Electron scattering by a vortex lattice in a mixed-state superconductor

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The effect of an ideal vortex lattice on the motion of normal electrons is considered near the critical field H_{c2} for the transition to the mixed state. It is shown that the presence of vortices leads to scattering of electrons such that the quasiparticle excitation of the electron branch changes into hole excitation and vice versa. Stationary normal electron states in a magnetic field are described in the Landau representation. The matrix elements of the pairing potential between stationary electron states are determined by perturbation-theory methods, and the probability W of the electron-hole transition in the mixed state near H_{c2} is calculated. The obtained value of W is employed to calculate the changes produced in the thermal conductivity and in the thermal Hall angle on going from the normal to mixed state. The results of the calculations are in agreement with experimental data for pure niobium.

Type-II superconductors, namely pure niobium single crystals, with electron mean free path $l \gg \xi$ (ξ is the coherence length), have made their appearance in recent years. In these superconductors, in the mixed state, the electrons can cover distances many times larger than the distances between the vortices. It is of interest to consider the influence of the vortex structure near the critical field H_{c2} of the transition to the mixed state on the motion of the normal electrons.

The presence of vortices leads to additional scattering of the electrons. This scattering has a peculiar character. The point is that when scattered by the vortex lattice, the quasiparticle excitation of the electron branch goes over to the hole branch and vice versa. Such transitions are analogous in their nature to the reflection, considered by Andreev [1], of electrons from the interface between a normal metal and a purely superconducting phase. For type-II superconductors, the peculiar character of the electron scattering by vortices was first noticed by Vinen in a discussion of changes of the thermal conductivity in the mixed state [2]. In a recent article [3] dealing with the calculation of the thermal conductivity near H_{c2}, Houghton and Maki also note that the calculation reflects the specifics of the interaction of the electrons with the vortex structure. In [3], however, the physical picture is obscured by the formalism of the calculation, which is carried out with the aid of temperature Green's functions.

This paper constitutes an attempt to describe directly, by simpler methods, the scattering of electrons by a vortex lattice near H_{c2}. Using the experimental observations of vortices with the aid of the decoration technique [4] or by neutron diffraction [5], the vortex lattice can be regarded as ideal over distances comparable with the electron mean free path. In the calculation, the vortex structure was assumed to be given. Its description near the critical temperature T_c was first presented by Abrikosov [6] within the framework of the Ginzburg-Landau theory. At the present time, there are known successful attempts to extend the Ginzburg-Landau theory to the entire temperature range $T < T_c$ (see, e.g., [7]). We therefore use here Abrikosov's description of the vortex structure and assume that is always valid at $T < T_c$.

The stationary state of the normal electrons in the magnetic field are described in the Landau representation. Perturbation-theory methods were used to determine the matrix elements of the pairing potential between the stationary states of the electrons and to calculate the electron-hole transition probability W in the mixed state near H_{c2}. The obtained value of W was used to calculate the changes in the thermal conductivity and the thermal Hall angle on going from the normal to the mixed state. The results of the calculations agreed with the experimental data for pure niobium [8,9].

ELECTRON-HOLE SCATTERING PROBABILITY

The excitations in a superconductor, without allowance for the spin, are described by the Bogolyubov-Gor'kov equations

$$i\hbar\frac{\partial\psi}{\partial t}=(\hat{\mathscr{H}}_0+\hat{\mathscr{H}}_1)\psi$$

For a two-component wave function $\psi = \binom{u}{v}$ (see, e.g., [1,10]). For stationary states we have

$$(\hat{\mathscr{H}}_0+\hat{\mathscr{H}}_1)\psi_\nu=\varepsilon_\nu\psi_\nu. \quad (1)$$

Here $\epsilon_\nu > 0$ is the excitation energy, u is the electron amplitude, v is the hole amplitude, the index ν classifies the stationary states, $\hat{\mathscr{H}}_0$ is the Hamiltonian for the metal in the normal state,

$$\hat{\mathscr{H}}_1=\begin{pmatrix}0 & \Delta\\ \Delta^* & 0\end{pmatrix}$$

and Δ is the pairing potential.

We shall consider henceforth a gas of free electrons for which the Hamiltonian $\hat{\mathscr{H}}_0$ is given by

$$\hat{\mathscr{H}}_0=\begin{pmatrix}\hat{T} & 0\\ 0 & -\hat{T}^*\end{pmatrix}\quad \hat{T}=\frac{1}{2m}\left(i\hbar\nabla+\frac{e}{c}\mathbf{A}\right)^2-E_f.$$

Here $e = -4.8 \times 10^{-2}$ cgs esu is the electron charge, m is its mass, c is the speed of light, **A** is the vector potential of the magnetic field, and E_f is the Fermi level. The fundamental character of the results that follow is unchanged when account is taken of effects occurring in a real metal. As will be shown below, allowance for the spin likewise does not change the results significantly.

1. In the normal state at $\Delta = 0$, the system of equations (1) breaks up into two independent equations describing the electronic and hole branches of the excitations. In a constant magnetic field **H** directed along the z axis, in a gauge $A_x = -Hy$, $A_y = 0$, and $A_z = 0$, we write down the solutions in the Landau representation in the

quasi-classical approximation. For electrons, the index includes the momentum along the field $p_z = \hbar k_z$, the quantum number n_1 that determines the excitation energy

$$\varepsilon_\nu = (n_1 + 1/2)\hbar\omega + p_z^2/2m - E_f,$$

and the degeneracy parameter k_1, which is analogous to the wave number.

