Some problems in the hydrodynamics of solutions of two superfluid liquids

V. P. Mineev

Institute of Solid State Physics, USSR Academy of Sciences
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Propagation of fourth sound in solutions of two superfluid liquids is investigated on the basis of the hydrodynamic equations for such solutions. The existence of two fourth sounds is demonstrated. The possibility of exciting fourth sounds by means of vibrations of a wall in a direction perpendicular to its plane is studied. Hydrodynamic equations for solutions of two superfluid liquids with dissipative terms are derived. These equations contain ten independent kinetic coefficients: one for the first viscosity, six for the second viscosity, and one each for diffusion, thermodiffusion, and thermal conductivity.

1. INTRODUCTION

The possibility of the existence of solutions of two superfluid liquids appeared with the discovery of the λ transition in liquid ^{3}He at a temperature of the order of a few millidegrees Kelvin.[1] Since the solution of liquid ^{3}He in superfluid ^{4}He does not separate into the pure components even at absolute zero temperature, up to concentrations of about 6% (see [2]), it is then evident that upon lowering of the temperature the ^{3}He in such a solution will also undergo a transition to the superfluid state and a solution of two superfluid liquids will be obtained. The equations of the hydrodynamics of such solutions were first obtained in Khalatnikov's paper [3], and the propagation of sound in them was studied in [4].

In this paper, we consider certain problems in the hydrodynamics of solutions of two superfluid liquids: fourth sound in such solutions, the excitation of fourth sound, and the set of hydrodynamic equations of solutions of two superfluid liquids with dissipative terms.[1)]

Before proceeding to the exposition of these problems, we write down several formulas which we shall need below, namely: the hydrodynamic equations of solutions of two superfluid liquids and expressions for the velocities of sound in such solutions (see [3,4]). There are six equations in all: two equations of continuity for each of the components of the solution

$$\dot\rho_1+\operatorname{div}(\rho_{s1}\mathbf{v}_{s1}+\rho_{n1}\mathbf{v}_n)=0,\qquad \dot\rho_2+\operatorname{div}(\rho_{s2}\mathbf{v}_{s2}+\rho_{n2}\mathbf{v}_n)=0; \tag{1.1}$$

here ρ is the density, c the concentration, $\mathbf{v}_{s1}$ and $\mathbf{v}_{s2}$ the superfluid velocities, ρ_{s1} and ρ_{s2} the superfluid densities, and ρ_{n1} and ρ_{n2} the normal densities of the first and second components, respectively; $\mathbf{v}_n$ the velocity of normal motion;

$$\rho_1=\rho c=\rho_{s1}+\rho_{n1},\qquad \rho_2=\rho(1-c)=\rho_{s2}+\rho_{n2}; \tag{1.2}$$

two equations of superfluid motion

$$\dot{\mathbf{v}}_{s1}+\nabla(\mu_1-{}^1/_2\mathbf{v}_n{}^2+\mathbf{v}_n\mathbf{v}_{s1}=0,\qquad \dot{\mathbf{v}}_{s2}+\nabla(\mu_2-{}^1/_2\mathbf{v}_n{}^2+\mathbf{v}_n\mathbf{v}_{s2})=0, \tag{1.3}$$

where μ_1 and μ_2 are the chemical potentials, which are defined by the identity for the energy ϵ:

$$d\varepsilon=TdS+\mu_1d\rho_1+\mu_2d\rho_2+\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)d(\mathbf{v}_{s1}-\mathbf{v}_n)+\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)d(\mathbf{v}_{s2}-\mathbf{v}_n); \tag{1.4}$$

the equation of conservation of the total momentum

$$\mathbf{j}=\rho_{s1}\mathbf{v}_{s1}+\rho_{s2}\mathbf{v}_{s2}+\rho_n\mathbf{v}_n, \tag{1.5}$$

$$\dot j_i+\partial\Pi_{ik}/\partial x_k=0, \tag{1.6}$$

here Π_{ik} is the momentum flux tensor

$$\Pi_{ik}=\rho_{s1}v_{s1i}v_{s1k}+\rho_{s2}v_{s2i}v_{s2k}+\rho_nv_{ni}v_{nk}+p\delta_{ik}, \tag{1.7}$$

