Changes in Cerenkov radiation caused by an external field

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(Submitted September 27, 1973)
Zh. Eksp. Teor. Fiz. **66**, 1258–1268 (April 1974)

In a number of earlier papers it has been asserted that it is possible for the Cerenkov radiation spectrum to be changed by moderate external fields. Both from general considerations and from two examples (helical motion of an electron in a medium and motion in the field of a plane electromagnetic wave) discussed in this paper, it can be seen that 1) the effect of a field on the Cerenkov radiation spectrum decreases as the energy of the particle increases and 2) even for nonultrarelativistic particles extremely strong fields must be applied for any considerable effect to be produced on the radiated spectrum if the intensity of the Cerenkov radiation is not excessively small.

1. INTRODUCTION

The intensity of the radiation of a classical particle moving in a constant magnetic field is described by a remarkably simple formula. The spectral and angular distributions which it gives have been thoroughly studied.[1-6] 1) It is interesting to generalize these formulas to include the case of motion of an electron in a medium by introducing the index of refraction $n(\omega)$. For the case of motion in a circle this has been done in [9-11]. The case of helical motion is considered in [12,13].

In the present paper we consider in detail the case of motion of an electron along a helix. The corresponding formula for the intensity of the radiation is studied for various ratios between the parameters of the problem, and the spectral distribution in each case is compared with that for Cerenkov radiation. This problem is interesting both in itself and also from a methodological point of view, as the simplest example of undulatory radiation, i.e., radiation of a particle with a vibratory motion superposed on its translatory motion. The study of radiation from various undulatory motions was initiated in papers by Ginzburg.[14] Closely related to the class of undulatory radiations is the radiation from a particle passing through a medium with a periodically varying index of refraction. Extensive studies in this field are well expounded in a monograph by Ter-Mikaelyan.[15]

In cases when the speed of motion of a particle in a medium exceeds that of light, $vn(\omega) > 1$, the question arises as to the change of the Cerenkov radiation under the influence of the oscillatory motion (the acceleration). Frequently the oscillatory motions caused by the external field are small. Accordingly there is little change of the velocity of translational motion of the particle and we can expect only small changes of the Cerenkov radiation. If, however, we are interested in fine details of the radiation, for example details of its angular distribution, they may easily turn out to be changed. The single cone of radiation now splits up into a system of cones, and it is quite possible that the radiation in each of the cones close to the central one is of the same order as that in the central cone. Unfortunately, only the radiation in the central cone is called Cerenkov radiation in the literature (cf., e.g., [13,16-18]). With this definition this radiation is indeed easily changed by external fields of moderate intensity. However, summation over the cones (integration over angles) as a rule reestablishes the Tamm-Frank formula.2)

From the experimental point of view it is precisely the rough characteristics of the radiation that it is convenient to deal with, because, for example, a smearing out of the cone of radiation can be caused by many other things (spread of momenta of the particles in the beam, multiple scattering, finite path length of the radiating particle, etc.). On the other hand, it is clear that when the intensity of the Cerenkov radiation is sufficiently small, i.e., when $vn(\omega) - 1 \ll 1$, even a moderate acceleration can cause a change of the radiation.

The condition for a field to affect the Cerenkov radiation can be obtained from simple qualitative considerations. In the idealized case the path of the radiating particle is straight and infinite, and the radiation at a given frequency goes out only at the Cerenkov angle ϑ_0, with $\cos\vartheta_0 = 1/vn(\omega)$. If, on the other hand, the path of the particle is finite, the diffraction spreading of the cone of radiation with frequency ω is (see Chap. 3, Sec. 8 in [21])3):

$$\Delta\vartheta = 1/\omega L \sin\vartheta_0. \tag{1}$$

For $\Delta\vartheta \ll \vartheta_0$ the situation is close to the idealized case $L = \infty$. For $\Delta\vartheta \sim \vartheta_0$ we must expect decided changes in the characteristics of the radiation, in particular to the spectrum. Accordingly, the minimum necessary length for the formation of Cerenkov radiation is

$$L_m = 1/\omega \sin^2\vartheta_0. \tag{2}$$

Owing to this it will be everywhere understood that the path of the radiating particle is much larger than L_m.