To simplify the notation, we introduce a parameter with dimension of length and a cyclotron frequency ω, such that $2\xi^2 = \hbar/m\omega = c\hbar/|e|H$. In addition, we introduce the functions

$$\rho_1(w) = (2n_1 - w^2)^{1/2},$$
$$\Phi_1(w) = \int_0^w \rho_1(w')\,dw' - n_1\pi/2. \quad (2)$$

The wave function of the electron then takes the form

$$u_\nu = \frac{\cos[\Phi_1(\zeta_1)]\exp[i(k_1x + k_zz)]}{(2\xi^2)^{1/4}[2\pi^3\rho_1(\zeta_1)]^{1/2}}, \quad 2n_1 - \zeta_1^2 \gg 1; \quad (3a)$$
$$u_\nu = \frac{(n_1!)^{1/2}\zeta_1^{n_1}\exp(-\zeta_1^2/2)}{2\pi(2\pi\xi^2)^{1/4}}, \quad \zeta_1^2 - 2n_1 \gg 1; \quad (4a)$$

here $\zeta_1 = y/2^{1/2}\xi - 2^{1/2}\xi k_1$.

For holes, the state index $\mu = \{k_z', n_2, k_2\}$, the excitation energy

$$\varepsilon_\mu = E_f - p_z'^2/2m - (n_2 + 1/2)\hbar\omega$$

and the functions $\rho_2(w)$ and $\Phi_2(w)$ are defined in analogy with (2), with the indices interchanged. Then

$$v_\mu = \frac{\cos[\Phi_2(\zeta_2)]\exp[i(k_2x + k_z'z)]}{(2\xi^2)^{1/4}[2\pi^3\rho_2(\zeta_2)]^{1/2}}, \quad 2n_2 - \zeta_2^2 \gg 1; \quad (3b)$$
$$v_\mu = \frac{(n_2!)^{1/2}\zeta_2^{n_2}\exp(-\zeta_2^2/2)}{2\pi(2\pi\xi^2)^{1/4}}, \quad \zeta_2^2 - 2n_2 \gg 1, \quad (4b)$$

where $\zeta_2 = y/2^{1/2}\xi + 2^{1/2}\xi k_2$.

The functions are normalized along the x and z axes to δ-functions of k_z, k_1, and k_2. The condition n_1, n_2 $\gg 1$ for applicability of the quasiclassical approximation is certainly satisfied for the overwhelming majority of the electrons in the metal in the magnetic fields of interest to us. Formulas (3) and (4) then fail to describe the solutions only in the relatively narrow region

$$|2n_i - \zeta_i^2|^{1/2} \approx 1, \quad i = 1, 2,$$

when it is necessary to use the exact solution in terms of Hermite polynomials. This region of states will henceforth be left out, assuming that its influence on the final results is small.

2. In the mixed state at $H \lesssim H_{c2}$, the pairing potential Δ gives rise to transitions between the stationary states of the normal metal, i.e., transitions between the electronic and hole branches of the excitations. To find the transition probability, we calculate the perturbation matrix element

$$\mathscr{H}_{\mu\nu} = \langle\mu|\hat{\mathscr{H}}_1|\nu\rangle = \int (u_\mu^*\Delta v_\nu + v_\mu^*\Delta^* u_\nu)\,dr.$$

Obviously $\mathscr{H}_{\mu\nu} = \mathscr{H}^*_{\nu\mu}$.

We are interested in transitions between "pure" stationary states. Let, for example, $u_\nu \neq 0$ and $v_\nu = 0$ in the initial state ν ("electron"), and $u_\mu = 0$ and $v_\mu \neq 0$ in the final state μ ("hole"). Then

$$\mathscr{H}_{\mu\nu} = \int v_\mu^*\Delta^* u_\nu\,dr.$$

To determine the pairing potential we assume that Δ is proportional in the usual manner to the Euler parameter. By starting from the form of the solution of the Ginzburg-Landau equations [6-11], we can assume that in our gauge

$$\Delta = C\sum_{n=-\infty} \exp[-t_n^2 + in(qx + n\pi/2)],$$

where

$$q = \pi^{1/2}3^{1/4}/\xi, \quad t_n = (y - n\pi^{1/2}3^{1/4}\xi)/2^{1/2}\xi,$$

n is an integer, and C is a constant such that $|C|^2 \propto H_{c2} - H$. Its connection with the experimentally determined magnetization will be written out below. We note that for $H \approx H_{c2}$ we can neglect the magnetization and assume the average induction in the metal to coincide with the applied field. The introduced quantity ξ is then none other than the coherence length in the superconductor, defined by the condition $2\pi\xi^2 H_{c2} = \Phi_0$ ($\Phi_0 = ch/2|e|$ is the flux quantum).

Reversing the order of summation and integration, we obtain for the matrix element

$$\mathscr{H}_{\mu\nu} = \frac{2^{1/4}C}{\pi\xi}\sum_n \exp\left(-\frac{in^2\pi}{2}\right) J_n \delta(k_1 - k_2 - nq)\,\delta(k_z - k_z'),$$
$$J_n = \int_{-\infty}^{+\infty} \frac{\cos[\Phi_1(\zeta_1)]\cos[\Phi_2(\zeta_2)]\exp(-t_n^2)}{[\rho_1(\zeta_1)\rho_2(\zeta_2)]^{1/2}}\,dy. \quad (5)$$

In view of the presence of exponentially decreasing factors in the function $\Delta(y)$ and in the functions u_ν and v_μ in formulas (4) the infinite series for $\mathscr{H}_{\mu\nu}$ actually is replaced by a sum with a limited number of terms. It can be shown that in (5) the maximum number $n_{max} \approx (2n_i)^{1/2}$ is of the order of the ratio of the Larmor radius in the field H_{c2} to the coherence length ξ. All the remaining terms of the series are vanishingly small. The same circumstance, incidentally, justifies the term-by-term integration of the series. We note once more that in formula (5) we have discarded a relatively small number of terms for which the conditions that ensure the description of the solutions by the quasiclassical formulas are violated. Calculation of these terms is cumbersome and unjustified, since it would not change the final result significantly.