and the pressure is determined by the expression

$$p=-\varepsilon+TS+\mu_1\rho_1+\mu_2\rho_2; \tag{1.8}$$

and the equation of continuity for the entropy

$$\dot S+\operatorname{div}S\mathbf{v}_n=0. \tag{1.9}$$

Khalatnikov[4] has shown that three different sounds can propagate in solutions of two superfluid liquids representing coupled oscillations of pressure, temperature and concentration. We write down the velocities of these sounds, assuming that $u_1 \gg u_2 \gg u_3$ and $c \ll 1$:

$$u_1^2=\frac{\partial p}{\partial\rho},\qquad u_2^2=\frac{\rho_s}{\rho_n}\sigma^2\frac{\partial T}{\partial\sigma},\qquad u_3^2=\frac{\rho_{s1}\rho_{s2}}{\rho_s\rho}\frac{\partial\zeta}{\partial c}; \tag{1.10}$$

Here $\sigma=S/\rho$, $\zeta=\mu_1-\mu_2$.

2. FOURTH SOUND IN SOLUTIONS OF TWO SUPERFLUID LIQUIDS

Fourth sound in an ordinary superfluid liquid is the oscillation of the superfluid component in narrow capillaries when the free path of the excitations exceeds the diameter of the tube and the normal component is immobile (see [5]). Since the transition of ^{3}He to the superfluid state in solutions of ^{3}He in superfluid ^{4}He will take place at much lower temperatures than for ^{4}He, the experiment might be performed more conveniently by measuring precisely the fourth sound in the solutions. Thus, we need to consider the linearized hydrodynamic equations (1.1), (1.3) and (1.9) at $\mathbf{v}_n=0$:

$$\dot\rho_1+\rho_{s1}\operatorname{div}\mathbf{v}_{s1}=0,\qquad \dot\rho_2+\rho_{s2}\operatorname{div}\mathbf{v}_{s2}=0,\qquad \dot{\mathbf{v}}_{s1}+\nabla\mu_1=0,\qquad \dot{\mathbf{v}}_{s2}+\nabla\mu_2=0,\qquad \dot S=0. \tag{2.1}$$

It is convenient to replace μ_1, μ_2 by new chemical potentials $\mu=c\mu_1+(1-c)\mu_2$ and $\zeta=\mu_1-\mu_2$; we then get the following from (1.8) and (1.7) ($\sigma=S/\rho$)

$$\rho^{-1}dp=\sigma dT+d\mu-\zeta dc. \tag{2.2}$$

Using this notation and also (1.2), we rewrite the set (2.1) in the form

$$\dot\rho c+\dot c\rho+\rho_{s1}\operatorname{div}\mathbf{v}_{s1}=0,\qquad \dot\rho(1-c)-\dot c\rho+\rho_{s2}\operatorname{div}\mathbf{v}_{s2}=0,$$

$$\dot{\mathbf{v}}_{s1}+(1-c)\nabla\zeta-\sigma\nabla T+\frac{\nabla p}{\rho}=0,\qquad \dot{\mathbf{v}}_{s2}-c\nabla\zeta-\sigma\nabla T+\frac{\nabla p}{\rho}=0, \tag{2.3}$$

$$\dot\sigma\rho+\sigma\dot\rho=0.$$

We now consider a plane sound wave in which all the variable quantities are proportional to $\exp\{i\omega(t-x/u\}$. Denoting the variable parts of the corresponding quantities by primes, we get the following set of algebraic equations from (2.3):

$$u\rho'c+uc'\rho-\rho_{s1}v_{s1}=0,\quad u\rho'(1-c)-uc'\rho-\rho_{s2}v_{s2}=0,$$
$$uv_{s1}+(c-1)\zeta'+\sigma T'-\frac{p'}{\rho}=0,\quad uv_{s2}+c\zeta'+\sigma T'-\frac{p'}{\rho}=0,\tag{2.4}$$
$$\sigma\rho'+\rho\sigma'=0.$$

We convert to the variables p, c, T in (2.4) by means of the formulas

$$\rho'=\frac{\partial\rho}{\partial p}p'+\frac{\partial\rho}{\partial c}c',\quad \sigma'=\frac{\partial\sigma}{\partial T}T'+\frac{\partial\sigma}{\partial c}c',$$
$$\zeta'=-\frac{1}{\rho^2}\frac{\partial\rho}{\partial c}p'-\frac{\partial\sigma}{\partial c}T'+\frac{\partial\zeta}{\partial c}c'.\tag{2.5}$$

In (2.5), we have used the smallness of the coefficient of thermal expansion $\partial\rho/\partial T$ [2]) and also the relations that follow from identity (2.2) for the derivatives of the thermodynamic quantities.