Suppose there is an external field causing oscillations of the particle with frequency Ω. We define a length L_F in which the field deflects the particle by an angle $\sim\vartheta_0$, namely:

$$\frac{\Delta p}{p} = \frac{F_\perp L_F}{p} = \sin\vartheta_0, \quad \mathbf{F} = e(\mathbf{E} + [\mathbf{v}\times\mathbf{H}]). \tag{3}$$

Here $F_\perp$ is the component of the force perpendicular to the motion and p is the momentum of the particle. It is clear that the field does not change the Cerenkov radiation much if

$$L_m = 1/\omega \sin^2\vartheta_0 \ll L_F = p\sin\vartheta_0/F_\perp, \quad \Omega L_F \lesssim 1. \tag{4}$$

Also in the case

$$\Omega L_F = \Omega p \sin\vartheta_0/F_\perp \gg 1 \tag{5}$$

the angular deflections of the particle in its periodic oscillations are always small in comparison with the angle ϑ_0. Then the change of the Cerenkov radiation is small simply because the amplitude of the oscillations is small.

We have so far tacitly assumed that the change of v^2 in a length L_m can be neglected in comparison with $v^2 n^2(\omega) - 1$. This condition is satisfied if

$$F_\perp/p_0\omega \sin^3\vartheta_0 \ll 1, \quad \Omega L_m < 1, \tag{6}$$

$$(1-v^2)F_\parallel/p_0\omega\sin^4\vartheta_0\ll 1,\quad \Omega L_m<1,$$
$$v=p/p_0,\quad F_\parallel=eE_\parallel. \qquad (7)$$

For $\Omega L_m \gg 1$ fields that cause considerable change of the Cerenkov radiation are so large that it is scarcely worth while discussing this case.

In a magnetic field the velocity of the particle does not change in magnitude and we can expect departure of the spectrum from the Cerenkov form if the condition (4) is violated. For $p_0/m\sim 1$, $\sin\vartheta_0\sim 1$, and $\omega/m\sim 10^{-5}$ this requires fields $H\gtrsim 10^8$ G.

The radiation of an electron moving in the field of a plane monochromatic wave in a medium is briefly discussed in Sec. 6.

2. THE INTENSITY OF THE RADIATION

The energy spectrum of the radiation is given by the expression

$$d\mathscr{E}_{\mathbf{k}}=\frac{\pi|e'\cdot j(k)|^2 d^3k}{n(\omega)[n(\omega)+\omega\, dn(\omega)/d\omega]},$$
$$d^3k=k^2\,d\varphi\,dt\left[n(\omega)+\omega\frac{dn(\omega)}{d\omega}\right]d\omega,\qquad t=\cos\vartheta,$$
$$k=|\mathbf{k}|=n(\omega)\omega; \qquad (8)$$
$$j_\mu(k)=\frac{e}{(2\pi)^2 m}\int_{-\infty}^{\infty}ds\,\pi_\mu(s)\exp[ik\cdot x(s)].$$

Here k_μ and e'_μ are the four-momentum and polarization of the photon; $\pi_\mu(s)=dx_\mu(s)/ds$ is the kinetic momentum of the particle, s being the proper time; the dot indicates products of four-vectors.

For a constant field the expression for $j_\mu(k)$ can be found in [8]. Assuming that there is a magnetic field only, directed along the 3 axis, and summing over the polarization e′, we get

$$\sum_{e'}|e'\cdot j(k)|^2=\frac{e^2}{(2\pi)^3}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{p_\parallel^2}{m^2}-\gamma^2\frac{\omega^2}{k^2}+\frac{n^2\eta^2}{k_\perp^2}\right)J_n^2(\rho)\right.$$
$$\left.+\frac{p_\perp^2}{m^2}J_n'^2(\rho)\right\}S\delta(a+n\eta),\qquad a=\frac{k_\parallel p_\parallel-\omega p_0}{m} \qquad (9)$$
$$\gamma=p_0/m,\quad \eta=eH/m,\quad \rho=k_\perp p_\perp/\eta m,\quad k_\perp^2=k_1^2+k_2^2,$$
$$k_\parallel=k_3,\quad 2\pi\delta(0)=S.$$