To calculate J_n, we expand the integrands in powers of t_n and make use of the fact that the main contribution to the integral is made by the region $t_n \lesssim 1$. At the same time, the quasiclassical approximation conditions enable us to simplify the expressions. The integration yields

$$J_n = \frac{\xi(\pi/2)^{1/2}\exp[-\beta^2/4(1+\alpha^2)]}{[\rho_1(\zeta_n)\rho_2(\zeta_n)]^{1/2}(\alpha^2+1)^{1/4}}\cos\left[\gamma - \frac{\alpha\beta^2}{4(1+\alpha^2)} - \frac{\operatorname{arctg}\alpha}{2}\right],$$

where

$$\zeta_n = 3^{1/4}(\pi/2)^{1/2}n - 2^{1/2}\xi k_1,$$
$$\alpha = \frac{\zeta_n}{2}\left[\frac{1}{\rho_1(\zeta_n)} + \frac{1}{\rho_2(\zeta_n)}\right], \quad \beta = \rho_1(\zeta_n) - \rho_2(\zeta_n),$$
$$\gamma = \int_0^{\zeta_n} [\rho_1(w) + \rho_2(w)]\,dw - \frac{n_1 - n_2}{2}\pi.$$

We simplify the expressions by using the smallness of the excitation energy:

$$\varepsilon_\nu \ll E_\perp \equiv E_f - p_z^2/2m \equiv (n_0 + 1/2)\hbar\omega.$$

We neglect here the small number of electrons with $E_\perp \approx \epsilon_\nu$. We put $\epsilon_\nu = l_1\hbar\omega$, and $\epsilon_\mu = l_2\hbar\omega$, where l_1 and $l_2 \ll n_0$. To simplify the notation we introduce an angle ϑ_n such that

$$\mathrm{tg}\,\vartheta_n=\zeta_n/(2n_0-\zeta_n^2)^{1/2}.$$

Then, after expanding the functions in series we obtain at angles ϑ_n that are not too close to $\pi/2$, in accord with the conditions of formulas (3),

$$\mathscr{H}_{\mu\nu}=\frac{C}{(2\pi n_0)^{1/2}}\exp\left[-\frac{(l_1+l_2)^2}{8n_0}\right]\cdot\times\sum_n\frac{\exp(-in^2\pi/2)\cos\gamma'}{(\cos\vartheta_n)^{1/2}}\delta(k_1-k_2-nq)\delta(k_z-k_z'),\qquad(6)$$

$$\gamma'=(2n_0+{}^1/_2+l_1-l_2)\vartheta_n+n_0\sin 2\vartheta_n-{}^1/_2(l_1-l_2)\pi.$$

In view of the condition l_1, $l_2 \ll n_0$ we shall henceforth put $\exp[-(l_1+l_2)^2/8n_0]\approx 1$.

3. The probability W(t) of the transition from the initial state ν within a time t is determined by the sum of the squares of the moduli of the amplitudes

$$\sum_{\mu\neq\nu}|\psi_\mu(t)|^2$$

of all the states μ at the instant t under the condition $\psi_{\mu\neq\nu}(t=0)=0$ and $\psi_\nu(t=0)=1$ (see, e.g., [12]). Near H_{c2}, so long as $\tau|\mathscr{H}_{\mu\nu}|\ll\hbar$, i.e., at

$$|C|\ll(E_\perp/\hbar\omega)^{1/2}\hbar/\tau,$$

where τ is the lifetime of the excitation (the relaxation time), this expression reduces to a sum of the squares of the matrix elements over all the final states

$$W(t)=\sum_\mu|\psi_\mu(t)|^2\approx\sum_\mu 4|\mathscr{H}_{\mu\nu}|^2\frac{\sin^2[(\varepsilon_\mu-\varepsilon_\nu)t/2\hbar]}{(\varepsilon_\mu-\varepsilon_\nu)^2}.$$

It is clear from (6) that the transitions proceed with conservation of the momentum along the field $p_z'=p_z$. We carry out the summation in the following sequence: first within the Landau level over k_2, and then over all the levels of the final states $\epsilon_\mu=\epsilon_\nu+l\hbar\omega$, where l takes on all the integer values including zero. For the free electrons, the g-factor is equal to 2 and allowance for the spin leads only to a redefinition of l, without influencing the transition probability, which is given by the formula

$$W(t)=\frac{2|C|^2}{\pi n_0\hbar^2\omega^2}\sum_{l=-\infty}^{+\infty}\left(\sum_n\frac{\cos^2\gamma'}{\cos\vartheta_n}\right)\frac{\sin^2(l\omega t/2)}{l^2}.$$

The factor $\cos\gamma'$ oscillates rapidly in coherent manner, and we can put in the subsequent expressions $\langle\cos^2\gamma'\rangle=1/2$.

The sum over n, where the summation limit is $n_{max}\gg 1$ can be replaced with sufficient accuracy by an integral, so that

$$\sum_n\cos^{-1}\vartheta_n\approx 2\pi^{1/2}3^{-1/4}n_0^{1/2}.$$

From this we obtain

$$W(t)=\frac{2|C|^2}{3^{1/4}\pi^{1/2}n_0^{1/2}\hbar^2\omega^2}\sum_{l=-\infty}^{\pm\infty}\frac{\sin^2(l\omega t/2)}{l^2}=\frac{2|C|^2}{3^{1/4}\pi^{1/2}\hbar\omega(E_\perp\hbar\omega)^{1/2}}\sum_{l=-\infty}^{\pm\infty}\frac{\sin^2(l\omega t/2)}{l^2}.$$

The sum of the series in this formula is expressed in terms of a Bernoulli polynomial [13]

$$B_2(x)=x^2-x+\frac{1}{6}=\frac{1}{\pi^2}\sum_{l=1}^{\infty}\frac{\cos 2\pi lx}{l^2},\quad 0\leqslant x\leqslant 1.$$

Hence

$$\sum_{l=-\infty}^{\pm\infty}l^{-2}\sin^2\frac{l\omega t}{2}=\frac{\pi\omega t}{2},\quad \omega t\leqslant 2\pi.$$

