave relative to the plasma. It turns out that this quantity consists of two parts

$$\delta\omega = \delta\omega_1 + \delta\omega_2, \tag{14}$$

where $\delta\omega_1$ is caused by the non-linear effects in a homogeneous plasma:[11]

$$\delta\omega_1 = -1.63\left[\frac{\partial(\varepsilon\omega)}{\partial\omega}\right]^{-1}\frac{1}{\tau}\frac{\omega_p{}^2\omega}{k^3}\frac{\partial^2 f_0}{\partial v^2}\bigg|_{v=\omega/k}, \tag{15}$$

while $\delta\omega_2$ is a correction to the frequency caused by the "interference" of the effects of nonlinearity and of inhomogeneity; we show in the Appendix that it is equal to

$$\delta\omega_2 = -\frac{16}{3\pi^2}\gamma_L\tau\omega \ln\frac{k}{k_0}. \tag{16}$$

Expression (16) is valid for those points where the condition[4] $\Delta k/k > 4/\pi(\omega\tau_0)$ is satisfied.

Assuming that the distribution function $f_0(v)$ of the unperturbed plasma is Maxwellian, we can transform Eq. (15) as follows:

$$\delta\omega_1 = \frac{3.26}{\pi}\gamma_L\left(\frac{\omega}{kv_T}\right)^2\frac{1}{\omega\tau}. \tag{17}$$

Comparing the quantities $\delta\omega_2$ and $\delta\omega_1$ given by Eqs. (16) and (17) we see, firstly, that they have different signs ($\delta\omega_2 > 0$, $\delta\omega_1 < 0$) and, secondly, that $\delta\omega_2$ increases as the wave is damped ($\delta\omega_2(0) = 0$), while $|\delta\omega_1|$ decreases. The quantity $\delta\omega$ can thus, due to the effect of the inhomogeneity, change its sign at sufficiently large distances from the point where the wave is injected (provided $\delta\omega_2$ increases relatively fast) and, moreover, can become of the order of $\delta\omega_2$ (the condition for this is given by Eq. (23)).

Substituting the expressions for $\delta\omega_1$ and $\delta\omega_2$ into Eq. (4) with $\partial\varphi/\partial t = 0$ we find the extra phase $\varphi(x)$ in the stationary regime

$$\varphi = -\int_0^x \frac{dx'}{v_g}\delta\omega(x') = \varphi_1 + \varphi_2.$$

The quantity φ_1 is determined by the quantity $\delta\omega_1$:

$$\varphi_1 = -\frac{3.26}{\pi}\frac{\gamma_L}{\omega v_g}\left(\frac{\omega}{kv_T}\right)^2\int_0^x \frac{dx'}{\tau(x')}.$$

In points where $E(x) \approx E_0$,

$$\varphi_1 = -\frac{3,26}{\pi}\frac{\gamma_L}{v_g}\left(\frac{\omega}{kv_T}\right)^2\frac{x}{\omega\tau_0},$$

while there where $E(x) \ll E_0$ we get, using (7)

$$\varphi_1 = a\left[(\omega\tau_0)^2\frac{1}{k^2}\frac{dk}{dx}\right]^{-1}\left(\frac{\omega}{kv_T}\right)^2\left|\frac{\gamma_L}{kv_g\omega\tau_0}\right|^{1/3}, \tag{18}$$

where the constant a is equal to

$$a = (3.26/2\sqrt{3})B(1/2, 1/3) \approx 0.8;$$

B is a beta function. As far as φ_2 is concerned it is determined by the quantity $\delta\omega_2$ and equals

$$\varphi_2 = \frac{16}{3\pi^2}\frac{\gamma_L\omega}{kv_g}\int_0^x \tau(x')\Delta k(x')dx'.$$

Substituting now the value of $\tau(x)$ from (7) we get

$$\varphi_2 = -\frac{1}{3}\left[(\omega\tau_0)^2\frac{1}{k^2}\frac{dk}{dx}\right]^{-1}\left[1-\frac{E(x)}{E_0}\right]. \tag{19}$$

It follows from Eqs. (18) and (19) that changing the value of the parameter $\omega\tau_0$, i.e., the amplitude of the wave injected into the plasma, we can obtain as a result essentially different values for the phase shift $\varphi(x)$. For instance, when $\omega\tau_0 > (\omega/kv_T)^4\gamma_L/kv_g$ the phase shift becomes negative (when $dk/dx > 0$) and is mainly determined by the magnitude of the inhomogeneity of the plasma column.