Here $p_\mu=\pi_\mu(0)$ is the four-momentum of the electron for $s=0$ (actually there are components independent of s). The intensity spectrum is determined from Eqs. (8) and (9) as $d\mathscr{E}_k/T$, where $T=\gamma S$ is the total time of motion of the particle. The integration over the azimuthal angle in Eq. (8) gives 2π, and the integration over the polar angle ϑ reduces to removing the δ function in Eq. (9); after this the summation over n [see Eq. (11) below] amounts to a summation over allowed values of ϑ. Using the well known relation (cf., e.g., page 269 in [4])

$$J_n'^2(\rho)=\left[\frac{1}{2}\left(\frac{1}{\rho}\frac{d}{d\rho}+\frac{d^2}{d\rho^2}\right)+\left(1-\frac{n^2}{\rho^2}\right)\right]J_n^2(\rho), \qquad (10)$$

we have finally

$$\frac{d\mathscr{E}_\omega}{T}=\frac{e^2}{4\pi}v\frac{V}{V_\parallel}\sum_{[n_-]}^{[n_+]}\left\{\left(1-\frac{1}{V^2}\right)+\frac{V_\perp^2}{2V^2}\left(\frac{1}{\rho}\frac{d}{d\rho}+\frac{d^2}{d\rho^2}\right)\right\}J_n^2(\rho)\,\omega\,d\omega, \qquad (11)$$

$$\mathbf{V}=\mathbf{v}n(\omega),\quad n_\pm=\frac{\omega}{\omega_0}(1\pm V_\parallel),\quad \rho=\frac{k_\perp p_\perp}{\eta m}=\frac{\omega}{\omega_0}V_\perp\sin\vartheta,$$
$$\omega_0=\eta/\gamma=eH/p_0,\qquad t=\cos\vartheta=\frac{1}{V_\parallel}\left[1-\frac{n\omega_0}{\omega}\right]. \qquad (12)$$

Here $[n_-]$ ($[n_+]$) is the next integer above (below) n_- (n_+), and ω_0 is the frequency of gyration of the electron in the magnetic field. Since the electron's velocity **v** is as a rule multiplied by $n(\omega)$, it is convenient to denote the product by **V**, keeping in mind that **V** depends on ω.

If the field is so large that $\omega/\omega_0\sim 1$, then there are only a few value s of n included in Eq. (11). We can then get all the information directly from this formula. At present, however, the most interesting case is $\omega/\omega_0\gg 1$, and we shall deal mainly with it. Then the important values of n in Eq. (11) can be those "in the neighborhood of zero" and also large positive values. These cases will be considered separately. We shall find that under the restrictions corresponding to Eqs. (4) and (5) the Tamm-Frank (hereafter written TF) formula follows from Eq. (11). [4)]

3. THE CASE OF SMALL $V_\perp$

By definition we regard $V_\perp$ as small if

$$V_\perp^2\ll V^2-1=V_\parallel^2-1+V_\perp^2\approx V_\parallel^2-1. \qquad (13)$$

Applying Eq. (5) for the case of a magnetic field, we see that it reduces to Eq. (13), since $\Omega=\omega_0$, $p\approx p_0$, $F_\perp=eHV_\perp/V$. Accordingly, Eq. (11) should reduce to the TF formula. This is rather natural, since small $V_\perp$ corresponds to weak interaction of the electron with the field (small Lorentz force). According to Eqs. (11) and (13) we have

$$-n_-=\frac{\omega}{\omega_0}\frac{V_\parallel^2-1}{V_\parallel+1}\approx\frac{\omega}{\omega_0}\frac{V^2-1}{V_\parallel+1}. \qquad (14)$$

Since $V^2-1>0$ (so that there can be Cerenkov radiation), $-n_->0$, i.e., the sum over n in Eq. (11) contains a term with $n=0$. According to Eq. (12),

$$\sin^2\vartheta=1-\frac{1}{V_\parallel^2}+\frac{2n\omega_0}{V_\parallel^2\omega}-\left(\frac{n\omega_0}{V_\parallel\omega}\right)^2. \qquad (15)$$

The effective values of n in Eq. (11) are determined by the effective values of ρ, i.e., the effective value of $\sin\vartheta$, and $n_{\rm eff}\sim\rho_{\rm eff}$. With the condition (13) it is natural to expect that

$$\sin^2\vartheta_{\rm eff}\approx V^2-1,\qquad \rho_{\rm eff}\approx\frac{\omega}{\omega_0}V_\perp(V^2-1)^{1/2}. \qquad (16)$$

Then

$$|n_{\rm eff}|\sim\rho_{\rm eff}\ll -n_- \qquad (17)$$

and substitution of $n_{\rm eff}$ in Eq. (15) shows that for the summation over such values of n the assumption (16) is justified. Then we have (cf. [19,20])

$$\sum_{[n_-]}^{[n_+]}J_n^2(\rho)\approx\sum_{n=-\infty}^{\infty}J_n^2(\rho)=1. \qquad (18)$$

We can neglect the term with V^2 in the curly bracket in Eq. (11), and moreover it vanishes when we differentiate (18) with respect to ρ. The result is that we get the TF formula [according to Eq. (13) $V_\parallel\approx V$].

