Convection of plasma in the tokamak

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It is shown that allowance for finite conductivity in current-carrying plasma may lead to the development of a nonlinear convective flow which is connected with local overheating of individual plasma filaments. The time necessary to remove heat by this convection mechanism is found to be of the order of the skin time.

1. INTRODUCTION

Among the various toroidal containment systems for high temperature plasma, the tokamak machines are the most highly developed at the present time. Not only have the best parameters been achieved for them, but a degree of clarity has been reached in the understanding of the various phenomena taking place in the plasma. However, the main question, i.e., what determines the characteristic energy-loss time for plasma in the tokamak, has not as yet been answered. It is known that the energy is removed along the "electron channel," i.e., the energy losses are due to a mechanism of the type of enhanced electronic thermal conductivity.[1] The effective thermal conductivity exceeds by one or two orders of magnitude the neoclassical value even when the electron-ion collision frequency is calculated from measured electrical conductivity.

Artsimovich has pointed out that the quantity $\beta_\theta = 8\pi p/B_a^2$ remains roughly constant in a broad range of plasma parameters (p is the mean plasma pressure and B_a is the poloidal magnetic field on the boundary of the plasma column). For Joule heating of plasma, this means that the lifetime is $\tau_E \sim a^2\sigma$ where a is the radius of the plasma column and σ is the electrical conductivity. It may be said that the quantity $c^2/2\pi\sigma$ can serve as a scale for the temperature diffusivity in the tokamak plasma. However, this quantity is also the coefficient of diffusion of the magnetic field, and the fact that the loss of plasma is either independent of or not very dependent on the intrinsic plasma parameters (Larmor radii, ranges, etc.), and is governed only by the rate of transport of lines of force due to the finite conductivity, would suggest that convective motion of the plasma may be possible and may be connected somehow with the 'breathing' of the magnetic configuration. This problem is discussed in the present paper. We shall present qualitative arguments suggesting that the development of conductivity inhomogeneities in the form of 'overheated' current filaments may result in the transport of such filaments across the plasma column, and in the convective mixing which may lead to thermal conductivity of the order of $c^2/2\pi\sigma$.

2. HELICAL CONDUCTING FILAMENT IN PLASMA

Let us suppose that, as a result of instability associated with overheating, a well-conducting filament is formed in the plasma. We shall investigate the fate of this filament during its evolution. For the sake of simplicity, we shall replace the toroidal discharge by a straight-line discharge of length $L = 2\pi R$, where R is the major radius of the torus, and will suppose that all the quantities are periodic along the column with the period L. In particular, if the conducting filament is helical then its pitch is $h = mL/n$, where m and n are integers and m measures the number of circuits completed by the helical purturbation from which the filament has developed. Let B_0 and B_θ be, respectively, the longitudinal and azimuthal components of the magnetic field. We note that the pitch of the lines of force is equal to Lq, where $q = 2\pi r B_0/RB_\theta$.

Let us begin by considering the simplest case, i.e., the case of a simple conducting filament in vacuum, for example, outside the plasma. Let E be the longitudinal electric field that produces the current in the plasma column. It is clear that, if a current flows through the conducting filament, it experiences the Ampere force in the radial direction and very rapidly begins to move in that direction. During this motion the filament cuts the lines of force and, as a result of the change in the magnetic flux linked with the filament, the current in the filament falls rapidly to zero. When this happens, the motion of the filament becomes slower: it is displaced in the radial direction with a velocity which is such that the induced emf compensates the external field:

$$\frac{1}{c}\dot{\Phi}=Eh, \qquad (1)$$

where h is the pitch of the helix, and

$$\Phi=\pi r^2 B_0-h\int_0^r B_\theta\, dr$$

is the magnetic field linked with the filament per turn. From (1) we can find the quantity $v = \dot{r}$:

$$v=\frac{cE}{B_\theta}\frac{m}{nq-m}. \qquad (2)$$

If the current $j = \sigma E$ is nearly uniform then (2) may be written in the form

$$v\cong\frac{c^2}{2\pi\sigma}\frac{m}{nq-m}. \qquad (3)$$

It is clear from (3) that the sign of v depends on the relationship between the pitch of the helix $h = mL/n$ and the pitch of the lines of force qL. When $h < qL$ the filament moves in the outward direction. The velocity of the filament increases as h approaches qL.

