Effect of scattering processes on the anomalies in the thermodynamic characteristics of superconductors under pressure

V. I. Makarov, V. G. Bar'yakhtar, and V. V. Gann

Physico-technical Institute, Ukrainian Academy of Sciences
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On the basis of the Eliashberg equations with account of the anisotropy of the electron-phonon interaction, it is shown that anomalies in T_c and $\vartheta T_c/\vartheta P$ due to variation in the Fermi-surface topology can be expressed in terms of the solution of the Éliashberg equations for an anisotropic superconductor without account of variation of the Fermi-surface topology. An analytic experession is obtained for ΔT_c and $\vartheta T_c/\vartheta P$ in first order in the small parameter $\delta v/\nu$ (δv is the addition to the density of states ν due to variation in the Fermi-surface topology) both for a pure superconductor and for a superconductor with impurities. It is shown that the effect of the impurities on ΔT_c is connected with two competing mechanisms: a decrease in the lifetime and isotropization of the quasiparticle energy gap. Depending on the nature of the anisotropy of the gap in the superconductor, the T_c and $\vartheta T_c/\vartheta P$ anomalies due to variation of the Fermi-surface topology induced by the impurities may smooth out, or the effect of electron scattering by impurities may be weaker.

INTRODUCTION

In the study of the simultaneous effects of impurities and pressure in certain superconductors, a nonlinear dependence of T_c and $\partial T_c/\partial P$ on the pressure P and the impurity concentration c has been observed. This dependence is due to variation of the electron concentration in the metal,[1-3] and the suggestion has been made that the nonlinear behavior of T_c and $\partial T_c/\partial P$ as functions of pressure and impurity concentration may be connected with a change of the Fermi-surface topology. The authors have constructed a theory of this phenomenon that explains qualitatively the experimental data on the nonlinear dependence of T_c and $\partial T_c/\partial P$ on P and c for indium and thallium[4] and explains them quantitatively for rhenium.[5]

In the study of solid solutions of indium with impurities, it was discovered that the value of the maximum of the derivative $\partial T_c/\partial P$ as a function of the impurity concentration depends weakly on the value of the residual resistance (i.e., the free path length).[6,7] Along with this, a decrease in the quasiparticle free path, due to scattering of electrons by impurities, leads in the simplest variant of the theory to a smearing of the anomalies in T_c and $\partial T_c/\partial P$ [8,9] and therefore the experimental results obtained in [6,7] have no interpretation at the present time.

It is shown in this paper that essential roles in the effect of scattering processes on the anomalies of T_c and $\partial T_c/\partial P$ are played not only by the free path of the electrons, but also by the anisotropy of the energy gap. In studying the effects of impurities on the thermodynamic characteristics of the superconductors, we started out from the Éliashberg equations with account of the anisotropy of the electron-phonon interaction. It is shown that the anomalies of T_c and $\partial T_c/\partial P$ due to variation of the Fermi-surface topology are completely explained by the solution of the Éliashberg equations for an anisotropic superconductor without change in the Fermi-surface topology. In the first approximation in the small parameter $\delta\nu/\nu$ ($\delta\nu$ is the increment to the density of states ν due to the change in the Fermi-surface topology), an analytic expression is obtained for ΔT_c and $\partial T_c/\partial P$ both for the pure superconductor and for the superconductor with impurities. It is shown that the effect of impurities on ΔT_c is due to two competing mechanisms: 1) a decrease in the lifetime; 2) isotropization of the quasiparticle energy gap.

If the gap in the direction in which the Fermi surface aligns itself is larger than the mean value of the gap, the two mechanisms mentioned lead to a "smoothing out" of the anomalies in T_c and $\partial T_c/\partial P$. But if the gap in the direction in which alignment of the Fermi surface takes place is less than its average value, then the two mechanisms act in opposite directions, so that a significant lessening of the effects of electron scattering by impurities on the anomalies of T_c and $\partial T_c/\partial P$ becomes possible.

BASIC EQUATIONS

1. In this section, we obtain a set of equations that describe the temperature of the superconducting transition and the energy gap in an anisotropic superconductor in the case in which the Fermi energy ϵ_F is close to the value ϵ_c at which the change in the Fermi-surface topology takes place. A set of coupled equations for description of a superconductor with strong coupling was obtained in [10-12]. In our case, the set of equations for the energy gap $\Phi(p, \omega)$ and the parameter $Z(p, \omega)$ can be presented in the following form:

$$\Phi(\mathbf{p},\omega)=-\int_0^\infty\frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\mathrm{Im}\left[\frac{\Phi(\mathbf{p}',\omega')}{Q(\mathbf{p}',\omega')}\right]K_+(\mathbf{p},\mathbf{p}',\omega,\omega',T),$$