It should now be noted that the quantity $\rho_{\rm eff}$ in Eq. (16) is allowed to be of the order of unity (or larger) if $-n_-\gg 1$. Therefore, if we define Cerenkov radiation as the term with $n=0$ in Eq. (11), we can say that it is essentially suppressed. However, it is precisely for $-n_-\gg 1$ that it is hard to distinguish channels with different n, because $\sin^2\vartheta$, according to Eq. (15), does not change in the summation over the effective values of n.

4. THE CASE OF MODERATE V

Here by hypothesis

$$V_\perp^2\sim V^2-1. \qquad (19)$$

Then $\omega_0 L_F \sim 1$, according to Eq. (4). The answer to the question of the effect of the field is given by the value of the quantity

$$\frac{\omega p \sin^3 \vartheta_0}{eHV_\perp} \sim \frac{\omega}{\omega_0 V_\perp}(V^2-1)^{3/2} \sim \frac{\omega}{\omega_0} V_\perp^2. \qquad (20)$$

According to (4), for $V_\perp^2 \omega/\omega_0 \lesssim 1$ the field does affect the radiation and for $V_\perp^2 \omega/\omega_0 \gg 1$ it does not. Let us examine these two cases.

1. $V_\perp^2 \omega/\omega_0 \lesssim 1$. Then $V_\perp^2 \lesssim \omega_0/\omega \ll 1$ and $V_\parallel \approx 1$, since $1 - V_\parallel^2 = V_\perp^2 - (V^2 - 1)$. It can be seen from Eq. (14) that $|n_-| \lesssim 1$. For $n \sim 1$ we have

$$\rho^2 \sim 2n\omega V_\perp^2/\omega_0 \sim 1, \quad n \neq 0. \qquad (21)$$

This means that n_{eff}, $\rho_{eff} \sim 1$. In fact, for $n \gg 1$ the function $J_n(\rho)$ falls off exponentially with increasing n, which can be verified easily by means of the formula [cf. Eq. (30) of Sec. 7.13 in [23]]:

$$J_n(\rho) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/3} \Phi(y) \exp\left[n\left(\operatorname{th}\alpha - \alpha + \frac{1}{3}\operatorname{th}^3\alpha\right)\right],$$

$$\Phi(y) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dx \exp\left[i\left(xy + \frac{x^3}{3}\right)\right], \quad y = \left(\frac{n}{2}\right)^{2/3} \operatorname{th}^2\alpha, \qquad (22)$$

$$\operatorname{th}^2\alpha = 1 - \frac{\rho^2}{n^2}, \quad \Phi(y)|_{y \gg 1} \approx \frac{1}{2y^{1/4}} \exp\left(-\frac{2}{3} y^{3/2}\right).$$

Here $\Phi(y)$ is the Airy function. The extremum of the function $\sin^2\vartheta/n^2$, which determines the dependence of ρ^2/n^2 on n, lies at

$$n_m = \frac{\omega}{\omega_0}(1 - V_\parallel^2) = (1 + V_\parallel) n_-,$$

so that $\rho^2/n^2 \ll 1$ for $n \gg 1$ both for $V_\parallel < 1$ and for $V_\parallel > 1$. Accordingly, by Eq. (11), the field does have an effect.