Let us now consider a well-conducting helical filament placed in a plasma column carrying a longitudinal current. Suppose that at the initial time the filament carries a current j_0. In that case, if the pitch of the lines of force at the given point is not equal to the pitch of the filament, the current-carrying filament will experience the Ampere force. Now, however, during the rapid radial displacement of the filament, the current through it will not vary because the field is frozen-in. Moreover, it is readily seen that a current with the opposite direction will be induced in the plasma during the motion of the filament. This phenomenon is illustrated qualitatively in Fig. 1 which shows the lines of force associated with a field which is transverse relative to

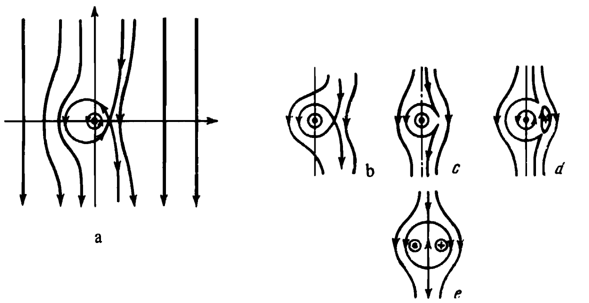

FIG. 1

the filament. Figure 1a shows the lines of force in the initial state and Figs. 1b–e illustrate how the reverse current is generated during the motion of the filament, and is subsequently smeared out over the cross section due to the finite conductivity so that the field distribution becomes almost uniform (Fig. 1e).

3. FLOWS WITH HELICAL SYMMETRY

To obtain a quantitative description of the motion of the current-carrying filament in plasma for $B_0 \gg B_\theta$ we can use the simplified set of equations of magnetic hydrodynamics for flows with helical symmetry.[2] All the quantities then depend only on the cylindrical variables r, θ, z in the following combinations: r, $\theta - \alpha z \equiv \theta - 2\pi nz/Lm$. The magnetic field is conveniently expressed in terms of the function ψ which is defined by

$$\frac{\partial\psi}{\partial r}=B_\theta-\alpha rB_0, \quad \frac{1}{r}\frac{\partial\psi}{\partial\theta}=-B_r. \tag{4}$$

It is shown in[2] that the variation in the magnetic field with time when $B_0 \gg B_\theta$ and the problem has helical symmetry is described by

$$\frac{4\pi\sigma}{c^2}\left(\frac{\partial\psi}{\partial t}+\mathbf{v}\nabla\psi\right)=\Delta\psi+2\alpha B_0-\frac{4\pi\sigma}{c}E, \tag{5}$$

where $\mathbf{v}$ is the velocity of the plasma across the column, which must be determined from the equations of motion

$$m_i n\frac{d\mathbf{v}}{dt}+\nabla p=\frac{1}{c}[\mathbf{j}\times\mathbf{B}], \tag{6}$$

m_i is the ion mass, and p the plasma pressure. However, in the case of slow plasma flows (in which we are mainly interested) the inertial term in (6) can be neglected, i.e., we should in fact be considering the evolution in time of the equilibrium states of plasma, which is due to the finite conductivity.

In the case of helical symmetry and $B_0 \gg B_\theta$, the equation of equilibrium is substantially simplified. In particular, it is shown in[2,3] that it reduces to the form

$$\nabla p_*=\Delta\psi\nabla\psi, \tag{7}$$

where p_* is a certain scalar function (the effective pressure). From (7) it follows that $p_* = p_*(\psi)$ and

$$\Delta\psi=F(\psi), \tag{8}$$

where F is an arbitrary function of ψ.