4. In[12] it was shown that the nonlinear frequency shift (15) causes in a homogeneous plasma a modulation instability of a plane monochromatic wave. As the inhomogeneity of the medium introduces an additional frequency shift (16) which, we saw, can be larger than (15) and which has the opposite sign we must reconsider the modulation instability problem in an inhomogeneous plasma.

Assuming that the wavelength of the perturbations of the envelope is larger than the nonlinear scale length l_N and appreciably smaller than the length x given by (13) we can for the analysis of the modulation instability use Eqs. (3) and (4) and choose the perturbed amplitude and phase proportional to $e^{iqx-i\Omega t}$. Linearizing Eqs. (3) and (4) with regard to the perturbations we find the following dispersion equation:

$$(\Omega-v_gq)^2+(\Omega-v_gq)\left[\frac{v_g'}{v_g}q\delta\omega+16\pi i\left[\frac{\partial(\varepsilon\omega)}{\partial\omega}\right]^{-1}\frac{\partial(\overline{j\mathcal{E}})}{\partial E^2}\right]=q^2v_g'\frac{\partial\delta\omega}{\partial E^2}E^2. \tag{20}$$

It is clear from Eqs. (5), (17), and (16) that

$$\frac{\partial(\overline{j\mathcal{E}})}{\partial E^2} = \frac{1}{4}\frac{\overline{j\mathcal{E}}}{E^2}, \quad \frac{\partial\delta\omega}{\partial E^2}E^2 = \frac{1}{4}(\delta\omega_1 - \delta\omega_2).$$

By virtue of inequalities (12) and (6) and also the condition $\omega\tau_0 > \omega/kv_T$ we can neglect the second term on the left-hand side of (20):

$$(\Omega-v_g q)^2={}^1/_4 q^2 v_g'(\delta\omega_1-\delta\omega_2). \tag{21}$$

We saw earlier that $\delta\omega_1 < 0$ and that sign $\delta\omega_2$ = sign (dk/dx) so that the modulation instability occurs certainly when[5] dk/dx and $v_g' > 0$. The growth rate of this instability is equal to

$$\Gamma={}^1/_2 q[v_g'(\delta\omega_2-\delta\omega_1)]^{1/2}. \tag{22}$$

It follows from (22) that, when dk/dx > 0, the inhomogeneity increases the growth rate of the modulation instability and, in the case when $\delta\omega_2 > |\delta\omega_1|$, will determine it.

The discussion given here by us of the evolution of the amplitude and phase of the wave field assumed that all quantities change little over the nonlinear scale length l_N. The expression we found for the growth rate (22) is thus valid only when $q < 2\pi/l_N$. The maximum value of the growth rate in the region where our theory is applicable is thus reached when $q \sim 2\pi/l_N$:

$$\Gamma_{max}\approx\frac{k}{2\omega\tau}[v_g'(\delta\omega_2-\delta\omega_1)]^{1/2}.$$

Substituting here the characteristic value of $\Delta k/k$ from (8) and the field amplitude E from (10) we find

$$\frac{\Gamma_{max}}{|\gamma_L|}\approx 0{,}2\frac{1}{(\omega\tau_0)^{3/2}}\left|\frac{kv_g}{\gamma_L\omega\tau_0}\right|^{3/4}\left[1+2\left(\frac{\omega}{kv_T}\right)^2\left|\frac{\gamma_L}{kv_g}\right|\right]^{1/2}.$$

If the condition

$$2\gamma_L/kv_g<(\omega/kv_T)^{-2}, \tag{23}$$

is satisfied, the main contribution to the modulation instability will come from the frequency shift $\delta\omega_2$ which arises when we take the inhomogeneity of the medium into account.

5. We now estimate the conditions under which one might observe the effects described above. We shall first start from the experimental data from[2]. Assuming that $\tau_0^{-1} = 7.37 \times 10^6\ s^{-1}$, $\gamma_L\tau_0 = 0.5$, $k = 3.64\ cm^{-1}$, $\omega = 2.54 \times 10^8\ s^{-1}$, $\gamma_L/v_g = 0.09\ cm^{-1}$, $v_T = 2 \times 10^7$ cm/s, we get from (8): $\Delta k/k = 3 \times 10^{-2}$, $\Delta n/n = 10^{-2}$. Substituting into (13) the length of the apparatus x = 50 cm we find $k^{-2}(dk/dx)(\omega\tau_0)^2 \approx 2 \times 10^{-2}$. According to (10) and (11) we have then $E_1/E_0 \approx 0.5$, $E_1 > E_2$. Thus, when the density decreases by 1% over the length of the column, the amplitude diminishes under the conditions mentioned here by roughly a factor 2. One must note that the sensitivity of the results with respect to even very small gradients in the concentration was noticed in[2].