In working with Eq. (11) it is convenient to use the representation (see p. 269 in [4])

$$J_n^2(\rho) = \sum_{s=0}^{\infty} (-1)^s \frac{(2n+2s)!\, \rho^{2(n+s)}}{s!\, 2^{2(n+s)} (2n+s)! [(n+s)!]^2}. \qquad (23)$$

Now even for $V_\perp^2 \omega/\omega_0 \ll 1$ it is not permissible to neglect the term with $V_\perp^2$ in the curly bracket in Eq. (11). In fact, here $V_\parallel^2 - 1 = (V^2 - 1) - V_\perp^2$ can be either larger or smaller than zero, but in any case

$$|n_-| = \frac{\omega}{\omega_0} \frac{|V_\parallel^2 - 1|}{V_\parallel + 1} \lessapprox \frac{\omega}{\omega_0} V_\perp^2 \ll 1.$$

For $V_\parallel > 1$ the terms with $n = 0$ and $n = 1$ survive:

$$\frac{d\mathscr{E}_\omega}{T} = \frac{e^2}{4\pi} v \left\{\left(1 - \frac{1}{V^2}\right) - \frac{V_\perp^2}{2}\right\} \omega\, d\omega, \quad V_\perp^2 < V^2 - 1. \qquad (24)$$

For $V_\parallel < 1$ there remains only the term with $n = 1$:

$$\frac{d\mathscr{E}_\omega}{T} = \frac{e^2}{4\pi} v \frac{V_\perp^2}{2} \omega\, d\omega, \quad V_\perp^2 > V^2 - 1. \qquad (25)$$

If indeed $V_\parallel = 1$, i.e., $V_\perp^2 = V^2 - 1$, then Eqs. (24) and (25) are identical. (V is replaced by unity wherever this is possible.)

2. Now let $V_\perp^2 \omega/\omega_0 \gg 1$. Then at $n \sim 1$ the quantity ρ^2 is already large [cf. Eq. (21)]. This means that ρ_{eff}, $n_{eff} \gg 1$. For $n \gg 1$ we can use the asymptotic form of the Bessel function $J_n(\rho)$ [see Eq. (28) of Sec. 7.13 in [23]]

$$J_n(\rho) = \frac{w}{\sqrt{3}}\left[J_{1/3}\left(\frac{nw^3}{3}\right)\cos\delta - Y_{1/3}\left(\frac{nw^3}{3}\right)\sin\delta\right] + O(n^{-1}),$$

$$\delta = n\left[w - \frac{w^3}{3} - \operatorname{arctg} w\right] + \frac{\pi}{6}, \quad w^2 = \frac{\rho^2}{n^2} - 1, \quad \rho > n. \qquad (26)$$

Since a large number of terms contributes to the sum over n in Eq. (11), we can change from the sum over n to an integral over $t = \cos\vartheta$ or a variable u related linearly to t:

$$n = \frac{\omega}{\omega_0}(1 - V_\parallel t) = \frac{\omega}{\omega_0}\left(\frac{V_\perp^2}{V^2} - V_\parallel u\right). \qquad (27)$$

In terms of u we have

$$w^2 = \frac{V^2(u_0^2 - u^2)}{(V_\perp^2/V^2 - V_\parallel u)^2}, \quad u_0 = \frac{V_\perp}{V^2}(V^2-1)^{1/2}. \qquad (28)$$

It will be seen from what follows that the values of u important for the integral are $|u| \lesssim u_0$. According to Eq. (19), in the effective range of u the quantity w^2 is not small compared with unity. Then the arguments of the functions $J_{1/3}$ and $Y_{1/3}$ are large and a further simplification is possible:

$$J_n(\rho) \approx \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/3} \frac{1}{(-y)^{1/4}} \sin\left[n(w - \operatorname{arctg} w) + \frac{\pi}{4}\right],$$

$$-y = \left(\frac{n}{2}\right)^{2/3} w^2, \quad -y \gg 1. \qquad (29)$$

Then

$$\sum_{n_-}^{n_+} J_n^2(\rho) \approx \frac{\omega V_\parallel}{\omega_0} \frac{1}{\pi} \int_{-u_0}^{u_0} du \left(\frac{2}{n}\right)^{2/3} \frac{1}{2(-y)^{1/2}} = \frac{V_\parallel}{V}. \qquad (30)$$

In Eq. (30) the square of the sine has been replaced with its mean value. In a narrow region near the chosen limits of integration the approximation (29) is violated, but the contribution of these regions to the integral is small. Similarly it is easily verified that the term in Eq. (11) with derivatives with respect to ρ gives no contribution in the present approximation. The result is that the TF formula is obtained. The integration over u with the limits $[-u_0, u_0]$ corresponds to integration over $t = \cos\vartheta$ from t_1 to t_2, where

$$t_{1,2} = \cos(\vartheta_1 \pm \vartheta_0), \quad \cos\vartheta_0 = \frac{1}{V}, \quad \cos\vartheta_1 = \frac{V_\parallel}{V},$$

i.e., the angle of the radiation is a combination of the angle ϑ_1 at which the particle moves and the Cerenkov angle ϑ_0. This was to be expected, if on each element of the trajectory (of length several times L_m) the radiation is Cerenkov radiation.