The velocity $\mathbf{v}$ in (5) must be chosen in accordance with (8), i.e., so that the Laplacian $\nabla^2\psi$ is always constant on the surfaces ψ = const. Moreover, we must satisfy the incompressibility condition

$$\operatorname{div}\mathbf{v}=0, \tag{9}$$

which follows from the fact that the magnetic field is frozen-in, and the perturbation of the strong longitudinal magnetic field for slow plasma motions is small, so that B_0 = const.

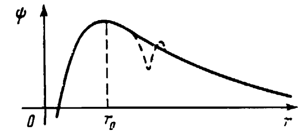

FIG. 2

Thus, plasma flows with helical symmetry are described by (5) and (8) together with the incompressibility condition. It is important to note that these equations are two-dimensional (all the quantities depend only on r and θ), and the velocity vector lies in the (r, θ) plane.

4. CONVECTIVE TRANSPORT OF A CURRENT-CARRYING FILAMENT

We shall now give a quantitative description of the plasma flow due to the motion of the overheated current-carying filament.

In the initial state, the plasma column has cylindrical symmetry and the function $\psi(r)$ is determined by the equation

$$\frac{d\psi}{dr}=B_\theta\left(1-\frac{n}{m}q(r)\right). \tag{10}$$

The function is illustrated schematically in Fig. 2.

We are considering the case where the singular point $q(r_0) = m/n$ lies inside the plasma column. In the case of a monotonic function $q = q(r)$, this corresponds to the hydrodynamically stable situation.

Let us now return to (5). In the equilibrium, cylindrically symmetric, state, the last three terms on the right of (5) cancel each other out. We now suppose that local overheating takes place in some small region in the plasma. This means that the right hand side of (5) now acquires a negative source $Q = 4\pi c^{-1}\Delta\sigma E$, where $\Delta\sigma$ is the change in σ due to the overheating. A large current will flow in the overheated region. However, as indicated above, such a current will necessarily be displaced in the transverse direction and will excite a reverse current (see Fig. 1). The corresponding local distortion of ψ is shown by the dashed lines in Fig. 2.

However, this formation cannot remain at rest either. The point is that (5) is a diffusion equation and therefore a 'bump' on the ψ curve (Fig. 2), which corresponds to the induced reverse current, will tend to spread out. However, the overheated filament can be in equilibrium with the plasma only in the presence of the induced reverse current, since it continues to move in order to compensate inductively the reverse-current dissipation. It is readily seen that the result of all this is that the current configuration in Fig. 3 will be displaced toward the right, cutting (and tending to close on itself) the lines of force. In the coordinate system attached to the current, the flux ψ will be incident from the right, and the outer part of the surfaces will break up and will jump to the left. At the same time, since the current filament is a source of ψ, it will generate the flux ψ between the filament and the boundary of the bundle, and these surfaces will join the outer surfaces on the separatrix. Conversely, on the right half of the bundle,

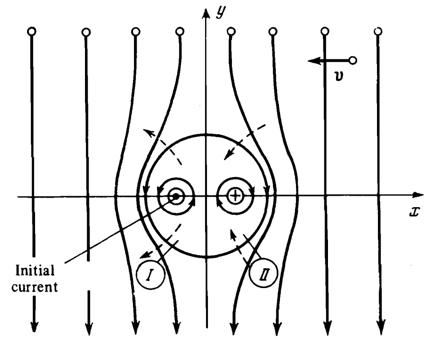

FIG. 3

the flux will be directed inward, as shown by the broken arrows in Fig. 3.

As the region on the right, which contains the colder plasma, is approached, the conductivity perturbation due to the overheating tends to increase. The excess of the conductivity above the background will increase, and the 'cavity' in Fig. 3 will tend to increase in size.