$$[1-Z(\mathbf{p},\omega)]\omega=+\int_0^\infty\frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\mathrm{Im}\left[\frac{\omega'Z(\mathbf{p}',\omega')}{Q(\mathbf{p}',\omega')}\right]K_-(\mathbf{p},\mathbf{p}',\omega,\omega',T), \qquad (1)$$

where

$$Q(\mathbf{p},\omega)=[\omega Z(\mathbf{p},\omega)]^2-\xi^2(\mathbf{p})-\Phi^2(\mathbf{p},\omega),$$

$$K_\pm(\mathbf{p},\mathbf{p}',\omega,\omega',T)=N_\pm(\mathbf{p},\mathbf{p}',\omega,\omega')\,\mathrm{th}\,(\omega'/2T)+P_\pm(\mathbf{p},\mathbf{p}',\omega,\omega')+f(\omega')\,[N_\pm(\mathbf{p},\mathbf{p}',\omega,\omega')\mp N_\pm(\mathbf{p},\mathbf{p}',\omega,-\omega')],$$

$$N_\pm(\mathbf{p},\mathbf{p}',\omega,\omega')=L_+(\mathbf{p},\mathbf{p}',\omega+\omega')\pm L_-(\mathbf{p},\mathbf{p}',\omega'-\omega),$$

$$L_\pm(\mathbf{p},\mathbf{p}',\omega)=\sum_\lambda\int_0^\infty d\nu\,B_\lambda(\mathbf{p}-\mathbf{p}',\nu)\,|g(\mathbf{p},\mathbf{p}',\lambda)|^2(\omega+\nu\pm i\delta)^{-1}.$$

$$P_\pm(\mathbf{p},\mathbf{p}',\omega,\omega')=\sum_\lambda\int_0^\infty d\nu\,B_\lambda(\mathbf{p}-\mathbf{p}',\nu)\,|g(\mathbf{p},\mathbf{p}',\lambda)|^2 n(\nu)\times\left[\frac{1}{\omega'+\omega+\nu+i\delta}\pm\frac{1}{\omega'-\omega+\nu-i\delta}\pm\frac{1}{\omega'-\omega-\nu-i\delta}+\frac{1}{\omega'+\omega-\nu+i\delta}\right].$$

Here $\xi(\mathbf{p}) = \epsilon_0(\mathbf{p}) - \epsilon_F$, $\epsilon_0(\mathbf{p})$ is the renormalized energy of the electron, $f(\omega)$ the Fermi distribution function, $n(\omega)$ the Bose distribution function, $g(\mathbf{p}, \mathbf{p}', \lambda)$ the parameter of electron-phonon interaction, $B_\lambda(\mathbf{p}, \nu)$ the phonon spectral density. (In the set of Equations (1), we have neglected the Coulomb interaction.)

2. We now find the superconducting transition temperature T_c from the set of Eqs. (1). Since the function $\Phi(\mathbf{p}, \omega)$ vanishes at $T = T_c$, the set of equations (1) splits into two uncoupled equations, which can be conveniently written in terms of the nondimensional quantities

$$\psi(\mathbf{p}, \omega) = \frac{\Phi(\mathbf{p}, \omega)}{\langle \Phi(\mathbf{p}, 0)\rangle_{\mathbf{p}}}, \quad \bar{L}_\pm(\mathbf{p}, \mathbf{p}', \omega) = \frac{L_\pm(\mathbf{p}, \mathbf{p}', \omega)}{2\langle\langle L_+(\mathbf{p}, \mathbf{p}', 0)\rangle_{\mathbf{p}}\rangle_{\mathbf{p}'}},$$

where $\langle\ldots\rangle_{\mathbf{p}} = \int(\ldots)dS_{\mathbf{p}}/\int dS_{\mathbf{p}}$, the integration being carried out over the Fermi surface.

We represent the set of equations (1) in the form

$$\psi(\mathbf{p}, \omega) = -\int_0^\infty \frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left[\frac{\psi(\mathbf{p}', \omega')}{Q(\mathbf{p}', \omega')}\right]K_+(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c); \quad (2)$$

$$[1 - Z(\mathbf{p}, \omega)]\omega = \frac{1}{\pi}\int_0^\infty d\omega'\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left[\frac{\omega' Z(\mathbf{p}', \omega')}{Q(\mathbf{p}', \omega')}\right]K_-(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c)$$

$$Q(\mathbf{p}', \omega') = [\omega' Z(\mathbf{p}', \omega')]^2 - \xi^2(\mathbf{p}'). \quad (2')$$