If we now substitute the above-mentioned values of the parameters into (19), we get $\varphi_2 \approx 10$ radians, $\varphi_1 \sim \varphi_2$. The effect of the inhomogeneity on the phase can be observed by measuring the phase shift at different distances from the source.

APPENDIX

DERIVATION OF THE EXPRESSION FOR THE NON-LINEAR FREQUENCY SHIFT $\delta\omega_2$

For the evaluation of $\delta\omega_2$ we can use the general formalism developed in[6]. We introduce the following notation

$$2\xi=\int_0^x k(x')dx'-\omega t+\varphi,$$

$$2\dot\xi=k(x)\left[v-\frac{\omega}{k}+\frac{1}{k}\left(\frac{\partial\varphi}{\partial t}+\frac{\omega}{k}\frac{\partial\varphi}{\partial x}\right)\right],$$

$$\alpha=-\frac{\omega^2}{2k^2}\frac{dk}{dx},\quad \sigma(x)=-\frac{\pi}{4}\omega\ln\frac{k}{k_0},\quad \varkappa=(\dot\xi^2\tau^2+\sin^2\xi)^{-1/2}. \tag{A.1}$$

The kinetic equation for the resonance particles then takes the form

$$\frac{\omega}{k(x)}\frac{\partial f}{\partial x}+\dot\xi\frac{\partial f}{\partial\xi}-\left[\frac{\sin 2\xi}{2\tau^2}+\alpha-\frac{1}{2}\left(\frac{\partial}{\partial t}+\frac{\omega}{k}\frac{\partial}{\partial x}\right)^2\varphi\right]\frac{\partial f}{\partial\dot\xi}=0; \tag{A.2}$$

where we have used the condition for resonance particles: $\dot\xi \lesssim 1/\tau \ll \omega$.

This equation differs from the earlier obtained equation[6] in that it contains derivatives of $\varphi(x,t)$. However, under the conditions considered in the present paper these terms are small compared to α. Indeed, it follows from Eqs. (17) and (4) that

$$\frac{1}{2}\left(\frac{\omega}{k}\right)^2\frac{\partial^2\varphi}{\partial x^2}=\frac{16}{3\pi^2}\frac{\gamma_L\omega\tau_0}{kv_g}\left(\frac{E_0}{E}\right)^{1/2}\alpha,$$

so that by virtue of conditions (6) and (10) we can neglect in the kinetic equation (A.2) the term containing $\varphi(x,t)$. We can thus for the determination of the distribution function of the resonance particles use our old results.[6]

We expand in Eq. (4) the distribution function in the resonance region in terms of the velocity and restrict ourselves to the first two terms in the expansion:[6]

$$f=f_0(\omega/k)+f_0'(\omega/k)(v-\omega/k), \tag{A.2'}$$

f_0 is the unperturbed distribution function. In the same approximation we can replace the linear distribution function f_L by f_0. Under condition (9), which we can write in the form

$$\left|\frac{d\sigma}{dx}\right|>\left|\frac{d}{dx}\left(\frac{1}{\tau}\right)\right|, \tag{A.3}$$

if we use (A.1), the distribution function of the drifting particles ($|\kappa| < 1$) is equal to[7]

$$f-f_0=\frac{2}{k}f_0'\left\{\frac{2}{\pi}\frac{E(\varkappa)}{\varkappa\tau}-\frac{1}{\varkappa\tau}(1-\varkappa^2\sin^2\xi)^{1/2}\right\} \tag{A.4}$$

when $-1 < \kappa < \kappa_{cr}$, and to

$$f-f_0=\frac{2}{k}f_0'\left\{\frac{2}{\pi}\left[\frac{E(\varkappa)}{\varkappa\tau}-\frac{2}{\tau(x')}-\sigma(x)\right]\times\left[\frac{d\sigma(x')}{dx'}-\frac{d}{dx'}\left(\frac{1}{\tau(x')}\right)\right]\Big/\left[\frac{d\sigma(x')}{dx'}+\frac{d}{dx'}\left(\frac{1}{\tau(x')}\right)\right]-\frac{1}{\varkappa\tau}(1-\varkappa^2\sin^2\xi)^{1/2}-\frac{2}{\pi}\sigma(x)\right\} \tag{A.5}$$

when $\kappa_{cr} < \kappa < 1$. The quantity κ_{cr} is determined from the relation

$$E(\varkappa_{cr})/\varkappa_{cr}=\tau(x)/\tau_0-\tau(x)\sigma(x)$$

($E(\kappa)$ is the a complete elliptical integral of the second kind).