Thus, the probability W of the transition from one branch to the other per unit time, for excitations with energy $\epsilon\ll(\hbar\omega E_\perp)^{1/2}$ for a sufficiently short excitation lifetime $\tau\leq 2/\pi$ is equal to

$$W=\pi^{1/2}3^{-1/4}|C|^2/\hbar(\hbar\omega E_\perp)^{1/2}.$$

We now determine the coefficient |C|. If we introduce the dimensionless quantity $\widetilde{C}=|C|/\Delta_T$, where Δ_T is the value of the gap in a zero field at the given temperature, then $\widetilde{C}$ can be easily related with the mean squared modulus of the order parameter $\langle f_0^2\rangle=|\widetilde{C}|^2\cdot 3^{-1/4}$. In turn, $\langle f_0^2\rangle$ is connected with the magnetization near H_{c2} [10]:

$$\langle f_0^2\rangle=\frac{2\varkappa_2^2}{(2\varkappa_2^2-1)\beta'}\frac{H_{c2}-H}{H_{c2}}.$$

In this formula, $\kappa_2=\kappa_2(T)$ is the second Maki parameter and $\beta'=1.16$ for a triangular vortex lattice. The values of κ_2 are determined from the experimental magnetization curves M from the known relations [10]. Substituting these expressions in the formula for W, we obtain ultimately

$$W=\frac{\pi^{1/2}|\Delta_T|^2}{\hbar(\hbar\omega E_\perp)^{1/2}}\frac{2\varkappa_2^2}{1{,}16(2\varkappa_2^2-1)}\frac{H_{c2}-H}{H_{c2}}.\qquad(7)$$

Let us make a few remarks. Naturally, the probabilities of the electron-hole and hole-electron transitions are the same. Further, W does not depend on the degeneracy parameter k_1 or on the energy for usual thermal excitations. For excitations with very high energy, the probability W decreases exponentially because of the appearance of the factor (see formula (6))

$$\exp(-\varepsilon^2/2\hbar\omega E_\perp).$$

Finally, let us estimate the order of magnitude of W from the known data. For pure niobium [11]

$$2\varkappa_2^2/(2\varkappa_2^2-1)\beta'\approx 1,\quad E_\perp\approx E_f\approx 2\times 10^{-12}\ \text{erg}.$$

Near T = 0°K, the field is $H_{c2}\approx 4000$ Oe, and assuming that the effective electron mass is $m^*\approx m$, we have $\omega\approx 5\times 10^{10}\ \mathrm{sec}^{-1}$ and $\hbar\omega\approx 5\times 10^{-17}$ erg. At absolute zero the gap is $\Delta_0\approx 2\times 10^{-15}$ erg. Substituting these values in (7), we obtain $W\approx 10^{12}(H_{c2}-H)/H_{c2}\ [\mathrm{sec}^{-1}]$. At $(H_{c2}-H)/H_{c2}=1\%$ we have $W\approx 10^{10}$ sec. Comparison with the cyclotron frequency shows that even near H_{c2} the transition from one branch of excitations to the other can occur within a time shorter than time of one revolution.

EXCITATION TRAJECTORIES

For a better insight in the electron-hole transition in the vortex structure, it is useful to consider localized states. Localization of quasiparticles and the description of their trajectories are effected in the usual manner by formation of suitable wave packets. For example, by direct integration it is easy to show that a wave packet of pure electronic states in a constant magnetic field

$$\psi(x,y)=\int_{-\infty}^{+\infty}\exp(-{}^1/_4k_1^2\delta^2)u_\nu dk_1$$

describes a classical circular trajectory of radius $R_1=2(n_1+{}^1/_2)^{1/2}\xi$ with center at the origin. This follows from the fact that $|\psi(x,y)|$ reaches a maximum precisely on this circle, with $|\psi(x,y)|$ noticeably different from zero in a ring of width $\delta\ll R_1$ (for more details see [14]).

We now examine the states into which the components of the initial wave packet go over as a result of scattering by a vortex lattice in a mixed state near H_{c2}. According to first-order perturbation theory [12], a wave packet of scattered states μ is given by

$$\psi^{(1)}(x, y) \propto \sum_{\mu} \left[\int \exp\left(-\frac{1}{4} k_1{}^2\delta^2 \right) \mathscr{H}_{\mu\nu} v_\mu dk_1 \right] \frac{\sin[(\varepsilon_\mu - \varepsilon_\nu) t/2\hbar]}{\varepsilon_\mu - \varepsilon_\nu}.$$

After a time interval $t \sim 1/\omega$ following the formation of the initial packet the states having an overwhelmingly large amplitude in the scattered packet $\psi^{(1)}(x, y)$ are those with energy $\epsilon_\mu = \epsilon_\nu$. We therefore confine ourselves for simplicity to the consideration of transitions with energy conservation. Then

$$\psi^{(1)}(x, y) \propto \int_{-\infty}^{+\infty}\int \exp(-{}^1/_4 k_1{}^2\delta^2) \mathscr{H}_{\mu\nu} v_\mu \, dk_1 \, dk_2.$$

We do not consider here localization along the z axis, which is carried out in the known manner.

Using the expressions for the matrix element $\mathscr{H}_{\mu\nu}$, we find for low-energy particles

$$\psi^{(1)}(x, y) \propto \sum_n \exp\left[-in\left(qx + \frac{n\pi}{2} \right) \right]$$

$$\times \int_{-\infty}^{+\infty} \frac{\exp(-k_1{}^2\delta^2/4 + ik_1x)}{[\rho_0(\zeta_n)\rho_2(\zeta)]^{1/2}} \cos\left[2\int_0^{\zeta_n} \rho_0(w)\,dw - l_1\pi - \frac{1}{2}\arcsin\frac{\zeta_n}{(2n_0)^{1/2}} \right]$$

$$\times \cos[\Phi_2(\zeta)]\,dk_1,$$

$$\zeta = y2^{-1/2}\xi^{-1} + 2^{1/2}\xi k_1 - (2\pi)^{1/2}3^{1/4}n.$$

The function $\rho_0(w)$ is determined from formula (2) with the subscript replaced. The exponential factor ensures a rapid decrease of the integrand function if the condition $k_1\delta \lesssim 1$ is violated.