Setting $\omega = 0$ in (2) and carrying out averaging over $\mathbf{p}$, we obtain an equation for T_c:

$$1 = -\int_0^\infty \frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left[\frac{\psi(\mathbf{p}', \omega')}{Q(\mathbf{p}', \omega')}\right]\langle K_+(\mathbf{p}, \mathbf{p}', 0, \omega', T_c)\rangle_{\mathbf{p}}. \quad (3)$$

The function $\psi(\mathbf{p}', \omega')$ is found from a solution of the homogeneous linear integral equation (2) of the type $\hat{L}\psi = 0$, where the operator $\hat{L}$ is singular. It is therefore convenient to represent Eq. (2) in the form of a linear inhomogeneous integral equation [13]

$$\psi(\mathbf{p}, \omega) = 2\langle \bar{L}_+(\mathbf{p}, \mathbf{p}', \omega)\rangle_{\mathbf{p}'} - \int_0^\infty \frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left[\frac{\psi(\mathbf{p}', \omega')}{Q(\mathbf{p}', \omega')}\right]F(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c), \quad (4)$$

where

$$F(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c) = [K_+(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c) - 2\langle K_+(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c)\rangle_{\mathbf{p}}\langle \bar{L}_+(\mathbf{p}, \mathbf{p}', \omega)\rangle_{\mathbf{p}'}].$$

Since the basic contribution to the superconducting characteristics of metals is made by electrons lying in the energy layer $-\tilde{\omega} < \xi < \tilde{\omega}$, where $\tilde{\omega}$ is the boundary frequency of the phonon spectrum, the effect of the change in Fermi-surface topology on T_c will be significant if the separation of the critical energy ϵ_c and the Fermi energy ϵ_F is of the order of $|\epsilon_F - \epsilon_c| \lesssim \tilde{\omega}$.

Let $\mathbf{p}_c$ belong to a star of wave vectors corresponding to those points of the Brillouin zone in which the change in the Fermi-surface topology takes place. Integration over $d^3\mathbf{p}'$ in Eqs. (2′), (3) and (4) is carried out in two regions: the region Ω_0, which does not contain the points $\mathbf{p}_c$, and the small region $\delta\Omega$ which contains $\mathbf{p}_c$, near which one can establish the form of $\epsilon(\mathbf{p})$.

We shall assume that we know the functions $\psi_0(\mathbf{p}, \omega)$ and $Z(\mathbf{p}, \omega)$ and the value of T_c^0, which are found from Eqs. (2′), (3) and (4) in the case in which integration over $d^3\mathbf{p}'$ is carried out over the main region Ω_0, and we write out the set of equations for the change in the superconducting transition temperature $\Delta T_c = T_c - T_c^0$ and the function $\delta\psi(\mathbf{p}, \omega) = \psi(\mathbf{p}, \omega) - \psi_0(\mathbf{p}, \omega)$, due to the presence of the small region $\delta\Omega$:

$$(\varkappa, (1-\hat{P})\delta\psi) + \Delta T_c\left(\frac{d\varkappa}{dT_c^0}, (1-\hat{P})\psi_0\right) = -(\varkappa, \hat{P}\psi_0) + (\varkappa, \lambda),$$

$$(1 + \hat{F}(1-\hat{P}))\delta\psi + \Delta T_c\frac{d\hat{F}}{dT_c^0}(1-\hat{P})\psi_0 = -\hat{F}\hat{P}\psi_0 + \hat{F}\lambda. \quad (5)$$

Here we have introduced the notation:

$$\varkappa(\mathbf{p}', \omega') = \langle K_+(\mathbf{p}, \mathbf{p}', 0, \omega', T_c^0)\rangle_{\mathbf{p}},$$

$$\lambda(\mathbf{p}, \omega) = \psi_0(\mathbf{p}, \omega)\left[1 - \frac{Q_0(\mathbf{p}, \omega)}{Q(\mathbf{p}, \omega)}\right], \quad Q_0(\mathbf{p}, \omega) = [\omega Z_0(\mathbf{p}, \omega)]^2 - \xi^2(\mathbf{p}), \quad (6)$$

$$(\chi, \psi) = \int_0^\infty \frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi(\mathbf{p}', \omega')}{Q_0(\mathbf{p}', \omega')}\right\}\chi(\mathbf{p}', \omega'),$$

$$\hat{F}\psi = \int_0^\infty \frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi(\mathbf{p}', \omega')}{Q_0(\mathbf{p}', \omega')}\right\}F(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c^0),$$

$$\hat{P}\psi(\mathbf{p}, \omega) = \begin{cases} \psi(\mathbf{p}, \omega), & \text{if } \mathbf{p} \text{ belongs to the region } \delta\Omega, \\ 0, & \text{if } \mathbf{p} \text{ does not belong to the region } \delta\Omega. \end{cases}$$

We also define the expression $\chi\hat{F}$ by the relation

$$\chi\hat{F} = \int_0^\infty \frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{F(\mathbf{p}', \mathbf{p}, \omega', \omega, T_c^0)}{Q_0(\mathbf{p}', \omega')}\right\}\chi(\mathbf{p}', \omega').$$

It is not difficult to establish the fact that $(\chi\hat{F}, \varphi) = (\chi, \hat{F}\varphi)$.

