Angular correlation of photons in superradiance

I. V. Sokolov and E. D. Trifonov

Leningrad State University
Leningrad Pedagogical Institute
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We consider the photon angular correlation, which leads to an angular structure of the light flux in a superradiance pulse emitted by a completely excited extended many-atom system. The energy distribution for multiphoton rays is obtained in the case of an elongated volume. The angular divergence of a multiphoton beam, as a function of the number of photons in the beam, is estimated.

1. INTRODUCTION

Under conditions when the emission time of inverted polyatomic systems is shorter than the reciprocal level width, the spontaneous emission has a collective character and is called superradiance. The superradiance of extended polyatomic systems was investigated in a number of papers[1-9]. Dicke [1] has shown that the spatial coherence of the states of the atoms in superradiance of extended systems leads to an angular correlation of the photons. The probability of photon emission in a definite direction depends on the directions of the emission of the preceding photons and is maximal if all the photons have identical wave vectors. The explicit form of the correlation dependence, which is determined by the shape of the volume and can be obtained as a result of summation over the positions of the atoms, has not been explained to date. Ernst and Stehle[2], in a paper devoted to the generalization of the Wigner-Weisskopf method to include polyatomic systems, have proposed that the entire radiation in the case of total initial inversion is produced in the form of a single beam with diffraction angular dimensions. Averaging over the directions of the beam, Ernst and Stehle obtained a qualitatively correct distribution of the average light flux for the superradiance of a nonisotropic system. Nonetheless, this assumption is not justified, since the probability of emission of all photons in one beam is vanishingly small, as was demonstrated by one of us[9].

We report in this paper a detailed investigation of the photon structure of a superradiance pulse for a volume of elongated form. We obtained the distribution for the number of l-photon beams in an optical pulse. We refined the angular dimensions of the beam as a function of the number of photons in the beam. Interest in this question is due, in particular, to the experimental observation of a grainy structure in the radiation of pulsed gas lasers[10, 11].

2. PROBABILITY OF EMISSION OF l-PHOTON BEAMS

Consider a polyatomic system contained in an elongated rectangular volume of length L and transverse dimensions D, with $L, D \gg \lambda$ (λ is the radiation wavelength). The atoms with resonant frequency ω_0 are fully excited at the initial instant of time. For the employed method to be applicable it is necessary to satisfy the conditions $n_0\lambda^3 > 1$ and $L/c < \tau$, where $n_0 = NV^{-1}$ is the density of the number of atoms and τ is the time of the development of the superradiance pulse. According to[8], the photons are emitted in beams with random directions. The possibility of formation of intense beams in the direction $\mathbf{k}$ is determined by the collective radiation constant $\gamma(\mathbf{k}) = \kappa(\mathbf{k}) l_{\mathbf{k}} \lambda^2 V^{-1}$, where $\kappa(\mathbf{k})$ is the angular density of the radiation of one atom and $l_{\mathbf{k}}$ is the average length of the volume in the $\mathbf{k}$ direction. The average photon-number flux $n(\mathbf{k})$ (see[9]) is maximal for directions in which the volume has the largest dimensions[1)]

$$n(\mathbf{k}) = \frac{\varkappa(\mathbf{k})}{\gamma(\mathbf{k})}\{\exp(N\gamma(\mathbf{k})\tau) - 1\}, \qquad \int n(\mathbf{k})\, d\Omega = N, \tag{1}$$

where $d\Omega$ is the solid-angle element. It therefore suffices to take into account the radiation into the solid angle $\Omega_0 \sim D^2/L^2 \ll 1$. We assume that the diffraction angle λ/D is much smaller than the angular dimensions D/L of the light beam. The probability density of photon emission in the directions $\mathbf{k}_1 \ldots \mathbf{k}_N$ ($k_i = \omega_0$) as the time $t \to \infty$ takes the form[8]

$$W_{\mathbf{k}_1\ldots\mathbf{k}_N} = \sum_{p(\mathbf{k}_1\ldots\mathbf{k}_N)} \left\{ \prod_{s=1}^{N} \left[\varkappa + \Gamma \sum_{i=1}^{s-1} \bar{\delta}(\Omega_s - \Omega_i) \right] [\varkappa\Omega_0 + (s-1)\Gamma]^{-1} \right\}, \tag{2}$$

where $\kappa = \kappa(\mathbf{k})$ and $\Gamma = \gamma(\mathbf{k})$. The angular δ-function $\bar{\delta}(\Omega)$ has a dispersion λ/D. It can be verified that the quantity (2) is normalized to unity (under the condition that the momenta $\mathbf{k}_1, \ldots, \mathbf{k}_N$ belong to the solid angle Ω_0). The probability that l photons will be emitted into the solid angle ϵ is represented in the form

$$W_{l,\varepsilon} = \frac{1}{l!(N-l)!} \int_{\varepsilon} d\Omega_1 \ldots d\Omega_l \int_{\Omega_0 - \varepsilon} d\Omega_{l+1} \ldots d\Omega_N W_{\mathbf{k}_1\ldots\mathbf{k}_N}. \tag{3}$$