Assuming $\xi \ll \delta \ll R_1$, we can expand all the integrands in terms of the small quantity $k_1\xi \ll 1$, retaining only first-order terms. As a result we get

$$\psi^{(1)}(x, y) \propto \sum_n \sum_{\pm} \exp(i\varphi_\pm) \int_{-\infty}^{+\infty} \exp\{-{}^1/_4 k_1{}^2\delta^2$$

$$+ i2^{1/2}\xi k_1 [x/2^{1/2}\xi \pm 2\rho_0(\zeta_{n0}) \mp \rho_2(\zeta_{ny})]\} dk_1;$$

$$\zeta_{n0} = (\pi/2)^{1/2}3^{1/4}n, \quad \zeta_{ny} = y/2^{1/2}\xi - (2\pi)^{1/2}3^{1/4}n.$$

In this formula, $\Sigma_\pm$ denotes the sum of four terms corresponding to four independent sets of pairs of signs in the exponentials, while $\varphi_\pm$ are certain phase factors.

The integral under the summation sign is proportional to

$$\exp\{-(2\xi^2/\delta^2)[x/2^{1/2}\xi \pm 2\rho_0(\zeta_{n0}) \mp \rho_2(\zeta_{ny})]^2\}.$$

Thus, $|\psi^{(1)}(x, y)|$ is exponentially small everywhere except in a strip of width δ near the circles

$$[x/2^{1/2}\xi \pm 2(2n_0 - n^2\pi 3^{1/4}/2)^{1/2}]^2 + [y/2^{1/2}\xi - n(2\pi)^{1/2}3^{1/4}]^2 = 2n_2.$$

Each such circle, specified by the number n, is the geometric locus of the relative maximum of $|\psi^{(1)}(x, y)|$ and the coherent change of the phase of $\psi^{(1)}(x, y)$, i.e., it corresponds to the classical trajectory of the scattered particle. Introducing the values of the Larmor radii $R_0 = 2(n_0 + 1/2)^{1/2}\xi$ and $R_2 = 2(n_2 + 1/2)^{1/2}\xi$, we can express the equation for the aggregate of the trajectories of the scattered particles in the form

$$[x \pm 2(R_0{}^2 - \pi 3^{1/2}n^2\xi^2)^{1/2}]^2 + (y - 2\pi^{1/2}3^{1/4}n\xi)^2 = R_2{}^2.$$

The limit imposed on n by the condition $\pi 3^{1/2}n^2 \lesssim 4n_2$ ensures a real value of the radical in the first bracket. The coordinates (x_0, y_0) of the centers of the trajectory satisfy the equation $x_0^2 + y_0^2 = 4R_0^2$, i.e., they lie on a circle of radius $2R_0$ (see Fig. 1). The wave packets of the scattered states overlap to a considerable degree and fill densely a broad ring around the trajectory of the initial packet.

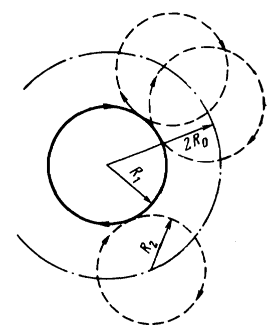

FIG. 1. Schematic representation of the trajectories of quasiparticles that go over into one another upon scattering by a vortex lattice.

In the stationary state in a constant magnetic field, the electron, strictly speaking, has no definite momentum in a plane perpendicular to the field. Nonetheless, in the quasiclassical approximation, as is well known, it is possible to introduce a momentum **p**, which is a sufficiently good quantum number and whose time variation, according to the usual classical equations, describes the motion of the particle. The Larmor radius R is connected with the component $p_\perp$ perpendicular to the field by the relation $R = cp_\perp/|e|H$. The wave-packet geometry described above corresponds to a quasiparticle scattering process in which the quasiparticle momentum is practically conserved. The longitudinal component is conserved rigorously, and the perpendicular momentum component of an excitation with energy ϵ changes by an amount $\delta p_\perp = |e|Hc^{-1}|R_1 - R_2| \approx 2\epsilon/v_f$, where v_f is the Fermi velocity of the particle, so that $\delta p/p \approx 2\epsilon/pv_f \approx \epsilon/E_f \ll 1$. A transition from one branch of excitations to another with conservation of the momentum **p** means a change in the sign of the velocity vector **v**. After being scattered by the vortex lattice, the quasiparticle moves along a tangent to the initial trajectory in the opposite direction. The situation is perfectly analogous to reflection of electrons from the boundary between a pure superconducting phase and a normal phase [1]. The difference is that in the mixed state this transition takes place on any section of the trajectory at any point inside the metal, and the transition probability depends strongly on the value of the magnetic field.

It is curious that the change of $p_\perp$ always occurs along the direction of motion of the quasiparticle. This means that the vortex lattice does not have a definite momentum. Indeed, the vortex lattice is described in fact by the same Ginzburg-Landau equations as the electron in the magnetic field. In the limit of small quantum numbers, there is no definite momentum in this case.

TRANSPORT PHENOMENA IN MIXED STATES

To consider transport phenomena, it is convenient to use a quasiclassical representation in which the excitations are described by an energy ϵ and a quasimomentum **p**. It is also convenient to introduce the quantity $\tilde{\epsilon} = (\mathbf{p} - \mathbf{p}_f) \cdot \mathbf{v}_f$, where $\mathbf{p}_f$ is the Fermi momentum and $\mathbf{v}_f$ is the velocity of the excitations on the Fermi surface. For electrons we have $\tilde{\epsilon} = \epsilon > 0$ and for holes $\tilde{\epsilon} = -\epsilon < 0$. In addition, we can neglect henceforth the difference between p and p_f ($|p - p_f| \ll p$) and assume that the excitation has a momentum situated on the Fermi surface. The electron distribution function $f(\mathbf{p})$, in the usual analysis, when the occupation of the energy states begins with the bottom of the band, is connected with the distribution function of the quasiparticle excitations n(**p**) by the relation[1]:

$$f(\mathbf{p},\tilde{\varepsilon})=\begin{cases} n(\mathbf{p}), & \tilde{\varepsilon}>0 \\ 1-n(-\mathbf{p}), & \tilde{\varepsilon}<0 \end{cases}.$$

This circumstance is illustrated in Fig. 2, which shows the deviations from the equilibrium function $n_0 = [\exp(\epsilon/k_B T)+1]^{-1}$ (k_B is the Boltzmann constant) in a constant electric field **E** and at a constant temperature gradient ∇T.

