Nonlinear effects in the problem of anomalous plasma resistance

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Nonlinear processes that occur in a plasma situated in a constant homogeneous electric field are considered on the basis of the asymptotic theory of the anomalous resistance. It is shown that in sufficiently strong field the nonlinear effects determine the time of the transition of the system into the asymptotic regime.

1. INTRODUCTION

It was established earlier[1] that if a collisionless plasma is placed in a homogeneous constant electric field **E**, then the appearance of anomalous (collisionless) resistance should cause, in final analysis, all the quantities characterizing the plasma to change with time in a universal manner. The corresponding solution of the problem was called "asymptotic." It can be obtained on the basis of quasilinear equations, inasmuch as in the asymptotic regime, as shown in[1], the oscillation energy increases in proportion to the time t, while the kinetic energy of the particles increases much more rapidly, in proportion to t^2. As a result, the increments of all the possible nonlinear processes must decrease with time and at t larger than a certain value t_0 they become small in comparison with the linear increments of the interaction of the oscillations with the electrons and ions (which are constant in the asymptotic regime).

The estimated time t_0 is not obvious beforehand, since in addition to the quantity $(1/\omega_{pe})^{-1}$ with the dimension of time (the reciprocal of the electron plasma frequency) the system has also a small parameter $\mu \equiv m/M$, which is the ratio of the electron and ion masses. To determine t_0 it is therefore necessary to know the probabilities of the different nonlinear processes that occur in the system. At large t, when the wave energy is already small in comparison with the particle energy, the principal role is obviously played by the nonlinearities that have the lowest order in the oscillation energy. These are three-wave processes and nonlinear Landau damping of the waves by the plasma particles.

Let us examine them in detail in the one-dimensional case, for which an exact asymptotic solution is known. We present for future use the particle distribution functions and the spectrum and energy of the oscillations for this case[1]:

$$f_{e,i}(v,t)=\left(\frac{mn}{eEt}\right)g_{e,i}(u);\qquad u=\frac{mv}{eEt},$$
$$g_e=[2\mu^{1/2}u/(u+\mu^2)]+\delta(u-1),\tag{1}$$
$$g_i=\delta(u)+\frac{2\mu^{1/2}(1-u)}{u+\mu^2},\qquad 0\leqslant u\leqslant 1.$$

From the dispersion relation

$$\varepsilon(\Omega,q)=1-\frac{1}{(\Omega-q)^2}-\frac{\mu}{\Omega^2}+\frac{2\mu^{1/2}}{\Omega q}-\frac{2\mu^{1/2}}{q(\Omega-q)}=0,$$
$$\Omega=\omega/\omega_{pe},\qquad q=eEtk/m\omega_{pe},\tag{2}$$

it follows that there are two oscillation modes:

$$\Omega_1={}^1/_2\{q-1-[(q-1)^2+4\mu^{1/2}q]^{1/2}\},$$
$$\Omega_2\doteq\mu^{1/2}q/(1+q),$$
$$\Omega_3=\begin{cases}\mu^{1/2}q/(1-q), & 1-q\gg\mu^{1/4},\\ q-1+\mu^{1/2}q/(q-1), & q-1\gg\mu^{1/4},\end{cases}\tag{3}$$
$$\Omega_4=q+1-\mu^{1/2}q/(q+1).$$

In the asymptotic regime, only modes 2 and 3 are excited. Their spectral density is

$$W(k,t)=eE^3t^2w(q)/8\pi^2m,$$
$$w(q)=\frac{\Omega}{q\mu^2}\left(1-\frac{\Omega}{q}\right)\left(\frac{\Omega}{q}+\mu^2\right)\left|\frac{\Omega}{q}-\frac{d\Omega}{dq}\right|.\tag{4}$$

2. THREE-WAVE PROCESSES

With such a distribution, out of all the possible three-wave processes satisfying the conservation laws

$$\Omega(q)=\Omega(q')+\Omega(q-q'),$$

we need consider only those in which at least two out of the three waves belong to modes 2 and 3 (since we are interested only in induced processes). It is easy to verify that there are only two such processes:

a) decay of wave 2 into waves 3 and 1:

$$\Omega_2(q')=\Omega_3(q)+\Omega_1(q'-q),\tag{5}$$

b) decay of wave 3 into 2 and 4:

$$\Omega_3(q)=\Omega_2(q')+\Omega_4(q-q').\tag{6}$$

It follows from the conditions (5) and (3) that the process (a) is possible at $2/3 < q < 2$. We confine ourselves to the region $q-1 \gg \mu^{1/4}$, since smaller q correspond to a very narrow part of the spectrum. Then $q' \approx 2$.