The solution of the set of Eqs. (5) has the form

$$\Delta T_c = -\frac{(\varphi_0, \hat{P}\psi_0) - (\varphi_0, \lambda)}{(d\varkappa/dT_c^0 - \varphi_0(1-\hat{P})d\hat{F}/dT_c^0, (1-\hat{P})\psi_0)}, \quad (7)$$

where the function φ_0 satisfies the equation

$$\varphi_0[1 + (1-\hat{P})\hat{F}] = \varkappa. \quad (8)$$

Using the definition (4) of the function F, we can proceed from the inhomogeneous equation for φ_0 to the homogeneous equation with prescribed normalization [1)]

$$\varphi_0(\mathbf{p}, \omega, T_c^0) = -\int_0^\infty \frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{K_+(\mathbf{p}', \mathbf{p}, \omega', \omega, T_c^0)}{Q_0(\mathbf{p}', \omega')}\right\}\varphi_0(\mathbf{p}', \omega', T_c^0),$$

$$\int_0^\infty \frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{1}{Q_0(\mathbf{p}', \omega')}\right\}\langle \bar{L}_+(\mathbf{p}', \mathbf{p}, \omega')\rangle_{\mathbf{p}}\varphi_0(\mathbf{p}', \omega', T_c) = -\frac{1}{2}. \quad (9)$$

Using Eqs. (9), we can transform Eq. (7) for ΔT_c to the form

$$\frac{\Delta T_c}{T_c^0} = -\frac{1}{B}\int_0^\infty d\omega\int_{\delta\Omega}\frac{d^3\mathbf{p}}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi_0(\mathbf{p}, \omega)}{Q_0(\mathbf{p}, \omega)}\right\}\varphi_0(\mathbf{p}, \omega, T_c^0), \quad (10)$$

where

$$B = T_c^0\int_0^\infty d\omega\int_0^\infty\frac{d\omega'}{\pi}\int_{\Omega_0}\frac{d^3\mathbf{p}}{(2\pi)^3}\int_{\Omega_0}\frac{d^3\mathbf{p}'}{(2\pi)^3}\operatorname{Im}\left\{\frac{\psi_0(\mathbf{p}', \omega')}{Q_0(\mathbf{p}', \omega')}\right\} \times \operatorname{Im}\left\{\frac{d}{dT_c^0}K_+(\mathbf{p}, \mathbf{p}'\omega, \omega', T_c^0)/Q(\mathbf{p}, \omega)\right\}\varphi_0(\mathbf{p}, \omega, T_c^0).$$

In the transition from (7) to (10), we have omitted the term (φ_0, λ), since it describes increments to ΔT_c that are smooth in $\epsilon_F - \epsilon_c$ and will not be considered here.

3. Equation (10) determines the change in the temperature of the superconducting transition in the case in which the topology of the Fermi surface changes with change in the Fermi energy. We shall consider the Fermi-surface changes near points $\mathbf{p}_c$ at which all three components of the velocity vanish: $v(\mathbf{p}_c) = 0$. In this case, the quadratic dispersion law is valid in the small region $\delta\Omega$. Moreover, one can replace the functions $\psi_0(\mathbf{p}, \omega)$ and $\varphi(\mathbf{p}, \omega, T_c^0)$ in this region by their values at the point $\mathbf{p}_c$. Then Eq. (10) takes the form

$$\frac{\Delta T_c}{T_c^0} = \frac{v}{2}\begin{cases}\mp I(\beta), \\ \mp I(-\beta),\end{cases} \quad (11)$$

where

$$v = \frac{1}{B}(2T_c^0 m_1 m_2 m_3)^{1/2}\frac{k^*}{2\pi^2},$$

m_1, m_2, m_3 are the effective masses and k^* is the number of equivalent points in the Brillouin zone. The upper line in Eq. (11) refers to the case in which a transition of a closed Fermi surface to an open one (with a minus sign) or vanishing of the closed parts of the Fermi surface (with a plus sign) takes place upon an increase in the Fermi energy, while the lower line refers to the case of the opposite processes. The function $I(\beta)$ has the following form:

$$I(\beta)=\int_0^\infty \frac{d\omega}{\omega}\operatorname{Re}\left\{\frac{\psi_0(\mathbf{p}_C\,\omega)}{Z_0(\mathbf{p}_C\,\omega)}J(\beta,\omega)\right\}\varphi_0(\mathbf{p}_C\,\omega,T_C^0), \qquad (12)$$