The foregoing discussion refers to the region $d\psi/dr < 0$ (on the right of the point r_0 in Fig. 2). In the region $d\psi/dr > 0$ the overheated bundle with enhanced current should move toward the center, i.e., into the region of higher temperature and greater current. Naturally, the difference between the temperature and current in the bundle and the ambient region will fall during this process, i.e., the perturbation will gradually disappear.

To calculate the resulting plasma flow we must simplify the problem. Thus we shall assume plane geometry with uniform magnetic field at infinity, lying along the y axis: $\psi = (d\psi_0/dx)x \equiv \Gamma x$. We shall transform to the set of coordinates in which the overheated region is at rest and the plasma is incident upon it with velocity $-v_0$ which is constant at infinity (see Fig. 3). In this coordinate frame, the initial set of equations assumes the form

$$\frac{4\pi\sigma}{c^2}(-v_0\Gamma+\mathbf{v}\nabla\psi)=\Delta\psi-g\delta(\mathbf{r}), \tag{11}$$

$$\Delta\psi=F(\psi), \tag{12}$$

$$\operatorname{div}\mathbf{v}=0. \tag{13}$$

We are assuming for the sake of simplicity that the linear size of the overheated region is small, and replace the corresponding source on the right of (11) by a δ function.

In general, the overheating will ensure that the local perturbation of the current will continue to increase, but we shall simplify the problem and consider the flow of a constant current over the plasma. The point is that the transport of the overheated current-carrying region in the outward direction, and its heating, take place in comparable intervals of time and, therefore, in the first approximation the heating effect can be neglected.

It is convenient to rewrite (11) in terms of dimensionless coordinates. This will be done by introducing characteristic quantities with the dimensions of velocity and length: $v_0 = \lambda(c^2/4\pi\sigma g)\Gamma$, $l = g/\Gamma$, so that in terms of the dimensionless variables $\mathbf{u} = \mathbf{v}/v_0$, $\nabla\psi = \nabla\psi'/\Gamma$, $\mathbf{r} = l\boldsymbol{\rho}$, Eq. (11) assumes the form

$$\lambda(-1+\mathbf{u}\nabla\psi')=\Delta\psi'-\delta(\boldsymbol{\rho}), \tag{14}$$

where λ is the characteristic parameter of the problem. We note that v_0 is equal to the velocity with which the plasma is incident on the overheated filament from infinity.

Let us return again to Fig. 3. There are three natural regions. There is the outer region (relative to the separatrix) and two internal regions I and II. The initial current lies in region I and the reverse current is generated in region II.

Let us begin by considering the region which is external to the separatrix. In this region, the lines of force depart to infinity, and at a sufficiently large distance from the filament we have $\Delta\psi = 0$ and, therefore, $F(\psi) = 0$ [see (12)]. However, in that case $F(\psi) \equiv 0$ on all the lines of force which are external to the separatrix and, consequently, in the external region too. Thus, in the external region, (14) reduces to

$$\lambda(-1+\mathbf{u}\nabla\psi)=0. \tag{15}$$

Its general solution is

$$\mathbf{u}=\frac{\nabla\psi}{(\nabla\psi)^2}+b[\mathbf{e}_z\times\nabla\psi]. \tag{16}$$

The second term in (16) describes the outflow or inflow of plasma along the lines of force, and does not lead to flow across the magnetic field. It is uniquely determined by the incompressibility conditions $\operatorname{div}\mathbf{v} = 0$ and $\mathbf{v} = \mathbf{v}_0$ at infinity.

In the internal regions I and II we may suppose that $\mathbf{u} = 0$. The point is that, in these regions, the quantity $\mathbf{u}\nabla\psi$ is a constant along the ψ = const lines in accordance with (14) and (12). When $\mathbf{u}\nabla\psi \neq 0$, the conclusion might be that the fluid flows into and vanishes in region II, whilst it is created in region I. However, this is in conflict with the condition $\operatorname{div}\mathbf{v} = 0$.

