Absolute parametric instability of inhomogeneous plasma

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Plasma subjected to a strong pump wave is unstable with respect to the growth of high-frequency electron waves (plasmons) due to the decay of the pump wave into two plasmons. This paper reports an analysis of the effect of spatial plasma inhomogeneity on this type of parametric instability. It is shown that allowance for the spatial inhomogeneity of the plasma leads to a substantial increase in the threshold for the decay of the pump wave into two high-frequency electron waves and to the trapping of these plasma waves which grow in time in the plasma. The latter effect corresponds to absolute instability.

Plasma subjected to a strong pump wave

$$E_y(x,t)=E_0(x)\sin\left(\omega_0 t-\int k_0(x)\,dx\right),$$
$$B_z(x,t)=-c\int_{-\infty}^{t}dt'\frac{\partial}{\partial x}E_y(x,t'),\tag{1}$$

is unstable against the growth of high-frequency electron waves (plasmons) due to the decay of the pump wave into two plasmons. This decay was observed by Krenz and Kino,[1] who gave an elementary explanation of this phenomenon in terms of hydrodynamic equations. Jackson[2] investigated the decay with the aid of the kinetic equations on the assumption of spatial homogeneity of the plasma. He showed that the $\omega_0\to 2\omega_{Le}$ decay is associated with the fact that the wave vector k_0 of the pump wave is finite. The influence of plasma inhomogeneity on the growth of the plasma wave and the corresponding growth rates when the frequency of these waves is not very different from half the frequency of the variation in the electric field is considered by Ramazashvili.[3] The latter paper is restricted by the use of a local geometric-optics approximation which assumes that the plasma density is a sufficiently smooth function of position. The importance of the $\omega_0\to 2\omega_{Le}$ decay as a mechanism for the absorption of radiation by plasma is emphasized by Sagdeev.[4] Rosenbluth[5] uses the truncated equations for plasmon amplitudes to consider their growth in space. These theoretical studies leave as an open question the possibility of the absolute instability of the plasma in a strong pump-wave field, accompanied by an increase in time of the plasmon amplitudes in the inhomogeneous plasma. This question can only be answered by studying the spatial dependence of the growing perturbations which are localized in a finite region of space in the case of absolute instability. This approach also yields the solution to the problem of the effect of spatial plasma inhomogeneity on the instability threshold which is found to increase under these conditions.

Since the motion of the ions is unimportant for the instability which we are considering, it will be assumed that their spatial distribution $n_i(\mathbf{r})$ is fixed, and that the distribution of electrons in the ground state is

$$f_0(\mathbf{p},x,t)=\left[1-\frac{\partial}{\partial x}\left\{p_y r_E(x)\sin\left(\omega_0 t-\int k_0(x)\,dx\right)\cdot-\frac{U_E(x)}{2\omega_0}\sin 2\left(\omega_0 t-\int k_0(x)\,dx\right)\right\}\frac{\partial}{\partial p_x}\right]F_0,$$

$$F_0=\frac{n_e(x)}{(2\pi m_e T_e)^{3/2}}\exp\left\{-\frac{1}{T_e}\left[\frac{\mathbf{p}^2}{2m_e}+e\varphi_0+U_E(x)\right]\right\},\tag{2}$$

where the field $\varphi(\mathbf{r},t)$ is determined by the inhomogeneous particle charge, i.e.,

$$\Delta\varphi_0(\mathbf{r},t)=-4\pi e_i n_i(r)-4\pi e\int d\mathbf{p} f_0(\mathbf{p},x,t),$$

$r_E(x)=eE_0(x)/m_e\omega_0^2$ is the amplitude of the electron oscillations in the field of the pump wave, and $U_E(x)=e^2E_0^2(x)/4m_e\omega_0^2$ is the high-frequency potential.

The perturbation $\delta f(\mathbf{p},\mathbf{r},t)$ of the electron distribution is described by the following kinetic equation if it is assumed that the associated nonequilibrium perturbation of the field, $\delta\varphi(\mathbf{r},t)$, is potential:

$$\frac{\partial\delta f}{\partial t}+\mathbf{v}\frac{\partial\delta f}{\partial\mathbf{r}}+e\left[E_y(x,t)-\frac{v_x}{c}B_z(x,t)\right]\frac{\partial\delta f}{\partial p_y}+e\left[-\frac{\partial\varphi_0}{\partial x}+\frac{v}{c}B_z(x,t)\right]\frac{\partial\delta f}{\partial p_x}-\frac{\partial f_0}{\partial\mathbf{p}}\frac{\partial\delta\varphi}{\partial\mathbf{r}}=0,\tag{3}$$

where the perturbation potential satisfies the Poisson equation

$$\Delta\delta\varphi(\mathbf{r},t)=-4\pi e\int d\mathbf{p}\,\delta f(\mathbf{p},\mathbf{r},t).\tag{4}$$