Let us see now how the electron-hole transitions affect the distribution function. In the language of quasiparticle excitations, scattering by a vortex lattice causes the vanishing of an electron in the state $\{\mathbf{p}, \tilde{\epsilon}\}$ and the appearance of a hole in the state $\{\mathbf{p}, -\tilde{\epsilon}\}$, i.e., in the language of the occupation numbers from the bottom of the band—condensation of two electrons from the states $\{\mathbf{p}, \tilde{\epsilon}\}$ and $\{-\mathbf{p}, -\tilde{\epsilon}\}$ in a state with zero momentum on the Fermi level (pairing). In the collision integral I, it is necessary to add to the term describing the ordinary scattering another term

$$I_{\mathrm{vort}}=\int[(1-f_{\mathbf{p}'})(1-f_{\mathbf{p}})-f_{\mathbf{p}'}f_{\mathbf{p}}]Q(\mathbf{p},\mathbf{p}')d\mathbf{p}'.$$

According to the results of the preceding section, the transition probability $Q(\mathbf{p}, \mathbf{p}')$ is given by

$$Q(\mathbf{p},\mathbf{p}')=W(\mathbf{p})\delta(\mathbf{p}+\mathbf{p}')\delta(\tilde{\varepsilon}+\tilde{\varepsilon}').$$

Then

$$I_{\mathrm{vort}}=W(\mathbf{p})[1-f(\mathbf{p},\tilde{\varepsilon})-f(-\mathbf{p},-\tilde{\varepsilon})].$$

Naturally, for an equilibrium function f_0 that does not depend on the direction **p** we have $f_0(-\tilde{\epsilon}) = 1 - f_0(\tilde{\epsilon})$ and $I_{\mathrm{vort}} = 0$. If we introduce in the usual manner an increment g to the equilibrium distribution function in accordance with the equation $f = f_0 - g\partial f_0/\partial\tilde{\epsilon}$ and introduce the relaxation time τ, then the collision integral is replaced in the kinetic equation at $H \leq H_{c2}$ by the term

$$I=g/\tau+W[g(\mathbf{p},\tilde{\varepsilon})+g(-\mathbf{p},-\tilde{\varepsilon})]. \qquad (8)$$

2. In the mixed state, it is meaningful to speak of the electric conductivity of the normal electrons in a plane perpendicular to the magnetic field, if the role of the superconducting component is taken into account by introducing the concentration $x_n = \overline{B}/H_{c2}$ of the normal phase ($\overline{B}$ is the average magnetic induction), in analogy with the intermediate state of a type-I superconductor. The conductivity σ of the normal electrons, i.e., the charge transported by the quasistatic excitations, is calculated with the aid of the kinetic equation.

The usual solution of the kinetic equation in a constant electric and magnetic field $g = e\mathbf{v}\mathscr{E}\tau$, where $\mathscr{E}$ is a certain constant vector (see, e.g., [15]), is valid also at $H \lesssim H_{c2}$, since the reversal of the sign of the velocity **v** leads to $g(\mathbf{p}, \tilde{\epsilon}) = -g(-\mathbf{p}, \tilde{\epsilon})$ and the term in the square brackets in (8) vanishes. This means that on going over to the mixed state near the critical field, the conductivity in a direction perpendicular to the magnetic field remains constant, as does the Hall angle. With further decrease of the field, the conductivity increases, namely $\sigma \propto 1/x_n$. These statements agree with the experimental data[16,17]. We note that we have in mind the conductivity of an ideal metal free of defects. In the existing superconductor samples, owing to the presence of vortex pinning, the conductivity of the normal electrons is determined by measuring the so-called resistance ρ_{flow} to the vortex flow.

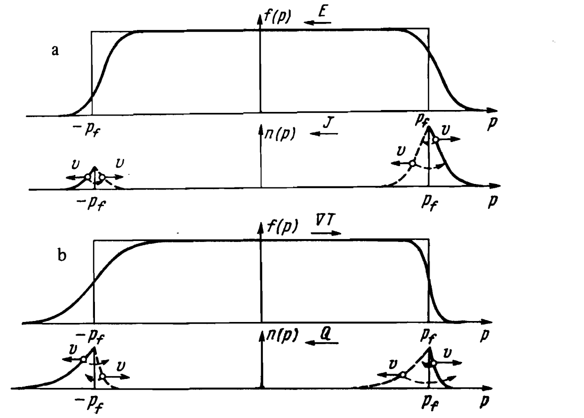

FIG. 2. Comparison of the electron distribution functions f(**p**) in the case of occupation from the bottom of the band and of the quasiparticle-excitation distribution function n(**p**) for the cases of the constant electric field E (a) and of a constant temperature gradient ∇T (b). The dashed curved arrows show the states connected with scattering by the vortex lattice.

Thus, in first-order approximation, the components of the electric conductivity tensor of the normal electrons remain unchanged at $H \leq H_{c2}$. At the same time, the thermal conductivity is greatly altered on going to the mixed state in pure type-II superconductors[2,8]. The superconducting electrons make no contribution to the heat transport, and in addition, in the temperature region of interest to us, the contribution of the phonons to the heat transport in metals near H_{c2} is very small[18], so that we can confine ourselves to electronic excitations. The abrupt decrease of the thermal conductivity at constant electric conductivity confirms the unique character of the scattering of electrons by a vortex lattice. This can be easily visualized with the aid of Fig. 2. Transitions from one branch of excitations to another, caused by the vortex lattice, occur at ϵ = const, i.e., at $|p - p_f|$ = const, and are shown by the curved arrows in Fig. 2. It is clear that electron-hole transitions do not affect the charge transport, since they do not change the excess of the quasiparticles moving in the corresponding directions. To the contrary, transitions of this kind radically decrease the heat transport, since they decrease the excess of quasiparticles moving against the temperature gradient.