5. THE CASE OF LARGE $V_\perp$

By large values of $V_\perp^2$ we mean those for which

$$V_\perp^2 \gg V^2 - 1, \quad 1 - V_\parallel^2 \approx V_\perp^2, \quad V \approx 1. \qquad (31)$$

Furthermore,

$$\omega L_F \sim \frac{(V^2-1)^{1/2}}{V_\perp} \ll 1, \quad n_- \approx \frac{\omega}{\omega_0} V_\perp^2. \qquad (32)$$

If $V_\perp^2 \omega/\omega_0 \lesssim 1$, then

$$\zeta^{-1} \equiv \frac{\omega}{\omega_0} \frac{(V^2-1)^{3/2}}{V_\perp} \lessapprox \frac{(V^2-1)^{3/2}}{V_\perp^3} \ll 1. \qquad (33)$$

According to Eq. (4), the field affects the radiation, and it is described by Eq. (11), with the main contributions from $n \sim 1$.

If, on the other hand, $V_\perp^2 \omega/\omega_0 \gg 1$, then the effect of the field is determined by the value of the parameter ζ in Eq. (33); for small ζ the effect is small. For $V_\perp^2 \omega/\omega_0 \gg 1$ we have $n_- \gg 1$, so that the approximation (26) holds for all the terms of the sum (11). For the important values of n, indeed, or for $u_{eff} \sim u_0$, we have $w^2 \ll 1$ under the condition (31). Then the term with $V_\parallel$

in Eqs. (27) and (28) can be neglected. Assuming also that $|nw^5| \ll 1$, we get

$$J_n(\rho) \approx \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/3}\Phi(y), \quad -y=\left(\frac{n}{2}\right)^{2/3}w^2; \tag{34}$$

$\Phi(y)$ is defined in Eq. (22). With this approximation we find

$$\sum_{n_-}^{n_+} J_n^2(\rho) \approx \frac{2^{5/3}}{\pi}\left(\frac{\omega}{\omega_0}\right)^{1/3}\frac{V_\parallel}{V_\perp^{4/3}}\int_0^\infty du\,\Phi^2(u)=\frac{1}{\sqrt{\pi}}V_\parallel\Phi_1(-b),$$
$$b=\left(\frac{\omega}{\omega_0 V_\perp}\right)^{2/3}(V^2-1)=\zeta^{-2/3}, \quad \Phi_1(x)=\int_x^\infty dz\,\Phi(z). \tag{35}$$

Analogously,

$$\sum_{n_-}^{n_+}\left(\frac{1}{\rho}\frac{d}{d\rho}+\frac{d^2}{d\rho^2}\right)J_n^2(\rho)\approx\frac{2^{7/3}}{\pi}\left(\frac{\omega_0}{\omega}\right)^{1/3}\frac{V_\parallel}{V_\perp^{2/3}}\int_{-\infty}^{\infty}du\frac{d}{dy}[\Phi'(y)\Phi(y)]$$
$$=-\frac{4V_\parallel}{\sqrt{\pi}}\frac{1}{V_\perp^2}\frac{V^2-1}{b}\Phi'(-b). \tag{35'}$$

In Eqs. (35) and (35′) we have used the equation (see [24])

$$\int_0^\infty\frac{dz}{\sqrt{z}}\Phi^2(z+x)=\frac{\sqrt{\pi}}{2}\Phi_1(2^{2/3}x)$$

and the expression obtained by differentiating it twice with respect to x. We finally get

$$\frac{d\mathscr{E}_\omega}{T}=\frac{e^2}{4\pi^{3/2}}\frac{1}{n(\omega)}\left(1-\frac{1}{V^2}\right)\left\{\Phi_1(-b)-\frac{2}{b}\Phi'(-b)\right\}\omega\,d\omega. \tag{36}$$

for $b \gg 1$ we again get the TF formula, since

$$\Phi_1(-b)-\frac{2}{b}\Phi'(-b)=\sqrt{\pi}\left[1+\frac{\cos(^2/_3\,b^{3/2}+\pi/4)}{b^{3/4}\sqrt{\pi}}+\dots\right].$$