Assuming that $\Omega_0 \gg \epsilon \gg \epsilon^{(l)}$, where $\epsilon^{(l)}$ is the divergence of the l-photon beam, which is of the order of the diffraction divergence, and will be defined below, we assume that the variance of the functions $\bar{\delta}(\Omega)$ is equal to zero, $\bar{\delta}(\Omega) \to \delta(\Omega)$. The permuted terms in (2) are then equal to each other. Integrating with respect to $\Omega_{l+1} \ldots \Omega_N$, we obtain

$$W_{l,\varepsilon} = \frac{N!}{l!(N-l)!} \left(\prod_{s=N-l}^{N-1} (\varkappa\Omega_0 + s\Gamma) \right)^{-1} \times \int_{\varepsilon} d\Omega_1 \ldots d\Omega_l \left\{ \prod_{j=1}^{l} \left(\varkappa + \Gamma \sum_{i=1}^{j-1} \delta(\Omega_j - \Omega_i) \right) \right\}. \tag{4}$$

Obviously, the probability of spontaneous (i.e., incoherent) emission of l photons into a solid angle ϵ is proportional to ϵ^l. The contribution linear in ϵ to the probability $W_{l,\epsilon}$ is connected with the stimulated emission of l photons; the general emission direction remains arbitrary within the confines of the solid angle ϵ. We shall use the term l-photon beam for the process of stimulated emission of l photons into a solid angle ϵ with probability $W_{l,\epsilon}^{(r)}$ proportional to ϵ. It follows from (4) that

$$W_{l,\varepsilon}^{(r)} = \frac{N!}{l!(N-l)!} \Gamma^{l-1} \left[\prod_{s=N-l}^{N-1} (\varkappa\Omega_0 + s\Gamma) \right]^{-1} \int_{\varepsilon} d\Omega_1 \ldots d\Omega_l \cdot \varkappa \prod_{j=2}^{l} \left(\sum_{i=1}^{j-1} \delta(\Omega_j - \Omega_i) \right) = \frac{N!}{l(N-l)!} \varkappa\varepsilon\Gamma^{l-1} \left[\prod_{s=N-l}^{N-1} (\varkappa\Omega_0 + s\Gamma) \right]^{-1}. \tag{5}$$

It is easy to show that

$$\sum_{l=1}^{N} l W_{l,\epsilon}^{(r)} = \frac{\varepsilon}{\Omega_0} N. \qquad (6)$$

The quantity $W_{l,\Omega_0}^{(r)} = \epsilon^{-1}\Omega_0 W_{l,\epsilon}^{(r)}$ can be interpreted as the probability of emission of an l-photon beam into a solid angle Ω_0. For the energy contribution of l-photon beams $lW_{l,\Omega_0}^{(r)}$ it follows from the relation

$$(l+1) W_{l+1,\Omega_0}^{(r)} = l W_{l,\Omega_0}^{(r)} \left(1+\frac{\varkappa\Omega_0-\Gamma}{(N-l)\Gamma}\right)^{-1} \qquad (7)$$

(at $l \ll N$) that

$$l W_{l,\Omega_0}^{(r)} \approx W_{1,\Omega_0}^{(r)} e^{-l/\bar{l}}, \quad W_{1,\Omega_0}^{(r)} = \frac{\varkappa\Omega_0}{\Gamma}, \quad \bar{l} = \frac{N\Gamma}{\varkappa\Omega_0}. \qquad (8)$$

From (7) and (8) we can draw the following conclusions:

1) Starting with $l \gtrsim \bar{l}$, the energy contribution of the l-photon beams decreases exponentially with increasing number of photons.

2) At $l < \bar{l}$, the l-photon beams carry equal numbers of photons.