where

$$J(\beta,\omega)=\frac{1}{(2T_C^0)^{1/2}}\{\sqrt{\rho_++a_+}+\sqrt{\rho_-+a_-}+i\sqrt{\rho_+-a_+}-i\sqrt{\rho_--a_-}\}, \qquad (12')$$

$$\rho_\pm=\{a_\pm^2+[\omega\operatorname{Im}\bar{Z}_0(\mathbf{p}_C\,\omega)]^2\}^{1/2},$$

$$a_\pm=\beta\pm\omega\operatorname{Re}\bar{Z}_0(\mathbf{p}_C,\omega),\quad \bar{Z}_0(\mathbf{p}_C,\omega)=Z_0(\mathbf{p}_C,\omega)/Z_0(\mathbf{p}_C\,0).$$

The function $Z_0(\mathbf{p},\omega)$, which is found from the second equation of the set (1), appears in Eq. (12).

We now calculate the derivative $dI/d\beta$, which is proportional to the experimentally observed value of dT_C/dP. The value of $dI/d\beta$ is determined by the behavior of the integrand at small values of ω. In the range of low frequencies ω, the ratio $\psi_0(\mathbf{p}_C,\omega)/Z_0(\mathbf{p}_0,\omega)$ is a smooth function of ω, and the function φ_0, as can be established on the basis of Eq. (9), has the form

$$\varphi_0(\mathbf{p}_C,\omega,T_C^0)\approx\operatorname{th}(\omega/2T_C^0)f(\mathbf{p}_C).$$

Therefore,

$$\frac{dI}{d\beta}\sim\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\operatorname{th}\frac{\omega}{2T_C}\{[(\beta+\omega)^2+[\omega\operatorname{Im}\bar{Z}_0(\mathbf{p}_C,\omega)]^2]^{1/2}+\beta+\omega\}^{-1/2}. \qquad (13)$$

This formula is valid at small values of β, i.e., for $\beta\ll\tilde{\omega}$. It is not difficult to see that the quantity $dI/d\beta$ as a function of β has a maximum at $\beta\sim 2T_C$.

We see from Eqs. (12) and (13) that for description of a superconductor with account of the change in the Fermi-surface topology it is important to know not only the isotropic parts of the functions φ_0, Z_0 and ψ_0, but also their values at the point $\mathbf{p}_C$. However, we note that in the description of the change of T_C under the influence of pressure, when only the value of β changes, Eqs. (12) and (13) correctly describe the dependence $\Delta T_C(P)$ even in the isotropic approximation for the functions ψ_0, Z_0 and φ_0.

A different situation arises if the change in the Fermi-surface topology occurs under the action of impurities. This is because the impurities not only change the Fermi energy, but also lead to isotropization of the gap in the superconductor.

4. We shall describe the scattering of conduction electrons by impurities by means of the Hamiltonian

$$\mathscr{H}_{ei}=\sum_{\mathbf{p}\mathbf{p}'}U(\mathbf{p},\mathbf{p}')\psi_{\mathbf{p}'}^{+}\hat{\tau}_3\psi_{\mathbf{p}},$$

where $U(\mathbf{p},\mathbf{p}')$ are the matrix elements of the interaction potential. Assuming the impurities to be randomly distributed throughout the crystal, we can find the contribution of the scattering of electrons by impurities to the mass operator of the electrons:[14]

$$\Sigma_{ei}(\mathbf{p},i\omega_n)=c\sum_{\mathbf{p}'}|U(\mathbf{p},\mathbf{p}')|^2\hat{\tau}_3G(\mathbf{p}',i\omega_n)\hat{\tau}_3,$$

where c is the impurity concentration and $G(\mathbf{p},i\omega_n)$ the Green's function of the superconductor with impurities.

Equations (2) and (9) for the functions $\psi_1(\mathbf{p},\omega)$ and $\varphi_1(\mathbf{p},\omega,T_C)$ in the presence of the interaction $\mathscr{H}_{ei}$ retain the same structure, but the kernel $K_+(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)$ in these equations should not be replaced by the kernel $K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)$, which is connected with the kernel K_+ by the equations

$$K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)=K_+(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)-c\int_{\Omega_0}\frac{d^3p_0}{(2\pi)^3}\frac{K_+'(\mathbf{p}_0,\mathbf{p}',\omega,\omega',T_C)}{Q_1(\mathbf{p}_0,\omega)}|U(\mathbf{p},\mathbf{p}_0)|^2, \qquad (14)$$

where

$$Q_1(\mathbf{p},\omega)=\omega^2Z_1^2(\mathbf{p},\omega)-\xi^2(\mathbf{p}).$$

The function for the renormalization of the electron mass in the presence of the impurities, $Z_1(\mathbf{p},\omega)$, is expressed in terms of the corresponding function for the pure metal $Z_0(\mathbf{p},\omega)$ in the following way:

$$\omega Z_1(\mathbf{p},\omega)=\omega Z_0(\mathbf{p},\omega)+{}^1/_2 i\Gamma(\mathbf{p}), \qquad (15)$$

where

$$\Gamma(\mathbf{p})=2\pi c\int\frac{dS_{\mathbf{p}'}}{v}|U(\mathbf{p},\mathbf{p}')|^2.$$