With (2.5), the conditions for the compatibility of Eqs. (2.4) give the equation for the sound velocity:

$$u^4-u^2\frac{\partial p}{\partial\rho}\Big\{\Big[c^2\frac{\rho_{s2}}{\rho}+(1-c)^2\frac{\rho_{s1}}{\rho}\Big]\Big[\frac{1}{\rho}\frac{\partial\rho}{\partial c}+\frac{\partial\rho}{\partial p}\Big(\frac{\partial\sigma}{\partial c}\Big)^2\frac{\partial T}{\partial\sigma}+\frac{\partial\rho}{\partial p}\frac{\partial\zeta}{\partial c}\Big]+2\Big[(1-c)\frac{\rho_{s1}}{\rho}-c\frac{\rho_{s2}}{\rho}\Big]\Big[\sigma\frac{\partial\rho}{\partial p}\frac{\partial\sigma}{\partial c}\frac{\partial T}{\partial\sigma}+\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big]$$
$$+\Big(\frac{\rho_{s1}+\rho_{s2}}{\rho}\Big)\Big(\frac{\partial\rho}{\partial p}\sigma^2\frac{\partial T}{\partial\sigma}+1\Big)\Big\}+\frac{\rho_{s1}\rho_{s2}}{\rho^2}\frac{\partial p}{\partial\rho}\Big\{\sigma^2\frac{\partial\rho}{\partial p}\frac{\partial\zeta}{\partial c}\frac{\partial T}{\partial\sigma}+\sigma^2\frac{\partial T}{\partial\sigma}\Big[\frac{1}{\sigma}\frac{\partial\sigma}{\partial c}+\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big]^2+\frac{\partial\zeta}{\partial c}\Big\}=0.\tag{2.6}$$

This equation can be solved by using the fact that one root is much smaller than the other in the concentration parameter $c\ll 1$. It is also necessary to take into account that terms of the order ρ_n/ρ are small at the temperatures considered, at least as small as c. With the foregoing, we obtain

$$(u_4')^2=\frac{\rho_{s2}}{\rho}\frac{\partial p}{\partial\rho},\quad (u_4'')^2=\frac{\rho_{s1}}{\rho}\frac{\partial\zeta}{\partial c}.\tag{2.7}$$

We can also rewrite Eq. (2.7) in the form (see (1.10))

$$(u_4')^2=\frac{\rho_{s2}}{\rho}u_1^2,\quad (u_4'')^2=\frac{\rho_s}{\rho_{s2}}u_2^2.\tag{2.8}$$

Thus, two "fourth sounds" can propagate in capillaries with a solution of two superfluid liquids, representing coupled oscillations of density and concentration.

3. EXCITATION OF FOURTH SOUNDS IN SOLUTIONS OF TWO SUPERFLUID LIQUIDS

As has been made clear in [4], a distinguishing property of solutions of two superfluid liquids is the possibility of propagation of a third type of sound wave in which concentration is the main oscillating property. Under conditions in which the free path length of the excitations are greater than the dimensions of the vessel, this sound corresponds to the second of the fourth sounds. It is interesting to establish how intense this specific sound will be under the usual method of excitation—by means of oscillations of the wall in the direction perpendicular to its plane. It is more convenient to carry out the treatment precisely for the fourth sound by virtue of its greater simplicity. The answer for third sound should not differ greatly from that obtained here. We shall proceed in the same fashion as in the investigation of the question of radiation of sound in an ordinary superfluid liquid (see [5]). Let the wall oscillate in the direction perpendicular to its plane with a velocity $v_0e^{-i\omega t}$. The velocities of the first superfluid component in the first and second of the radiated fourth sounds will be

$$v_{s1}'=A_1e^{-i\omega(t-x/u)},\quad v_{s1}''=A_2e^{-i\omega(t-x/u)},\tag{3.1}$$

while the velocities of the second component will be, respectively,

$$v_{s2}'=A_1a_1e^{-i\omega(t-x/u)},\quad v_{s2}''=A_2a_2e^{-i\omega(t-x/u)}.\tag{3.2}$$