Formula (A.5) refers to those drifting particles which have changed the sign of their relative velocity $\dot\xi$ during the time of flight (in the notation use by us in[6]—negative particles), and also to those which left the trapping region of phase space due to the decrease in the wave amplitude. The quantity $\tau(x')$ in (A.5) is the value of τ in the point where the particle is reflected or where it leaves the trapping region

$$E(\varkappa)/\varkappa\tau(x)=1/\tau(x')+\sigma(x')-\sigma(x). \tag{A.6}$$

For trapped particles ($|\kappa| > 1$) we have

$$f-f_0=-\frac{2}{k}f_0'\left[\frac{1}{\varkappa\tau}(1-\varkappa^2\sin^2\xi)^{1/2}+\frac{2}{\pi}\sigma(x)\right]. \quad (A.7)$$

Substituting (A.4), (A.5), and (A.7) into (4) we get

$$\delta\omega_2=\frac{64}{\pi^3}\gamma_L\tau\left\{\frac{1}{3}\sigma(x')\right.$$
$$\left.+\tau(x)\int_0^x\left[\frac{2}{\varkappa^2}\left(1-\frac{E(\varkappa)}{K(\varkappa)}\right)-1\right]\frac{d}{dx'}\left(\frac{\sigma(x')}{\tau(x')}\right)dx'\right\}. \quad (A.8)$$

The function of κ occurring in the integral in (A.8) is a single-valued function of x′ according to formula (A.6). Moreover, it is positive definite and changes from

$$\frac{\pi^2}{32}\tau^{-2}(x)\left[1/\tau(x')+\sigma(x')-\sigma(x)\right]^{-2} \quad (A.9)$$

for $x' \ll x$ to 1 for $x' = x$.

If the condition $|\tau(x)\sigma(x)| > 1$ is satisfied, which, if we bear in mind the definition of the quantity $\sigma(x)$ in (A.1), is equivalent to the condition $\Delta k/k > 4\pi^{-1}(\omega\tau_0)^{-1}$, we can replace the integrand in (A.8) by (A.9) and approximately evaluate that integral

$$\delta\omega_2=\frac{64}{\pi^3}\gamma_L\tau\left\{\frac{1}{3}\sigma(x)-\frac{\pi^2}{32}\frac{1}{\tau(x)}\right\}\approx\frac{64}{3\pi^3}\gamma_L\tau\sigma(x).$$

Substituting then the expression from (A.1) for $\sigma(x)$ we get (16).

1)It is clear from (5) that, at least when $k-k_0 \ll k_0$ the sign of the nonlinear decay-rate is independent of the sign of the gradient of k and determined merely by the sign of γ_L (a more detailed analysis is given elsewhere [6-8]).

2)It is necessary [9] that $\omega\tau_0 > (\omega/kv_T)^2$ in order that no satellites will be generated, as that would change the evolution of the wave in an essential way.

3)When the condition $(\omega\tau_0/k)^2\, dk/dx \geqslant 1$ is satisfied the change in the phase velocity due to the inhomogeneity is so large that there is no particle trapping at all. Karpman and Shklyar [10] obtained for that case a general expression for the effective damping rate.

4)It is clear from Eq. (A.8) in the Appendix that $\delta\omega_2 \to 0$ as $\Delta k \to 0$.

5)These conditions are satisfied, in particular, when the radius of the plasma column is appreciably larger than the Debye radius so that

$$v_g=\frac{3v_T^2k}{\omega},\qquad \frac{1}{k}\frac{dk}{dx}=-\frac{1}{6}\left(\frac{\omega}{kv_T}\right)^2\frac{1}{n}\frac{dn}{dx}$$

and the display diminishes when x increases ($dn/dx < 0$).

6)Taking the third term in the expansion, which is proportional to the second derivative of the distribution function $\partial^2 f_0/\partial v^2|_{v=\omega/k}$, into account leads to the expression for the nonlinear frequency shift in a homogeneous medium, $\delta\omega_1$ found by Morales and O'Neil. [11]

7)We have given explicitly only the distribution function in the region in front of the maximum of the wave packet. Expressions (A.4) to (A.7) refer to the region beyond the maximum. One can obtain them by the method developed in [6] taking into account the differences of the regions at different sides of the maximum which we noted there.

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184