The change in the temperature of the superconducting transition under the action of impurities is expressed by Eq. (11) as before, but the function $I(\beta)$ now has the form

$$I(\beta)=\int_0^\infty\frac{d\omega}{\omega}\operatorname{Re}\left\{\frac{\psi_1(\mathbf{p}_C,\omega)}{Z_1(\mathbf{p}_C,\omega)}J_1(\beta,\omega)\right\}\varphi_1(\mathbf{p}_C,\omega,T_C^0). \qquad (16)$$

The function $J_1(\beta,\omega)$ is determined by Eq. (12′) with the replacement of $Z_0(\mathbf{p},\omega)$ by $Z_1(\mathbf{p},\omega)$.

In order to determine the functions $\psi_1(\mathbf{p},\omega)$ and $\varphi_1(\mathbf{p},\omega,T_C^0)$ that enter into Eq. (16), it is first necessary to solve Eq. (14) for the kernel $K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)$. For this purpose, we represent the kernel $K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)$ and the function $|U(\mathbf{p},\mathbf{p}')|^2$ in the form

$$K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)=\langle\langle K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)\rangle_{\mathbf{p}}\rangle_{\mathbf{p}'}[1+\alpha(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)],$$

$$|U(\mathbf{p},\mathbf{p}')|^2=\langle\langle|U(\mathbf{p},\mathbf{p}')|^2\rangle_{\mathbf{p}}\rangle_{\mathbf{p}'}[1+\theta(\mathbf{p},\mathbf{p}')].$$

The functions $\alpha(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)$ and $\theta(\mathbf{p},\mathbf{p}')$ are so chosen in these expressions that $\langle\langle\alpha(\mathbf{p},\mathbf{p}')\rangle\rangle=0$ and $\langle\langle\theta(\mathbf{p},\mathbf{p}')\rangle\rangle=0$. Then, neglecting terms of the type

$$c\langle[\alpha(\mathbf{p}_0,\mathbf{p}')-\langle\alpha(\mathbf{p}_0,\mathbf{p}')\rangle_{\mathbf{p}_0}][\theta(\mathbf{p},\mathbf{p}_0)-\langle\theta(\mathbf{p},\mathbf{p}_0)\rangle_{\mathbf{p}_0}]\rangle_{\mathbf{p}_0},$$

in Eq. (14), we obtain

$$K_+'(\mathbf{p},\mathbf{p}',\omega,\omega',T_K)=K_+(\mathbf{p},\mathbf{p}',\omega,\omega',T_K)+\frac{i}{2}\Gamma(\mathbf{p})\frac{\langle K_+(\mathbf{p},\mathbf{p}',\omega,\omega',T_C)\rangle_{\mathbf{p}}}{\omega\langle Z_0(\mathbf{p},\omega)\rangle_{\mathbf{p}}}. \qquad (17)$$

Substituting (15) and (17) in Eqs. (2) and (9), we get the following equations for the functions $\psi_1(\mathbf{p},\omega)$ and $\varphi_1(\mathbf{p},\omega,T_C)$, (in the linear approximation in the anisotropy) in terms of the corresponding functions for the pure superconductor:

$$\psi_1(\mathbf{p},\omega)=\psi_0(\mathbf{p},\omega)+i\frac{\Gamma(\mathbf{p})}{2}\frac{\langle\psi_0(\mathbf{p},\omega)\rangle_{\mathbf{p}}}{\omega\langle Z_0(\mathbf{p},\omega)\rangle_{\mathbf{p}}}, \qquad (18)$$

$$\varphi_1(\mathbf{p},\omega,T_C^0)=\varphi_0(\mathbf{p},\omega,T_C^0).$$

With (15) and (18), Eq. (16) takes the form

$$I(\beta)=\int_0^\infty\frac{d\omega}{\omega}\operatorname{Re}\left\{\langle D(\mathbf{p},\omega)\rangle_{\mathbf{p}}\left[1+\frac{a_0(\mathbf{p}_C\,\omega)}{1+i(\omega\tau)^{-1}}\right]J_1(\beta,\omega)\right\}\varphi_0(\mathbf{p}_C,\omega,T_C^0), \qquad (19)$$

where

$$D(\mathbf{p},\omega)=\frac{\psi_0(\mathbf{p},\omega)}{Z_0(\mathbf{p},\omega)},\quad a_0(\mathbf{p},\omega)=\frac{D(\mathbf{p},\omega)-\langle D(\mathbf{p},\omega)\rangle_{\mathbf{p}}}{\langle D(\mathbf{p},\omega)\rangle_{\mathbf{p}}}$$

$$1/\tau=\Gamma(\mathbf{p}_C)/2Z(\mathbf{p}_C,0).$$

The function $a_0(\mathbf{p},\omega)$ characterizes the value of the gap anisotropy. It is not difficult to see that is $1/\tau=0$, then Eq. (19) transforms into (12).