The solution of the kinetic equation for the case of a constant temperature gradient at $H \lesssim H_{c2}$ can be sought in the form

$$g=-\tilde{\varepsilon}(k_B T)^{-1}\mathbf{v}\mathscr{T}\tau'$$

($\mathscr{T}$ is a certain constant vector), which is perfectly analogous to the form of the solution in the normal state. The proposed form of the solution means that $g(-\mathbf{p}, -\tilde{\epsilon}) = g(\mathbf{p}, \tilde{\epsilon})$, since **v** and $\tilde{\epsilon}$ reverse sign simultaneously.

According to (8), the collision integral is placed by the expression g/τ', where

$$1/\tau'=1/\tau+2W. \qquad (9)$$

The solution of the kinetic equation of $H \lesssim H_{c2}$ can be obtained here from the solution for the normal metal, with τ replaced by τ'. Since W, according to (7), depends on $E_\perp = E_f - p_z^2/2m$, it is necessary to use a solution in a form in which the dependence of the relaxation time on p_z is admitted. To the contrary, τ is regarded as constant.

Omitting the well known calculations[15], we obtain for the relative change in the components of the thermal conductivity tensor of a metal with spherical Fermi surface

$$\frac{K_{xx}}{K_n}=\frac{3}{4}\int_0^\pi \frac{\tau'}{\tau}\frac{\sin^3\theta}{1+\omega^2\tau'^2}d\theta,$$

$$\frac{K_{yx}}{K_n}=\frac{3}{4}\int_0^\pi \frac{\tau'}{\tau}\frac{\omega\tau'}{1+\omega^2\tau'^2}\sin^3\theta\, d\theta,$$

$$\frac{K_{zz}}{K_n}=\frac{3}{2}\int_0^\pi \frac{\tau'}{\tau}\sin\theta\cos^2\theta\, d\theta.$$

Here K_n is the thermal conductivity of the normal metal without a magnetic field, θ is the angle between **H** and the vector **p**, and ∇T lies in the plane y = 0.

We simplify the expressions by assuming that $(\omega\tau)^2 \ll 1$. This condition is satisfied in certain experiments [8,9,19]. Neglecting $(\omega\tau')^2 \ll 1$ in the denominator of the integrands and recognizing that

$$W=W_0(E_\perp/E_f)^{-1/2}=W_0/\sin\theta,$$

where

$$W_0=\frac{\pi^{1/2}|\Delta_T|^2}{\hbar(\hbar\omega E_f)^{1/2}}\frac{2\varkappa_2^2}{1.16(2\varkappa_2^2-1)}\frac{H_{c2}-H}{H_{c2}}, \tag{10}$$

we obtain for the relative changes of the thermal conductivity along the field $K_\parallel$, for the transverse thermal conductivity $K_\perp$, and for the tangent of the thermal Hall angle α_H the following formulas:

$$\frac{K_\parallel}{K_n}=3\int_0^{\pi/2}\frac{\sin^2\theta\cos^2\theta}{\sin\theta+2W_0\tau}d\theta,$$

$$\frac{K_\perp}{K_n}=\frac{3}{2}\int_0^\pi\frac{\sin^4\theta}{\sin\theta+2W_0\tau}d\theta, \tag{11}$$

$$\mathrm{tg}\,\alpha_H=\omega\tau\left[\int_0^{\pi/2}\frac{\sin^5\theta\, d\theta}{(\sin\theta+2W_0\tau)^2}\right]\left[\int_0^{\pi/2}\frac{\sin^4\theta\, d\theta}{\sin\theta+2W_0\tau}\right]^{-1}$$

The integrals in (11) can be calculated in elementary fashion. We do not write out the integrated functions, but present their plots in Figs. 3 and 4. Figure 3a shows plots of the function $(1 + 2W_0\tau)^{-1}$, which describes the change in the thermal conductivity and of $\tan\alpha_H$ at $H \le H_{c2}$, if the dependence of W on p_z is neglected. This dependence is quite weak, and the approximate description is quite close to the exact one for a spherical Fermi surface; for the same reason the plot for $(\tan\alpha_H)/\omega\tau$ and $K_\perp/K_n$ near H_{c2} practically coincide.

3. We can carry out the comparison with experiment by using data on the thermal effects in pure niobium in the mixed state. The Fermi surface of niobium is far from spherical, and we must therefore use certain averaged quantities. The denominator of (10) contains in the case of an isotropic metal, the combination $\hbar\omega E_f = |e|\hbar v_f^2/2c$. We use the average value of the Fermi velocity $v_f = 6.2 \times 10^7$ cm/sec, obtained by calculating the band structure of niobium [20], which agrees with the experimental data on the de Haas–van Alphen effect [21]. The ratio Δ_T/Δ_0 can be calculated by the BCS theory, and the experimental value of the gap at 0° can be taken from the data of [22], namely $\Delta_0 = 1.83\, k_B T_c = 2.33 \times 10^{-15}$ erg, which agrees with the direct measurements on the tunnel effect [23]. The values of the Maki parameter κ_2 were determined from measurements of the magnetization in [22,24].