The total number of beams with less than l photons, where $l < \bar{l}$, is estimated at

$$\sum_{s=1}^{l} W_{s,\Omega_0}^{(r)} \approx \int_1^l \frac{\varkappa\Omega_0}{s\Gamma} e^{-s/\bar{l}}\, ds \sim \frac{\varkappa\Omega_0}{\Gamma} \ln l \qquad (9)$$

Therefore the number of beams in which the number of photons is of the same order at $l < \bar{l}$ is approximately equal to $(\kappa\Omega_0/\Gamma)\ln 10$. The total number of "bright" beams with more than l photons, where $l > \bar{l}$, can be estimated at

$$\sum_{s>l} W_{s,\Omega_0}^{(r)} \approx \int_l^\infty \frac{\varkappa\Omega_0}{s\Gamma} e^{-s/\bar{l}}\, ds \lesssim \frac{\bar{l}}{l}\frac{\varkappa\Omega_0}{\Gamma} e^{-l/\bar{l}}. \qquad (10)$$

At $l > \bar{l}$, the number of l-photon beams in the pulse decreases rapidly, so that $\bar{l}$ determines the order of magnitude of the maximum beam intensity.

3. ANGULAR DIVERGENCE OF BEAM

The distribution of the photons in an l-photon beam is determined by the integrand in formula (5). Since the angular divergence of the beam is connected with the uncertainty of the emitted photon momenta, it is necessary to take into account the finite variance of the functions $\tilde{\delta}(\Omega)$. We fix the wave vector $\mathbf{k}_1$ of the first photon and direct the axis 3 of the coordinate system along $\mathbf{k}_1$. The angular deviations of the photons $\mathbf{k}_i$ in the beam will then be determined by the momentum components $k_i^{(1)}$ and $k_i^{(2)}$ ($k_i^{(3)} = \omega_0$). Symmetrizing the distribution of the photons in the beam $W_{\mathbf{k}_1\ldots\mathbf{k}_l}^{(r)}$ with respect to the photon momenta, we obtain

$$W_{\mathbf{k}_1\ldots\mathbf{k}_l}^{(r)} = \frac{1}{(l-1)!} \sum_{p(\mathbf{k}_1\ldots\mathbf{k}_l)} \left\{ \delta(\mathbf{k}_1) \prod_{j=2}^{l} \left(\sum_{i=1}^{j-1} \tilde{\delta}(\mathbf{k}_j-\mathbf{k}_i) \right) \right\},$$
$$\mathbf{k}_i = \{k_i^{(1)}, k_i^{(2)}\}. \qquad (11)$$

The constant factor is chosen such that the probability of emission of l photons in the beam is equal to unity

$$\frac{1}{l!}\int d\mathbf{k}_1\ldots d\mathbf{k}_l W_{\mathbf{k}_1\ldots\mathbf{k}_l}^{(r)} = 1. \qquad (12)$$

In the case of a volume of elongated form, when the photons are emitted in a direction close to the direction of the larger edge of the volume L, the momentum δ function $\delta(\mathbf{k})$ takes the form

$$\tilde{\delta}(\mathbf{k}) = \frac{1}{(2\pi D)^2}\int_{D^2} \exp[i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)]\, d\mathbf{r}_1\, d\mathbf{r}_2, \qquad (13)$$

where the integration is carried out over the cross section of the volume, and $\mathbf{r}_i = \{r_i^{(1)}, r_i^{(2)}\}$. The photon-number density $f_l(\mathbf{k})$ in an l-photon beam is defined by

$$f_l(\mathbf{k}) = \frac{1}{l!}\int d\mathbf{k}_1\ldots d\mathbf{k}_l \left(\sum_{i=1}^{l}\delta(\mathbf{k}-\mathbf{k}_i)\right) W_{\mathbf{k}_1\ldots\mathbf{k}_l}^{(r)}. \qquad (14)$$

It is easy to verify that satisfaction of the normalization conditions

$$\int f_l(\mathbf{k})\, d\mathbf{k} = \int f_l(k^{(1)}, k^{(2)})\, dk^{(1)}\, dk^{(2)} = l. \qquad (15)$$

It follows from (11) and (14) that the density of the number of photons in the beam satisfies the recurrence relation

$$f_{l+1}(\mathbf{k}) - f_l(\mathbf{k}) = \frac{1}{l}\int f_l(\mathbf{k}')\tilde{\delta}(\mathbf{k}-\mathbf{k}')\, d\mathbf{k}'. \qquad (16)$$

To estimate the divergence of the beam at large photon numbers, $l \gg 1$, we change over from the difference equation (16) to the differential equation

$$\frac{\partial}{\partial l} f_l(\mathbf{k}) = \frac{1}{l}\int f_l(\mathbf{k}')\tilde{\delta}(\mathbf{k}-\mathbf{k}')\, d\mathbf{k}'. \qquad (17)$$