Here

$$a_1=v_{s2}'/v_{s1}',\quad a_2=v_{s2}''/v_{s1}''.\tag{3.3}$$

The quantities v_{s1} and v_{s2} should coincide with the surface velocity on a solid surface: $A_1+A_2=v_0$, $A_1a_1+A_2a_2=v_0$, whence $A_2/A_1=(1-a_1)/(1-a_2)$. The mean energy density radiated in each of the sounds is obtained by means of (3.1) and (3.2):

$$E=\rho_{s1}\overline{v_{s1}^2}+\rho_{s2}\overline{v_{s2}^2}={}^1/_2A^2(\rho_{s1}+\rho_{s2}a^2),$$

and the intensity ratio of the radiated waves of first and second fourth sounds has the form

$$\frac{I_2}{I_1}=\frac{u_4''}{u_4'}\Big(\frac{A_2}{A_1}\Big)^2\frac{\rho_{s1}+\rho_{s2}a_2^2}{\rho_{s1}+\rho_{s2}a_1^2}=\frac{u_4''}{u_4'}\Big(\frac{1-a_1}{1-a_2}\Big)^2\frac{\rho_{s1}+\rho_{s2}a_2^2}{\rho_{s1}+\rho_{s2}a_1^2}.\tag{3.4}$$

Thus, to find the final answer, it is necessary to find a_1 and a_2 (see (3.3)). From (2.4) and (2.5), we can obtain

$$\frac{v_{s2}}{v_{s1}}=\Big\{-\rho_{s1}\Big\{(1-c)^2\Big[\Big(\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big)^2+\frac{\partial\rho}{\partial p}\Big(\frac{\partial\sigma}{\partial c}\Big)^2\frac{\partial T}{\partial\sigma}+\frac{\partial\rho}{\partial p}\frac{\partial\zeta}{\partial c}\Big]+2(1-c)\Big[\frac{\partial\rho}{\partial p}\frac{\partial\sigma}{\partial c}\sigma\frac{\partial T}{\partial\sigma}-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big]+\sigma^2\frac{\partial\rho}{\partial p}\frac{\partial T}{\partial\sigma}+1\Big\}+u^2\rho\frac{\partial\rho}{\partial p}\Big\}$$
$$\times\Big\{-\rho_{s2}\Big\{c(1-c)\Big[\Big(\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big)^2+\frac{\partial\rho}{\partial p}\Big(\frac{\partial\sigma}{\partial c}\Big)^2\frac{\partial T}{\partial\sigma}+\frac{\partial\rho}{\partial p}\frac{\partial\zeta}{\partial c}\Big]+2c\Big[\frac{\partial\rho}{\partial p}\frac{\partial\sigma}{\partial c}\sigma\frac{\partial T}{\partial\sigma}-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big]-\frac{\sigma}{\rho}\frac{\partial\sigma}{\partial c}\frac{\partial\rho}{\partial p}\frac{\partial T}{\partial\sigma}+\frac{1}{\rho}\frac{\partial\rho}{\partial c}-\sigma^2\frac{\partial\rho}{\partial p}\frac{\partial T}{\partial\sigma}-1\Big\}\Big\}^{-1}.$$

Keeping only the principal terms in the concentration, we have

$$a_1=\frac{v_{s2}'}{v_{s1}'}=(u_4')^2\rho\frac{\partial\rho}{\partial p}\Big/\rho_{s2}\Big(1-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big),$$
$$a_2=\frac{v_{s2}''}{v_{s1}''}=\Big\{-\rho_{s1}\Big[\frac{\partial\rho}{\partial p}\frac{\partial\zeta}{\partial c}+\Big(1-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big)^2\Big]+(u_4'')^2\rho\frac{\partial\rho}{\partial p}\Big\}\Big/\rho_{s2}\Big(1-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big).$$

Using (2.7), we can rewrite these expressions in the form

$$a_1=\Big(1-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big)^{-1},\quad a_2=-\frac{\rho_{s1}}{\rho_{s2}}\Big(1-\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big).\tag{3.5}$$