Detailed data on the change of $K_\perp/K_n$ at $H \lesssim H_{c2}$ were obtained in [8] on samples with resistance ratio R(300°K)/R(4°K) ≈ 700. We use the data given in [8] for two temperatures, 1.98 and 5.54°K. The values $\kappa_2(1.98°K) = 1.8$ and $\kappa_2(5.54°K) = 1.45$ are taken from [24], where the magnetization of niobium samples of quality close to the samples of [8] was measured. We substitute in (11) the value $\tau = l/v_f = 3.2 \times 10^{-12}$ sec, where $l = 2 \times 10^{-4}$ cm was estimated in [8] from the thermal conductivity of samples in the normal state. Using the indicated values of the parameters, we can compare the experimental data on the dependence of $K_\perp/K_n$ on H at $H \le H_{c2}$ with the results of the calculation in Fig. 3b. At T = 5.54°K the agreement is satisfactory, in spite of the rough approximations employed. For T = 1.98°K, the agreement is much worse. It is possible that the dependence of the pairing potential on $H_{c2} - H$ in these measurements does not agree with the theoretical one. It is known [22,24], for example, that the value of the parameter κ_2 depends strongly on the purity of the sample at low temperatures.

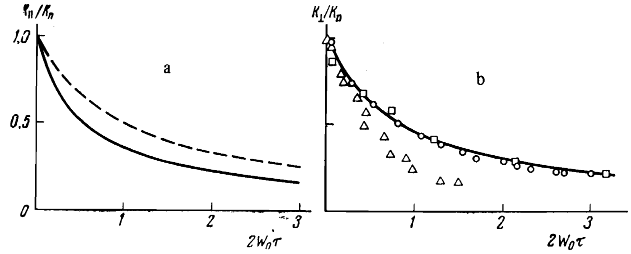

FIG. 3. Change of the thermal conductivity of a pure superconductor on going to the mixed state, $2W_0\tau \sim (H_{c2}-H)/H_{c2}$. Solid curves—calculation by formula (11), dashed—the function $(1 + 2W_0\tau)^{-1}$. The experimental points were obtained for niobium: O—data of [8] at T = 5.54°K, Δ—data of [8] at T = 1.98°K, □—data of [19] at T = 4.75°K.

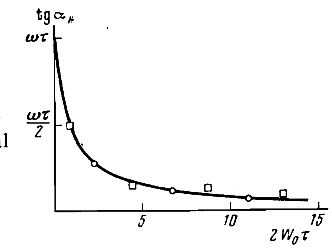

FIG. 4. Change of the thermal Hall angle in a pure superconductor on going to the mixed state: $2W_0\tau \sim (H_{c2}-H)/H_{c2}$. Solid curve—calculation by formula (11). Experimental points—data for niobium obtained in [9]: O—at T = 3°K, □—at T = 505°K.

It is interesting to compare the change of the thermal conductivity on going to the next state for samples with different mean free paths. The experimental results of [19] of the measurement of the thermal conductivity of niobium samples with resistance ratio 6500 are shown in Fig. 3b for $\kappa_2 = 1.76$ from [22] and $\tau = 2.2 \times 10^{-11}$ sec. The value of τ was chosen for best agreement with the experimental data [19] with the calculated plot, since it could not be determined from the experimental data on the thermal conductivity. The ratio of the relaxation times for the samples from [8,19] is ≈ 7 and conforms only approximately to their purity as determined from the resistance ratio 6500/700 ≈ 9. This discrepancy can be attributed to the inaccuracy of the measurements of R(4°K), which is determined by extrapolation. Deviations from the Wiedemann-Franz law are also significant, and cause the values of τ obtained from the thermal and electric measurements to disagree.

In [8] are given data on the ratio $K_\parallel/K_\perp$ in the mixed state. At $H \approx H_{c2}$, the ratio $K_\parallel/K_\perp$ is close to unity, in agreement with the results of the calculations. A more serious comparison would be meaningless, since it is obvious that this ratio depends strongly on the anisotropy of the Fermi surface. Moreover, $K_{\parallel}/K_{\perp}$ can depend on the temperature, if electrons from small sections of the Fermi surface with large effective mass take part in the thermal conductivity. For such electrons, which travel along the field and make the principal contribution to $K_{\parallel}$, the probability of scattering at high temperatures by the vortex lattice can be greatly weakened by the factor $\exp(-\epsilon^2/\hbar\omega E_f)$ and this leads to an increase of the ratio $K_{\parallel}/K_{\perp}$ at $H \le H_{c2}$. This was precisely the situation observed in experiment [8].

It should be noted that the theory of the thermal conductivity of pure superconductors in the mixed state, developed by Haughton and Maki [3], yields qualitatively the same results as our calculations. Only the ratio $K_{\parallel}/K_{\perp}$, calculated in [3], differs too much from unity at $H \le H_{c2}$. No quantitative comparison with experiment was made in [3].

We proceed now to the experimental results on the thermal Hall angle. In [9] they observed the Righi-Leduc effect in niobium samples with resistance ratio $R(300°K)/R(4°K) \approx 4000$. For the samples used there, the value $\tau = 1.7 \times 10^{-11}$ sec can be determined from the measured value $\omega\tau = 0.31$ in the normal state at $T = 5.5°K$ in the field H_{c2}, if one assumes as a result of averaging of the band-structure calculation [20] that $m^*/m = 2.0$. According to [22], it can be assumed that $\kappa_2(5.5°K) = 1.58$ and $\kappa_2(3°K) = 2.24$. The experimental data on the thermal Hall angles are plotted in Fig. 4, using the indicated values in accordance with formula (11). Unfortunately, notice should be taken of the paucity of experimental data near H_{c2}. Nonetheless, the agreement between calculation and experiment is satisfactory.

We note in conclusion that in dirty superconductors at $l \lesssim \xi$ the role of the scattering by the vortex lattice decreases. Accordingly, the changes of the thermal conductivity and of the Hall angle on going to the next state differ from the case of pure niobium. The thermal-conductivity coefficient for alloys changes much less than for pure niobium [8], and the Hall angle can even increase when the alloy goes over to the mixed state [25]. At the same time, all the data on the thermal effects in the mixed state near H_{c2} in pure niobium are in satisfactory agreement with formulas (10) and (11), and by the same token confirm the assumption concerning the character of the scattering of the electrons by the vortex lattice.

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