Greatest interest attaches to the nonlinear increment for the mode 3, which covers practically the entire phase-velocity interval where there are oscillations. To find it, we use the kinetic equation for the waves[2,3], expressed in the form

$$\frac{\partial N_q^{(3)}}{\partial t}=\frac{8\pi^2m\omega_{pe}}{eE^3t^3}\int\frac{N_q^{(3)}N_{q'}^{(2)}|U_{q,q'}|^2}{(\partial\varepsilon/\partial\Omega)_q(\partial\varepsilon/\partial\Omega)_{q'}(\partial\varepsilon/\partial\Omega)_{q''}}$$
$$\times\,\delta(\Omega_q-\Omega_{q'}-\Omega_{q-q'})\,dq',\qquad N_q=W_k(\partial\varepsilon/\partial\omega)_k.\tag{7}$$

The main contribution to the matrix element

$$U_{q,q'}=\int\frac{du\,\partial(g_e-\mu^2g_i)/\partial u}{(\Omega-qu)(\Omega'-q'u)(\Omega''-q''u)}$$

is made by electrons, and this element is approximately equal to $3\,[{}^1/_2+(q-1)^{-2}]$. From (7), using the dispersion relation (2) and knowing the energy of the oscillations (4), we easily obtain the nonlinear increment

$$\gamma_n=\frac{1}{2N_q^{(3)}}\frac{\partial N_q^{(3)}}{\partial t}=\left(\frac{\mu^{-1/2}}{18t}\right)[{}^1/_2+(q-1)^{-2}]^2.\tag{8}$$

The second of the possible decays, (b), was considered earlier[4]. It occurs at $q > 2$. Calculations show that the matrix elements for both processes coincide, and therefore formula (8) is valid in the entire wavenumber interval $q - 1 \gg \mu^{1/4}$. How is the asymptotic solution (1)–(4) altered by the nonlinear wave interaction? Instead of the condition $\gamma_e + \gamma_i \approx 0$ (where $\gamma_{e,i}$ are the electronic and ionic linear increments) we now must write

$$\gamma_e+\gamma_i+\gamma_n\approx 0. \tag{9}$$

As a result, the particle distribution functions $f_{e,i}$ will differ from (1). The difference remains small when $\gamma_n \ll \gamma_e$. Recognizing that for mode 3 the linear increment is $\gamma_e \sim \mu^{5/2}\omega_{pe}$, we find that this occurs at $t \gg \mu^{-3}/\omega_{pe}$.

Nonlinear excitation of modes 1 and 4, where there are no oscillations in the asymptotic regime, leads to the appearance of particles in the velocity regions $u < 0$ and $u > 1$. It is important that these are oscillations with positive energy, for only in this case can the condition (9) be satisfied for them. Indeed, the regions $u < 0$ and $u > 1$ the electrons and ions are lost through quasilinear diffusion. Therefore $\partial f_{e,i}/\partial v > 0$ at $u < 0$ and $\partial f_{e,i}/\partial v < 0$ at $u > 1$. Then γ_e and γ_i are of the same sign and are negative only for waves with positive energy. The sign of the energy of mode-3 oscillations (their energy is negative) is of no fundamental significance[1)], inasmuch as the linear increment of any wave can be either positive or negative in the interval $0 < u < 1$.

3. NONLINEAR DAMPING

We proceed to consider the nonlinear interaction of the oscillations via the plasma particles—the nonlinear Landau damping. It is described by the following equation[3]:

$$\frac{\partial N_k}{\partial t}=N_k\int N_{k'}k''\left[\frac{W_v^{(e)}(k,k')}{m}\frac{\partial f_e}{\partial v}+\frac{W_v^{(i)}(k,k')}{M}\frac{\partial f_i}{\partial v}\right]dk'\,dv,$$

$$W_v^{(e)}(k,k')=\frac{16\pi^2e^4}{m^2}\delta(\omega''-k''v)\left(\frac{\partial\varepsilon}{\partial\omega}\right)_k^{-1}\left(\frac{\partial\varepsilon}{\partial\omega}\right)_{k'}^{-1}\left|\frac{1}{(\omega_k-kv)^2}-\frac{4\pi e^2}{mk''\varepsilon(k'',\omega'')}\int\frac{dv\,\partial(f_e-\mu^2f_i)/\partial v}{(\omega-kv)(\omega'-k'v)(\omega''-k''v)}\right|^2,$$