We substitute (3.5) in (3.4). Using (2.7), we obtain, to the principal order in the concentration,

$$\frac{I_2}{I_1}=\Big(\frac{\partial\zeta}{\partial c}\frac{\partial\rho}{\partial p}\Big)^{1/2}\Big(\frac{\rho_{s1}}{\rho_{s2}}\Big)^{3/2}\Big(\frac{1}{\rho}\frac{\partial\rho}{\partial c}\Big)^2.\tag{3.6}$$

Thus the ratio of the intensities of the second to the first of the fourth sounds in the radiation of a wall vibrating in a direction perpendicular to its plane contains the concentration parameter to the 3/2 power. This quantity may also prove to be not so small, which allows both sounds to be excited in this fashion.

4. HYDRODYNAMIC EQUATIONS OF A SOLUTION OF TWO SUPERFLUID LIQUIDS WITH DISSIPATIVE TERMS

In order to obtain the equations of hydrodynamics of a solution of two superfluid liquids with account of dissipation, in complete analogy to what was done for pure helium (see [5]), we introduce additional terms in Eqs. (1.1), (1.3) and (1.6):

$$\dot{\rho}_1+\operatorname{div}(\rho_{s1}\mathbf{v}_{s1}+\rho_{n1}\mathbf{v}_n+\mathbf{g}_1)=0, \qquad \dot{\rho}_2+\operatorname{div}(\rho_{s2}\mathbf{v}_{s2}+\rho_{n2}\mathbf{v}_n+\mathbf{g}_2)=0,$$
$$\dot{\mathbf{v}}_{s1}+\nabla\left(\mu_1-\frac{\mathbf{v}_n^2}{2}+\mathbf{v}_n\mathbf{v}_{s1}+h_1\right)=0, \qquad \dot{\mathbf{v}}_{s2}+\nabla\left(\mu_2-\frac{\mathbf{v}_n^2}{2}+\mathbf{v}_n\mathbf{v}_{s2}+h_2\right)=0, \tag{4.1}$$
$$\dot{j}_i+\frac{\partial(\Pi_{ik}+\tau_{ik})}{\partial x_k}=0.$$

Π_{ik} is given by Eq. (1.7), and $\mathbf{g}_1+\mathbf{g}_2=0$, since the continuity equation $\dot{\rho}+\operatorname{div}\mathbf{j}=0$ holds for the entire liquid. We then have $\mathbf{g}_2=-\mathbf{g}_1$ everywhere below. The total energy in the coordinates bound to the normal motion has the form

$$E={}^1/_2\rho\mathbf{v}_n^2+(\mathbf{p}_1+\mathbf{p}_2)\mathbf{v}_n+\varepsilon. \tag{4.2}$$

Here ϵ is determined by expression (1.4), and $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta of the two superfluid motions, taken in the coordinates in which the normal component is at rest (see [3]):

$$\mathbf{p}_1=\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n), \qquad \mathbf{p}_2=\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n). \tag{4.3}$$

Differentiating (4.2) with respect to time and applying (1.1)–(1.9), (4.2), and (4.3), we can establish (see [3]) the law of energy conservation in the form [3)]

$$\partial E/\partial t+\operatorname{div}\mathbf{Q}=0, \tag{4.4}$$

where

$$\mathbf{Q}=(\mathbf{p}_1+\rho_1\mathbf{v}_n)(\mu_1-{}^1/_2\mathbf{v}_n^2)+(\mathbf{p}_2+\rho_2\mathbf{v}_n)(\mu_2-{}^1/_2\mathbf{v}_n^2)+\mathbf{v}_n(\mathbf{j}\mathbf{v}_n)+\mathbf{p}_1(\mathbf{v}_n\mathbf{v}_{s1})+\mathbf{p}_2(\mathbf{v}_n\mathbf{v}_{s2})+TS\mathbf{v}_n. \tag{4.5}$$

The same calculations, carried out with the equations of hydrodynamics with the additional terms of (4.1), give

$$\partial E/\partial t+\operatorname{div}(\mathbf{Q}+\mathbf{Q}')=0. \tag{4.6}$$

Here Q is determined from formula (4.5),

$$\mathbf{Q}'=\mathbf{q}+h_1\mathbf{p}_1+h_2\mathbf{p}_2+(\mathbf{v}_n\tau). \tag{4.7}$$