We now note that the change from summation over n to integration over t (or u) in Eqs. (30), (35), and (35′) presupposed that $n_+ - n_- \gg 1$. If $V_\parallel$ is too small, this condition may be violated. This is due to the fact that for $V_\parallel \to 0$ the spectrum is discrete: $\omega \to n\omega_0$. The replacement of the sum with the integral smooths out this discreteness, and in this sense Eq. (36) admits passage to the limit $V_\parallel = 0$. The result agrees with [9,10] and corresponds to Eq. (74,13) in [1] for $n(\omega) = 1$. The inequality (4) begins to be violated when the intensity of the Cerenkov radiation becomes comparable with that of the bremsstrahlung, whose spectrum has its maximum in the region of the Cerenkov radiation.

6. ELECTRON IN THE FIELD OF A PLANE WAVE

Let us now consider the case when the external field is a plane monochromatic wave. Let it not be intense enough to deflect the particle by an angle $\sim\vartheta_0$ during a half-period of the wave. In such a wave there should not be much change in the characteristics of the Cerenkov radiation. Still, because of increasing interest among experimenters, it seems useful to consider this case at least qualitatively, to estimate the feasibility of the experiment and its difficulty. For simplicity and brevity we confine ourselves to the case of a circularly polarized wave, described by the vector potential

$$A_\mu=a_{1\mu}\sin\varphi+a_{2\mu}\cos\varphi, \quad \varphi=k'\cdot x=\mathbf{k}'\cdot\mathbf{x}-\omega' t,$$
$$a_1^2=a_2^2=a^2, \quad a_1\cdot a_2=a_1\cdot k'=a_2\cdot k'=0, \quad k'^2=\mathbf{k}'^2-\omega'^2>0. \tag{37}$$

The solution of the classical equations of motion is the following expression for the momentum:

$$\pi_\mu=p_\mu-\frac{k_\mu'}{k'^2}k'\cdot p-eA_\mu+\frac{k_\mu'}{k'^2}k'\cdot\pi, \tag{38}$$

$$k'\cdot\pi=\pm[(k'\cdot p)^2-e^2a^2k'^2+2k'^2e(a_1\cdot p\sin\varphi+a_2\cdot p\cos\varphi)]^{1/2}, \tag{39}$$

$$m\frac{d\varphi}{ds}=k'\cdot\pi=\beta[1-\varkappa^2\sin^2\psi]^{1/2}. \tag{40}$$

We note that if

$$k'^2\frac{ea_i\cdot p}{(k'\cdot p)^2}\ll 1, \quad i=1,2,$$

then Eq. (38) is of the same form as in the case of a plane wave with $k'^2=\mathbf{k}'^2-\omega'^2=0$; the ultrarelativistic particle "sees" the wave as a wave in a vacuum. On this basis we can in a number of cases reduce the calculation of the undulatory radiation to that of radiation in the field of a plane wave (cf. [25]).

The external field (37) possesses axial symmetry with respect to an axis along the vector $\mathbf{k}'$ (to within an unimportant initial phase). Therefore without loss of generality we can set $a_1\cdot p=0$, which makes the formalism more compact. We then get

$$2\psi=\varphi, \quad \varkappa^2=4\beta^{-2}k'^2ea_2\cdot p,$$
$$\beta=\pm[(k'\cdot p)^2+2k'^2ea_2\cdot p-e^2a^2k'^2]^{1/2}. \tag{41}$$

The sign of the square root in the definition of β corresponds to the sign of $\pi_\parallel$ in the system in which the field is purely magnetic (for definiteness we assume that $\kappa^2 < 1$). According to Eqs. (40) and (41), the phase φ is connected with the proper time s by the relation

$$\varphi/2=\psi=\operatorname{am}\tau, \quad \tau=\beta s/2m. \tag{42}$$

Corresponding to this, there are expressions for π_μ as functions of τ in terms of elliptic functions

$$\frac{\beta}{2}\frac{dx_\mu}{d\tau}=\pi_\mu=p_\mu-\frac{k_\mu'}{k'^2}(k'\cdot p)-eA_\mu+\frac{k_\mu'}{k'^2}\beta\operatorname{dn}\tau,$$
$$\sin\varphi=2\operatorname{sn}\tau\operatorname{cn}\tau=-\frac{2}{\varkappa^2}\frac{d}{d\tau}\operatorname{dn}\tau, \quad \operatorname{dn}\tau=\frac{d}{d\tau}\operatorname{am}\tau,$$
$$\cos\varphi=1-2\operatorname{sn}^2\tau. \tag{43}$$