$$W_v^{(i)}(k,k')=\frac{16\pi^2e^4}{m^2}\delta(\omega''-k''v)\left(\frac{\partial\varepsilon}{\partial\omega}\right)_k^{-1}\left(\frac{\partial\varepsilon}{\partial\omega}\right)_{k'}^{-1}\left|\frac{\mu}{(\omega_k-kv)^2}+\frac{4\pi e^2}{mk''\varepsilon(k'',\omega'')}\int\frac{dv\,\partial(f_e-\mu^2f_i)/\partial v}{(\omega-kv)(\omega'-k'v)(\omega''-k''v)}\right|^2 \tag{10}$$

$$k''=k-k',\qquad \omega''=\omega_k-\omega_{k'},$$

$$\varepsilon(k,\omega)=1+\frac{4\pi e^2}{mk}\int\frac{dv\,\partial(f_e+\mu f_i)/\partial v}{\omega-kv}.$$

At large t, the nonlinear damping of the oscillations at the resonant frequencies is small in comparison with the linear damping, and the principal contribution in (10) is that of the nonresonant particles—the electron and ion cores. Their interaction with the oscillations depends essentially on the particle velocity distribution in the cores, which is not determined in the asymptotic solution (it is simply a δ-function). The final widths of the cores are the result of two processes: adiabatic quasilinear diffusion of the particle[2] and nonlinear wave damping. The first causes the scatter of the particle energy inside the cores to increase in proportion to the wave energy, i.e., in proportion to t.[2)]

As will be shown below, nonlinear damping turns out to be more effective in this sense. The distribution functions $f_{e,i}$ of the core particles then satisfy the equations[3]

$$\frac{\partial f_e}{\partial t}=\frac{1}{2m^2}\frac{\partial}{\partial v}\int N_kN_{k'}(k'')^2W_v^{(e)}\,dk\,dk'\frac{\partial f_e}{\partial v}, \tag{11}$$

$$\frac{\partial f_i}{\partial t}=\frac{1}{2M^2}\frac{\partial}{\partial v}\int N_kN_{k'}(k'')^2W_v^{(i)}\,dk\,dk'\frac{\partial f_i}{\partial v}. \tag{12}$$

It is convenient to investigate (11) in a reference frame that moves with the electronic core. In this frame

$$\Omega_2(q_2)=-q_2+\frac{\mu^{1/2}q_2}{q_2+1},\qquad \Omega_3(q_3)=-1+\frac{\mu^{1/2}q_3}{q_3-1}.$$

The nonlinear resonance condition

$$\Omega(q)-\Omega(q')=(q-q')mv/eEt$$

can be satisfied for core electrons with $mv/eEt \ll 1$ if:

a) one wave belongs to mode 2 and the other to mode 3, with $q_2 \approx 1$;

b) both waves belong to mode 3, with

$$q,q'\gg 1,\qquad qq'=-\mu^{1/2}eEt/mv\qquad (v<0).$$

The probability $W_v^{(e)}(k,k')$ that the electron will absorb a plasmon k and emit k' depends, in turn, on f_e, as seen from (10). This greatly complicates the determination of the exact solution of Eq. (11). We shall therefore confine ourselves henceforth to only an estimate of the width of the electronic core, characterizing this width by a certain effective temperature T_e. In the calculation of $W^{(e)}$ it must be recognized that the phase velocity of the waves is large in comparison with the velocity of the core electrons that make the main contribution to the integral with respect to the velocities.

In the zeroth approximation, the principal terms cancel out and, retaining the small thermal corrections we obtain

$$W^{(e)}\sim\frac{e^2T_e}{m\omega_{pe}^3E^2t^2}\delta\left(\Omega''-\frac{q''mv}{eEt}\right)(q+q')^2\Big/\left(\frac{\partial\varepsilon}{\partial\Omega}\right)_q\left(\frac{\partial\varepsilon}{\partial\Omega}\right)_{q'}.$$

In case (a) we then obtain for the electron diffusion coefficient

$$D_n^{(e)}=\frac{1}{2m^2}\int N_kN_{k'}(k'')^2W^{(e)}\,dk\,dk'$$

the estimate

$$D_n^{(e)}\sim\mu^{-5/2}\frac{T_e}{m\omega_{pe}t^2}\int\frac{(q+q')^2(q-q')^2}{q^2}\delta\left(\Omega''-\frac{q''mv}{eEt}\right)dq\,dq'\sim\mu^{-3}\frac{T_e}{m\omega_{pe}t^2}q_{max}^3, \tag{13}$$

where q_{max} is the maximum value of the wave number of the oscillations of mode 3. It must exist, because the phase velocity of the waves tends to zero as $q \to \infty$ and ultimately falls in the region of the velocities of the electronic core, where there are no oscillations because of the strong damping.