Using a Fourier transformation, we obtain a solution of this equation

$$f_l(\mathbf{k}) = \frac{1}{(2\pi)^2}\int \exp(-i\mathbf{k}\mathbf{s} + 2\pi\tilde{\delta}(\mathbf{s})\ln l)\, d\mathbf{s}, \qquad (18)$$

where $\mathbf{s} = \{s^{(1)}, s^{(2)}\}$; $\tilde{\delta}(\mathbf{s})$ is the Fourier transform of $\tilde{\delta}(\mathbf{k})$:

$$\tilde{\delta}(\mathbf{s}) = \delta'(s^{(1)})\delta'(s^{(2)}), \quad \delta'(s) = \begin{cases} (2\pi)^{-1/2}(1+s/D), & 0>s>-D \\ 0, & s\geqslant D,\ s\leqslant -D. \\ (2\pi)^{-1/2}(1-s/D), & D>s>0 \end{cases} \qquad (19)$$

To estimate the angular dimension of the light spot it suffices to consider the simultaneous density of the number of photons in the beam

$$f_l(k^{(1)}) = \int f_l(k^{(1)}, k^{(2)})\, dk^{(2)}. \qquad (20)$$

At $l \gg 1$ we obtain

$$f_l(k^{(1)}) \approx \frac{l}{\pi}\frac{D^{-1}\ln l}{(k^{(1)})^2 + (D^{-1}\ln l)^2}. \qquad (21)$$

Thus, the main intensity flux in an l-photon beam propogates within the confines of the solid angle $\epsilon^{(l)} \sim (\lambda D^{-1}\ln l)^2$. This angle determines the characteristic divergence of the beam. From the condition $\bar{l} \sim N\lambda^2 L^2 D^{-4}$ assumed by us, it follows that $\Omega_0 \gg \epsilon^l$, i.e., it is possible to obtain angular resolution of light spots in the cross section of the light beam.

We note that a similar method can be used to investigate also the angular divergence of superradiance of atoms excited by a coherent pulse of a plane wave. If the relative level of the excitation of the atoms N_{exc}/N is close to unity, then the emission solid angle increases in comparison with the diffraction relation of the type (13) (cf.[4]) by a factor $\{\ln[N/(N-N_{exc})]\}^2$. This can be attributed to a certain "dephasing" of the initially produced state of the atoms in the course of development of the superradiance pulse.

4. CONCLUSION

Owing to the angular correlation of the photons in superradiance of extended polyatomic systems, the brightness distribution in the cross section of the light flux is not uniform. The radiation is produced in the form of multiphoton beams in random directions. If the volume has an elongated form, then the most probable is the production of beams in longitudinal directions for which the dimension of the volume is maximal. This leads to an angular dependence of the average light flux on the shape of the volume. A typical characteristic of the photon structure of superradiance of an elongated volume $\bar{l} \sim N\lambda^2 L^2 D^{-4}$ have low probability. Beams with a small number of photons (i.e., smaller than $\bar{l}$), make equal energy contributions and produced the brightness background. The most intense beams in the light flux, which give rise to the grainy structure, are made up of $\sim \bar{l}$ photons. Assuming by way of estimate L = 10 cm, D = 0.3 cm, $\lambda = 10^{-4}$ cm, and $NV^{-1} = n_0 = 10^{15}$ cm^{-3}, we find that the beam with the largest intensity has $l \sim \bar{l} = 10^{11}$ photons. The number of intense beams, according to (10), does not exceed $D^4 L^{-2} \lambda^{-2} \sim 10^4$. The angular dimension of an l-photon beam is of the order of $\lambda D^{-1} \ln l$, i.e., it is determined mainly by the diffraction divergence and increases slowly with increasing number of photons. This increase is due to the uncertainty of the momentum of each succeeding photon in the beam,an uncertainty caused by the finite dimension of the volume.

Superradiance, meaning also angular correlation of the photons in extended systems, can appear if the duration of the superradiance pulse is shorter than the transverse relaxation time. One can expect this to be satisfied for pulsed gas lasers at low pressures ($p \lesssim 1$ Torr). Under optimal conditions, the reciprocal emission time agrees well with the reciprocal radiation-coherence time and can exceed the Doppler width of the spectrum[10,11]. It was observed that the emission has a grainy structure and that spatial coherence takes place only within the limits of each grain, while the positions of the grains are random. When the temporal coherence condition is satisfied, this structure can be regarded as a consequence of the angular correlation of the photons in the superradiance. Each grain constitutes the trail of a multiphoton beam produced by enhancement of the initial spontaneous photon. We note that the experimental angular dimensions of the grains agree with our estimate (21) of the angular divergence of the beam. Additional arguments might be obtained on the basis of a more detailed experimental investigation of the grainy structure (for example the statistics of the grain brightnesses), and also by investigating the probability of the transverse relaxation in collisions with electrons or atoms in a gas discharge.

[1)]We use a system of units in which $\hbar = 1$ and $c = 1$.

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