In this case, the rate of change of entropy can be written in the form

$$T\left\{\dot{S}+\operatorname{div}\left[S\mathbf{v}_n+\frac{\mathbf{q}}{T}-\frac{\mathbf{g}_1\zeta}{T}\right]\right\}=-h_1\operatorname{div}\mathbf{p}_1-h_2\operatorname{div}\mathbf{p}_2-\tau_{ik}\frac{\partial v_{ni}}{\partial x_k}-\mathbf{g}_1T\nabla\frac{\zeta}{T}-\mathbf{q}\frac{\nabla T}{T^2}. \tag{4.8}$$

In the expressions for the energy flux and entropy fluxes, we have inserted the additional unknown term $\mathbf{q}$. The right side of Eq. (4.8) determines the dissipative function R. From its positiveness condition we have

$$h_1=-\zeta_1\operatorname{div}\mathbf{p}_1-\zeta_2\operatorname{div}\mathbf{p}_2-\zeta_3\operatorname{div}\mathbf{v}_n, \tag{4.9}$$
$$h_2=-\zeta_4\operatorname{div}\mathbf{p}_1-\zeta_5\operatorname{div}\mathbf{p}_2-\zeta_6\operatorname{div}\mathbf{v}_n, \tag{4.10}$$
$$\tau_{ik}=-\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{ni}}{\partial x_i}\right)-\delta_{ik}(\zeta_7\operatorname{div}\mathbf{p}_1+\zeta_8\operatorname{div}\mathbf{p}_2+\zeta_9\operatorname{div}\mathbf{v}_n), \tag{4.11}$$
$$\mathbf{g}_1=-\alpha\nabla\frac{\zeta}{T}-\beta\frac{\nabla T}{T^2}, \tag{4.12}$$
$$\mathbf{q}=-\gamma\nabla\frac{\zeta}{T}-\delta\frac{\nabla T}{T^2}. \tag{4.13}$$

By virtue of the Onsager symmetry principle, we have the following relations for the kinetic coefficients:

$$\zeta_2=\zeta_4, \quad \zeta_3=\zeta_7, \quad \zeta_6=\zeta_8, \quad \beta=\gamma. \tag{4.14}$$

In order to introduce the diffusion, thermal-diffusion, and thermal-conductivity coefficients, we shall proceed in the same fashion as in the case of ordinary helium with a nonsuperconducting impurity (see [5]). We express the heat flux $\mathbf{q}$ in terms of the diffusion flux $\mathbf{g}_1$ and ∇T. We get from (4.12) and (4.13)

$$-\mathbf{q}=-\frac{\gamma}{\alpha}\mathbf{g}_1+\left(\delta-\frac{\beta\gamma}{\alpha}\right)\frac{\nabla T}{T^2}.$$

The coefficient of thermal conductivity κ is defined in such a way that at zero flux $\mathbf{g}_1$ the heat flux would be equal to $-\kappa\nabla T$, i.e.,

$$\varkappa=\left(\delta-\frac{\beta\gamma}{\alpha}\right)\frac{1}{T^2}. \tag{4.15}$$

We now transform, as usual, to the variables p, T, c and introduce the notation

$$D=\frac{\alpha}{\rho}\frac{\partial}{\partial c}\frac{\zeta}{T}, \quad D\rho k_T=\alpha T\frac{\partial}{\partial T}\frac{\zeta}{T}+\frac{\beta}{T}, \quad k_p=p\frac{\partial}{\partial p}\frac{\zeta}{T}\Big/\frac{\partial}{\partial c}\frac{\zeta}{T}. \tag{4.16}$$

In this notation, the fluxes (4.12) and (4.13) take on the form

$$-\mathbf{g}_1=\rho D\left(\nabla c+k_T\frac{\nabla T}{T}+k_p\frac{\nabla p}{p}\right),$$
$$-\mathbf{q}=T^2\left[\frac{\partial}{\partial T}\frac{\zeta}{T}-\frac{k_T}{T}\frac{\partial}{\partial c}\frac{\zeta}{T}\right]\mathbf{g}_1+\varkappa\nabla T. \tag{4.17}$$

The quantity D is the diffusion coefficient, k_TD is the coefficient of thermal diffusion and k_p the coefficient of pressure diffusion. The pressure diffusion coefficient is not a kinetic coefficient, since it is completely expressed in terms of derivatives of thermodynamic quantities.