To estimate q_{max}, we formulate the problem rigorously. The equation (11) must be solved with the following boundary conditions: $\partial f_e/\partial v = 0$ as $v \to +\infty$ and, at $v = v_0 = -eEt/mq_{max}$ its solution must match the asymptotic solution

$$f_e(v_0)=\frac{2\mu^{1/2}mn}{eEt},\quad\left(D_n^{(e)}\frac{\partial f_e}{\partial v}\right)_{v=v_0}=\frac{2\mu^{1/2}n}{tq_{max}}. \tag{14}$$

This yields the velocity v_0 itself. Inasmuch as at $v \sim (T_e/m)^{1/2}$ we have

$$f_e\sim n/(T_e/m)^{1/2}\gg\mu^{1/2}mn/eEt,$$

the velocity v_0 is large compared with the average thermal velocity of the core:

$$v_0=-A(T_e/m)^{1/2}, \quad A\gg 1.$$

The exact value of A depends on the shape of the "tail" of the electron distribution function. In the region of the core, the diffusion coefficient $D_n^{(a)}(v)$ depends little on the velocity, and therefore the function $f_e(v)$ decreases rapidly at $v > (T_e/m)^{1/2}$, as a result of which v_0 is of the order of several thermal velocities, and A varies slowly with time. Assuming it therefore to be constant, we find that

$$q_{max}\sim\frac{eEt}{m}\left(\frac{T_e}{m}\right)^{-1/2}.$$

We then get from (13)

$$D^{(e)}\sim\frac{\mu^{-3}(T_e/m)^{-1/2}}{\omega_{pe}t^2}\left(\frac{eEt}{m}\right)^3 \quad (15)$$

The variation of the electron-core temperature with time can be estimated at $dT_e/dt \sim mD_H^{(e)}$. Hence

$$T_e\sim m\left(\frac{eEt}{m}\right)^2\mu^{-2}(\omega_{pe}t)^{-3/2}. \quad (16)$$

In the interaction of waves of mode 3, the corresponding diffusion coefficient, which differs from zero only at $v < 0$, turns out to be of the same order as (15). The estimated temperature of the electronic core (16) therefore remains unchanged.

The structure of the ionic core is determined by the following processes:

a) the interaction of the waves of mode 2 with the low-frequency part of mode 3 ($q_3 \approx 1/2$);

b) both waves belong to mode 3:

$$q,\ q'\gg 1, \quad qq'=\mu^{1/2}eEt/mv \quad (v>0).$$

In perfect analogy with the procedure for the electrons, we can obtain an estimate of the ionic-core temperature:

$$T_i\sim M\left(\frac{eEt}{m}\right)^2\mu^{3/2}(\omega_{pe}t)^{-3/2}. \quad (17)$$

Knowing the core temperatures, we estimate the decrement of the nonlinear Landau damping of the oscillations from (10). For waves of mode 3, it is determined by the damping by the electronic core:

$$\gamma_n=\frac{1}{2m}\int N_{k'}k''W_v^{(e)}\frac{\partial f_e}{\partial v}dk'\,dv. \quad (18)$$

We find therefore that in the interval of phase velocities on the order of unity and

$$\gamma_n\sim\frac{\mu^{-1/2}}{t}(\omega_{pe}t)^{1/2}. \quad (19)$$

From a comparison of (19) and (8) it follows that at large t the principal nonlinear process is the nonlinear Landau damping.

4. CONCLUSION

We have thus found that at large t, when the oscillation energy density $\sim\mu^{-2}E^2(\omega_{pe}t)$ is already small in comparison with the kinetic particle energy $\sim E^2(\omega_{pe}t)^2$, the nonlinear processes in the system still play an appreciable role. As a result, the time necessary to reach the asymptotic regime is very large: $t \gg t_0 \sim\mu^{-9/2}/\omega_{pe}$.

This stringent criterion is the consequence of the specific features of the considered one-dimensional solution. One can therefore expect the asymptotic solution to establish itself much more rapidly in the three-dimensional problem. In addition, in a sufficiently weak electric field, when the time of the transition of the plasma from the initial state to the asymptotic regime in the quasilinear problem, $\tau \sim 1/E$, is larger than t_0, the nonlinear effects appear to be generally inessential.

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[1)]In contradiction to the statement made by Kingsep [4].

[2)]An exact calculation shows that owing to the peculiarities of the oscillation spectrum in this problem the particle energy scatter is proportional to t ln t.

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