Let us see how allowance for electron scattering by impurities alters the character of the dependence of $I(\beta)$ and $dI/d\beta$ on β. If the superconductor is isotropic, i.e., $a_0(\mathbf{p}, \omega)=0$, then only the function $J_1(\beta, \omega)$ depends on τ and account of the scattering processes makes the function $I(\beta)$ smoother and the value of the maximum of the derivative $dI/d\beta$ decrease monotonically with increase in $1/\tau$ (see the figure, b).[8,9]

Account of the anisotropy leads to a significant change in the $I(\beta)$ dependence. This is due to the presence of the factor

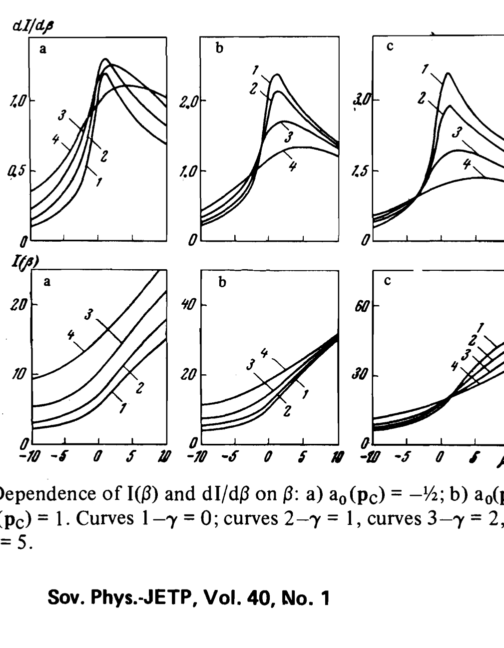
$$A(\omega, \tau)=1+\frac{a_0(\mathbf{p}_c, \omega)}{1+i(\omega\tau)^{-1}},$$

in Eq. (19); it describes the effect of isotropization of the gap under the action of the impurities.

If $a_0(\mathbf{p}_c, \omega)>0$, i.e., the energy gap of the pure superconductor $D(\mathbf{p}, \omega)$ at the point $\mathbf{p}_c$ is greater than the mean value $\langle D(\mathbf{p}, \omega)\rangle_\mathbf{p}$, then the factor $A(\omega, \tau)$ falls off with increase in $1/\tau$. This leads to the result that the maximum of the function $dI/d\beta$ falls off with increase in $1/\tau$ more rapidly than in the isotropic superconductor (see figure, c).

A more interesting situation arises for $a_0(\mathbf{p}_c, \omega)<0$, i.e., in the case in which the value of the gap at the point $\mathbf{p}_c$ is smaller than its mean value. In this case, the factor $A(\omega, \tau)$ increases with increase in $1/\tau$. Thus, one of the functions under the integral, $A(\omega, \tau)$, increases and the other, $J_1(\beta, \omega)$, becomes smeared out with increase in $1/\tau$, as a consequence of the increase in the damping of the quasiparticles. The competition of these two mechanisms can lead to a significant decrease in the effect of the scattering processes on the value of the maximum of the function $dI/d\beta$, which is proportional to the experimentally observed value of dT_c/dP. Thus, at $a_0(\mathbf{p}_c, \omega)=-\frac{1}{2}$, the value of the maximum of the function $dI/d\beta$ varies by not more than 10% over a wide range of $1/\tau$ (see the figure, a).