We can now write down the hydrodynamic equations of the solution of two superfluid liquids with dissipative terms in their final form:

$$\dot{\rho}+\operatorname{div}\mathbf{j}=0,$$
$$\dot{\rho}_1+\operatorname{div}(\rho_{s1}\mathbf{v}_{s1}+\rho_{n1}\mathbf{v}_n)=\operatorname{div}\left[\rho D\left(\nabla c+\frac{k_T}{T}\nabla T+\frac{k_p}{p}\nabla p\right)\right],$$
$$\dot{\mathbf{v}}_{s1}+\nabla(\mu_1-{}^1/_2\mathbf{v}_n^2+\mathbf{v}_n\mathbf{v}_{s1})=\nabla[\zeta_1\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)+\zeta_2\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)+\zeta_3\operatorname{div}\mathbf{v}_n],$$
$$\dot{\mathbf{v}}_{s2}+\nabla(\mu_2-{}^1/_2\mathbf{v}_n^2+\mathbf{v}_n\mathbf{v}_{s2})=\nabla[\zeta_4\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)+\zeta_5\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)+\zeta_6\operatorname{div}\mathbf{v}_n], \tag{4.18}$$
$$\dot{j}_i+\frac{\partial\Pi_{ik}}{\partial x_k}=\frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{ni}}{\partial x_i}\right)+\delta_{ik}[\zeta_7\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)+\zeta_8\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)+\zeta_9\operatorname{div}\mathbf{v}_n]\right\}.$$

The equation for increase in entropy takes the form

$$T\left\{\dot{S}+\operatorname{div}\left[S\mathbf{v}_n+\frac{\mathbf{q}}{T}-\mathbf{g}_1\frac{\zeta}{T}\right]\right\}=R. \tag{4.19}$$

The fluxes $\mathbf{q}$ and $\mathbf{g}_1$ are determined by the expressions (4.17)

$$R=\zeta_1(\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n))^2+\zeta_5(\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n))^2+\zeta_9(\operatorname{div}\mathbf{v}_n)^2$$
$$+2\zeta_2\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)+2\zeta_3\operatorname{div}\rho_{s1}(\mathbf{v}_{s1}-\mathbf{v}_n)\operatorname{div}\mathbf{v}_n$$
$$+2\zeta_6\operatorname{div}\rho_{s2}(\mathbf{v}_{s2}-\mathbf{v}_n)\operatorname{div}\mathbf{v}_n+\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{ni}}{\partial x_i}\right)^2 \tag{4.20}$$
$$+\rho D\frac{\partial\zeta}{\partial c}\left(\nabla c+\frac{k_T}{T}\nabla T+\frac{k_p}{p}\nabla p\right)^2+\varkappa\frac{(\nabla T)^2}{T}.$$

The coefficients ζ_i (i = 1, 2, ..., 9) have the meaning of second-viscosity coefficients. Because of the Onsager symmetry relations, only six of these are independent. The second-viscosity coefficient ζ_2 is specific for the solution of two superfluid liquids. It corresponds to the friction between the two superfluid components. The quantity η is the coefficient of first viscosity, which is essentially connected with the normal motion. Just as in the case of ordinary superfluidity, the coefficient that is analogous to the first viscosity does not appear in superfluid motions. Thus, there are in all ten independent kinetic coefficients for a solution of two superfluid liquids: one first viscosity, six second viscosity and one each of diffusion, thermal diffusion and thermal conductivity.

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1) Just as in [3,4], we shall ignore the possible anisotropy arising because of the anistropy of the gap in the ^3He spectrum, and study the case of isotropic superfluidity.

2) As is well known, the coefficient of thermal expansion in solutions of ^3He in superfluid ^4He is not small. This is because there is always a finite amount of the normal component in such solutions above the λ point of ^3He—all ^3He atoms are in the normal component. But in the case considered, both ^4He and ^3He are in the superfluid state and therefore, just as in the case of the pure superfluid ^4He, $\rho_n \ll \rho_s$ and the coefficient of thermal expansion of the solution of the two superfluid liquids should be a small quantity.

3) An error was made in the calculations in [3]–the last term in the expression for the energy flux (4.5) was omitted.

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