Since $\mathbf{u} = 0$, we find that in the internal regions I and II, Eq. (11) reduces to the following:

$$\Delta\psi=\lambda-\delta(\boldsymbol{\rho}) \tag{17}$$

in region I, and

$$\Delta\psi=\lambda \tag{18}$$

in region II.

To ensure that regions I and II are in equilibrium with the plasma, the total currents flowing through I and II must be equal and opposite, i.e.,

$$-\int(\lambda-\delta(\boldsymbol{\rho}))\,dS=\int\lambda\,dS.$$

This leads to the condition

$$\lambda S=1, \tag{19}$$

where S is the total area of regions I and II. Therefore, to find the characteristic value λ we must find S. In general, the shape and size of regions I and II are determined by (11) and (12) together with the boundary conditions which ensure that the magnetic pressure inside and outside of I and II are equal. It may be supposed that the shape and size of S are not very sensitive to the current distribution inside the overheated region. We shall therefore simplify the problem somewhat by replacing the distributed currents in (17) and (18) by equal point currents.

The shape and size of the separatrix, bounding regions I and II, can readily be found for such currents. It turns out to be very nearly circular, and its area is $S = 1/16\pi$. In other words, its diameter in terms of our dimensionless coordinates is $d = 1/2\pi$. From (19) we then find the characteristic value

$$\lambda = 16\pi. \tag{20}$$

Substituting this into the expression for the velocity at infinity, we can find v_0 from the formula $v_0 = \lambda(c^2/4\pi\sigma g)\Gamma$.

It is more convenient, however, to introduce a quantity which has the dimensions of the coefficient of temperature diffusivity $\chi = v_0 d_0$, where d_0 is the diameter of the nearly circular region of localization of currents in the usual units, i.e., $d_0 = l d$, so that

$$\chi = 2c^2/\pi\sigma. \tag{21}$$

It may be supposed that the temperature diffusivity will reach a value of the same order in the plasma column when the latter is sufficiently densely packed with the overheated filaments.

As can be seen, χ is independent of the local current perturbation and hence of the diameter d_0 of the filament. A large current moves more slowly but occupies a larger area, whilst a small current moves more rapidly but its size is smaller.

So far, in order to be specific, we considered an overheated bundle of plasma with enhanced current. The analysis will of course be also valid for a bundle in which the temperature is lower than that of the ambient medium, except that the bundle will move in the opposite direction. Thus, overheated bundles will travel in the outward direction, whilst underheated ones will travel from the surface of the plasma in the inward direction. The result of all this will be that the plasma will contain formations which are very similar to knots in wood. The only difference is that ordinary knots grow in the transverse direction, whilst ours lie along the "trunk." We shall use the work "knot" for the sake of brevity henceforth.

As we have seen, the formation of a knot leads to the breaking up of the magnetic surfaces. As a result, a plasma column containing moving knots is only very remotely similar to the idealized symmetric plasma column which is usually investigated theoretically.

We shall now consider what happens when two knots, one overheated and one underheated, encounter one another. If the currents in the two of them are equal and opposite, and they move along the same radius, the resulting configuration will be stationary. The lines of force will assume the same configuration as in Fig. 3, except that regions I and II will contain external sources of equal strength but opposite sign. Such a configuration will, of course, be very unstable. As soon as one of the currents undergoes a slight change, the entire configuration will move in the inward or outward direction, depending on the sign of the resultant source. However, it is convenient for explaining the conditions which are necessary for the existence of the knots. We must therefore consider it in somewhat greater detail.

Firstly, it is more convenient to express the source $Q = 4\pi c^{-1}\Delta\sigma E$ connected with overheating $(\Delta\sigma > 0)$ or underheating $(\Delta\sigma < 0)$ of a knot in terms of the current density $Q = 4\pi\Delta\sigma/c\sigma) j_z$ in the column. Next, since we have a source in both regions I and II (with different signs), we now have the following equations instead of (17) and (18):

$$\Delta\psi = \lambda - \delta(\rho + l_0/2) \tag{22}$$

and

$$\Delta\psi = \lambda + \delta(\rho - l_0/2), \tag{23}$$

where l_0 is the distance between the centers of the currents, which we again replace by δ functions.