The results of numerical calculations of the functions $I(\beta)$ and $dI/d\beta$ are given in the drawing for different values of the damping parameter $\gamma=1/(2T_c^0\tau)$ and the anisotropy $a_0(\mathbf{p}_c)$. The calculations were carried out for an anisotropic model of a superconductor



Dependence of $I(\beta)$ and $dI/d\beta$ on β: a) $a_0(\mathbf{p}_c)=-\frac{1}{2}$; b) $a_0(\mathbf{p}_c)=0$; c) $a_0(\mathbf{p}_c)=1$. Curves $1-\gamma=0$; curves $2-\gamma=1$, curves $3-\gamma=2$, curves $4-\gamma=5$.

with a multiplicative kernel of the electron-phonon interaction:

$$K_\pm(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c)=[1+a_0(\mathbf{p})][1+a_0(\mathbf{p}')]K_\pm(\omega, \omega')\,\mathrm{th}(\omega'/2T_c), \quad (20)$$

where the functions $K_\pm(\omega, \omega', T)$ were calculated according to the isotropic model of Debye-Frohlich.[15]

The solutions of the equations (4) and (9) with the kernel (20) have the form

$$\psi_0(\mathbf{p}, \omega)=[1+a_0(\mathbf{p})]\psi_0(\omega), \quad \varphi_0(\mathbf{p}, \omega)=[1+a_0(\mathbf{p})]\varphi_0(\omega). \quad (21)$$

Expression (19) in this model takes the form

$$I(\beta)=[1+a_0(\mathbf{p}_c)]\int\limits_0^\infty \frac{d\omega}{\omega}\,\mathrm{th}\frac{\omega}{2T_c^0}\cdot \quad (22)$$
$$\times\,\mathrm{Re}\left\{\frac{\psi_0(\omega)}{Z_0(\omega)}\left[1+\frac{a_0(\mathbf{p}_c)}{1+i(\omega\tau)^{-1}}\right]J_1(\beta, \omega)\right\}\varphi_0(\omega).$$

The functions $\psi_0(\omega)$ and $\varphi_0(\omega)$ were found as a result of numerical solution of the equations corresponding to the isotropic model, in the case $\tilde{\omega}/T_c=30$, $\rho=K_+(0, 0)=0.3$.

Thus, the change in the superconducting transition temperature due to the change in the Fermi-surface topology under the action of impurities has a most nontrivial character in an anisotropic superconductor: in some metals the effect of the scattering processes on the dependence of ΔT_c and dT_c/dP on c can be greater than that on the thermodynamic characteristics of the metal in the normal state, and in other metals the dependence of ΔT_c and dT_c/dP on c can be practically insensitive to the scattering of electrons by impurities. The insensitivity of the value of the maximum of dT_c/dP in indium to significant changes in the relative residual resistance is evidently connected with this latter circumstance.[6,7]

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APPENDIX

Equation (2) can be represented in the following symbolic form:

$$(1+\hat{H})\hat{\psi}=0, \quad \hat{\psi}=\begin{pmatrix}\mathrm{Re}\,\psi(\mathbf{p}, \omega)\\ \mathrm{Im}\,\psi(\mathbf{p}, \omega)\end{pmatrix}, \quad (A.1)$$

where $\hat{H}$ is an integral operator with a kernel of the form

$$H(\mathbf{p}, \omega; \mathbf{p}', \omega')=\frac{1}{\pi(2\pi)^3}\begin{pmatrix}\mathrm{Re}\,\overline{K}_+(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c)\\ \mathrm{Im}\,\overline{K}_+(\mathbf{p}, \mathbf{p}', \omega, \omega', T_c)\end{pmatrix}$$
$$\otimes\left(\mathrm{Im}\frac{1}{Q(\mathbf{p}, \omega)};\ \mathrm{Re}\frac{1}{Q(\mathbf{p}, \omega)}\right).$$

Along with Eq. (A.1), we can consider the transposed equation

$$\hat{\varphi}(1+\hat{H})=0,$$

where

$$\hat{\varphi}=(\varphi_1(\mathbf{p}, \omega);\ \varphi_2(\mathbf{p}, \omega)).$$

It is easy to see that $\hat{\varphi}$ has the following form:

$$\hat{\varphi}=\left(\mathrm{Im}\frac{1}{Q(\mathbf{p}, \omega)};\ \mathrm{Re}\frac{1}{Q(\mathbf{p}, \omega)}\right)\varphi(\mathbf{p}, \omega),$$

where the function $\varphi(\mathbf{p}, \omega)$ is found from the equation

$$\varphi(\mathbf{p}, \omega)=-\int\limits_0^\infty\frac{d\omega'}{\pi}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\mathrm{Im}\left\{\frac{\overline{K}_+(\mathbf{p}', \mathbf{p}, \omega', \omega, T_c)}{Q(\mathbf{p}', \omega')}\right\}\varphi(\mathbf{p}', \omega'). \quad (A.2)$$

This equation is identical with Eq. (9). We note that

Eq. (A.2) can serve for calculation of the temperature of the superconducting transition, and that it can turn out to be more convenient for the determination of T_c than Eq. (1). This circumstance is connected with the fact that the kernel of Eq. (A.2) at $\omega' \approx 0$ does not contain a negative-frequency part.

[1]It is shown in the Appendix that Eq. (9) is transposed with respect to Eq. (2) and that the condition for its solvability is identical with the the condition for solvability of Eq. (2).

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