By means of Eq. (43) we find $j_\mu(k)$ [see Eq. (8)]. For this purpose we represent the τ-dependent part of the function $k\cdot x(\tau)$, obtained from Eq. (43), in the form $\tau\cdot\text{const}+f(\tau)$, where $f(\tau)$ is a periodic function. Its Fourier expansion is known, since the expansions of the elliptic functions are known. The expansion

$$e^{if(\tau)}=\sum_{n=-\infty}^{\infty}A_ne^{in\Omega\tau}, \quad \Omega=\pi/K, \tag{44}$$

defines functions A_n, which play the role of the J_n in Eq. (11); $K=K(\kappa)$ is the complete elliptic integral. Instead of the parameter ρ there are now the parameters

$$\varkappa^2, \quad ea_ik/\beta, \quad i=1,2, \tag{45}$$

on which the functions A_n depend.

Multiplying Eq. (44) by the complex conjugate equation and integrating over a period, we get

$$\sum_{n=-\infty}^{\infty}|A_n|^2=1. \tag{46}$$

Since $|\beta|\approx|k'\cdot p|=p_0\omega'\epsilon$, for sufficiently small ϵ we can have $\kappa^2\sim 1$. But if $\epsilon \ll 1$, $ea \ll p$, then according to Eq. (43), in the coefficient of the exponential in the definition of $e'\cdot j(k)$ [see Eq. (8)] we can replace $e'\cdot\pi$ by $e'\cdot p$. This means that the polarization of the radiation is still that of Cerenkov radiation. In the coordinate system with polar axis along the vector $\mathbf{p}$ we get

$$\frac{d\mathscr{E}_k}{T}=\frac{e^2}{8\pi^2}\sum_n\frac{|e'\cdot p|^2}{pp_0}|A_n|^2\omega\,d\omega\,d\varphi. \tag{47}$$

The summation over the two polarizations e' reduces to replacing $|e'\cdot p|^2$ with $|\mathbf{p}|^2\sin^2\vartheta$. Inspection shows that ω and $\cos\vartheta = \cos(\widehat{\mathbf{pk}})$ are approximately connected by the relation

$$\omega \approx \frac{n\omega'\varepsilon\pi/2K}{1 - vn(\omega)\cos\vartheta}; \qquad (48)$$

$\pi/2K \sim 1$, if κ^2 is not too close to unity. The fact that for $\epsilon \ll 1$ the effective frequency of the undulator is $\epsilon\omega'$, and not ω', can be seen already from Eqs. (42) and (43), since $|\beta| \approx p_0\omega'\epsilon$; in the system with a stationary magnetic wave $p_{\parallel}$ is very small, i.e., the particle is incident at a small glancing angle on a magnetic grating. It follows from Eq. (48) that the summation over the effective values $n \sim 1$ leaves $\cos\vartheta$ almost unchanged if $vn(\omega) - 1$ is not too small. Consequently Eq. (47) reduces to the TF formula in virtue of Eq. (46). The spectrum of the emitted photons is quasicontinuous in the scale of ω' for $\epsilon \ll 1$. The effect of neglected factors such as ionization losses of the particle becomes much more important, since the effective period of the undulator and the formation length for the process are increased by a factor ϵ^{-1}.

In conclusion we thank V. I. Ritus for important comments, and also B. M. Bolotovskiĭ and M. N. Yakimenko for helpful discussions.

1) Equally simple expressions are obtained for the intensity in the case of a constant electric field [7] and in that of the combined action of constant electric and magnetic fields. [8]

2) This is a typical infrared situation. The electron can effectively interact with the soft quanta of the external field. This leaves the main process almost unchanged, if we are not concerned with fine details. (cf. [19, 20]).

3) Even in a classical treatment the quantum language is convenient. For example, Eq. (1) is an example of the uncertainty principle. We use in this paper units with $c = \hbar = 1$, $e^2/4\pi = 1/137$.

4) The development of the theory of Cerenkov radiation and its many applications are rather fully expounded in the monographs [21, 22].

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Translated by W. H. Furry
129