Since the currents are equal we have

$$-\int(\lambda-\delta)\,dS = \int(\lambda+\delta)\,dS$$

from which it follows at once that $\lambda = 0$. This is to be expected because this result signifies that the velocity of the knot is zero.

Transforming back to dimensional variables, we can find the relation between the knot diameter d_0 (in ordinary units) and the strength of the source Q:

$$d_0 = 4\frac{d\psi/dr}{Q} = 4B_\theta(m-nq)\left(\frac{4\pi}{c}j_z\frac{\Delta\sigma}{\sigma}m\right)^{-1}. \tag{24}$$

Since

$$\frac{1}{r}\frac{d}{dr}rB_\theta = \frac{4\pi}{c}j_z,$$

and assuming that $B_\theta \simeq 2\pi c^{-1} j_z r$, we can rewrite this condition in the form

$$\frac{d_0}{r} \simeq 2\frac{m-nq}{m}\frac{\sigma}{\Delta\sigma}. \tag{25}$$

The knot diameter cannot of course exceed the radius of the column, i.e., $d_0/r < 1$. This condition imposes the following restriction on conductivity fluctuations for which the appearance of the knot is possible:

$$\frac{\Delta\sigma}{\sigma} \gg 2\frac{m-nq}{m}. \tag{26}$$

Thus the knot can appear only near resonance surfaces, in the neighborhood of which condition (26) reduces to the following: $\Delta\sigma/\sigma \gg 4rq'/q$. Their subsequent motion in the radial direction is entirely connected with overheating (or cooling) which is described by the heat balance equation.

5. CONVECTION OF PLASMA

The appearance of regions with enhanced or reduced electrical conductivity can thus lead to the formation of locally closed magnetic surfaces (knots) which can move in the transverse direction in the plasma. This mechanism naturally leads to thermal conduction and diffusion for which the scale is the magnetic viscosity $c^2/2\pi\sigma$. The associated possible convection mechanism is the most natural variant of diffusion when its magnitude is determined by the rate of transport of the lines of force. The possibility of such diffusion was considered some time ago by Grad.[4]

This type of convection would appear to correspond to the observed loss of plasma in tokamaks. However, it is not at all obvious that this type of convection does in fact occur. It was shown above that the knots exist only for sufficiently strong relative perturbations $\Delta\sigma/\sigma$. On the other hand, for small perturbations, the magnetic surfaces tend to self-heal, i.e., there is a restoration of configurations of the form of toroidal surfaces inserted one into another. Even near singular points, where the lines of force close, the appearance of the knots requires sufficiently strong overheating instability. One cannot, of course, exclude the possibility that strong knot-type perturbations may develop on the periphery of the plasma in the region where the plasma is in contact with the diaphragm. In this case, the loss of plasma is very dependent on the conditions on the plasma boundary (which is, by the way, observed in practice).

6. CONCLUSION

We have shown that conventional ideas on magnetic surfaces in tokamaks, in which they are looked upon as circular toroidal surfaces inserted one into another, are not always correct. In particular, overheating-type instabilities, or strong conductivity perturbations in the region where the plasma is in contact with the diaphragm, may lead to knot-type perturbations, i.e., closed bundles forming an isolated system of surfaces inserted into each other. Such bundles should be transported across the plasma and, when the cross section is filled with them to a sufficient density, they may lead to thermal conductivity and diffusion with the scale $c^2/2\pi\sigma$. The experimentally observed transport is precisely of this order of magnitude, but further detailed and largely experimental studies will be necessary before any firm conclusion can be drawn as to whether this plasma-current convection mechanism is in fact the determininig factor.

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