The self-consistent set of equations given by Eqs. (3) and (4) yields the following set of equations for the harmonics of the potential $\delta\varphi$:

$$\begin{aligned}&k_\perp^2\varepsilon(\omega)\Phi_0-\frac{\partial}{\partial x}\left(\varepsilon(\omega)\frac{\partial\Phi_0}{\partial x}\right)-2ik_yK\frac{\partial\Phi_{-1}}{\partial x}\left[1-\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\right]\\&-ik_y\frac{\partial K}{\partial x}\Phi_{-1}\left[1-\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\right]+ik_yK\Phi_{-1}\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\frac{\partial\ln n_e(x)}{\partial x}=0,\\&k_\perp^2\varepsilon(\omega-\omega_0)\Phi_{-1}-\frac{\partial}{\partial x}\left(\varepsilon(\omega-\omega_0)\frac{\partial\Phi_{-1}}{\partial x}\right)-2ik_yK^*\frac{\partial\Phi_0}{\partial x}\left[1-\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\right]\\&-ik_y\frac{\partial K^*}{\partial x}\Phi_0\left[1-\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\right]+ik_yK^*\Phi_0\frac{\omega_{Le}^2(x)}{\omega(\omega-\omega_0)}\frac{\partial\ln n_e(x)}{\partial x}=0,\end{aligned}\tag{5}$$

where

$$K(x)=\frac{i}{2}\frac{d}{dx}\left(r_E(x)\exp\left\{i\int k_0(x)\,dx\right\}\right),\quad k_\perp^2=k_y^2+k_z^2$$
$$\varepsilon(\omega)=1-\frac{\omega_{Le}^2(x)}{\omega^2}\left(1-i\frac{\nu_{ei}}{\omega}\right),\quad \omega_{Le}^2(x)=\frac{4\pi e^2 n_e(x)}{m_e},$$

and the potential $\delta\varphi$ is written in the form

$$\delta\varphi(\mathbf{r},t)=\sum_{n=-\infty}^{+\infty}e^{-i\omega t-in\omega_0 t}\Phi_n(\mathbf{r}).$$

In deriving Eq. (5), we neglect thermal corrections to the plasmon spectrum and restrict our attention to the plasmon wavelength region in which Landau damping is smaller than the damping due to electron-ion collisions. In the limit of the homogeneous plasma, Eq. (5) becomes identical with the equations given in[6].

We now consider the situation in which the homogeneity of the pump-wave amplitude at the plasmon wavelength is small. It is convenient to rewrite Eq. (5) in the form of a fourth-order differential equation for the amplitude of one of the plasmons:

$$\Phi_0^{IV}+2\varepsilon_1\Phi_0'''+2p\Phi_0''+2\varepsilon_2\Phi_0'+\Phi_0=0, \qquad (6)$$

where

$$\varepsilon_1=\frac{2z+ib}{z^2+b^2}, \quad \varepsilon_2=-\varepsilon_1+\frac{1}{2}\frac{\alpha^2}{z^2+b^2}\frac{1}{z+ib}, \quad z=k_\perp x,$$

$$p=-1+\frac{1}{2}\frac{\alpha^2}{z^2+b^2}, \quad b=k_\perp L\left(\frac{2}{i}\frac{\delta\omega}{\omega_{Le}(0)}+\frac{\nu_{ei}}{\omega_{Le}(0)}\right), \quad \alpha^2=16k_y^2L^2|K|^2.$$

In deriving Eq. (6), we assume that the plasma electron density is a linear function of the coordinate x, i.e.,

$$\omega_{Le}^2(x)=\omega_{Le}^2(0)(1+x/L), \qquad (7)$$

and that $\omega = \omega_{Le}(0) + \delta\omega$, where $\delta\omega$ is a small correction to the plasmon frequency.

It will be shown below that plasmons which appear as a result of the decay of the pump wave are trapped in the plasma and that there is the possibility of their mutual transformation. Effects associated with the coupling of solutions corresponding to different branches of the oscillations are discussed in great detail in[7] which gives a classification of the various types of crossing of wave solutions in inhomogeneous media. According to this classification, Eq. (6) describes the type III transformation since the coefficients of this equation near the points of crossing of the solutions (they will be called branching points) are of the order of unity.

When the conditions

$$|k_\perp x+ib|\gg 1, \qquad 16k_y^2L^2|K|^2\gg 1, \qquad (8)$$

are satisfied, the geometric-optics approximation is valid and, in this approximation, the functions ϵ_1 and ϵ_2 are small because in the plasmon localization region discussed below the dimensionless coordinate z is large. We shall seek the solution Φ_0 in the form

$$\exp\left\{i\int k_x(x)\,dx\right\}, \qquad k_x=k+\delta k.$$

From Eq. (6), we obtain the following expressions for k and δk:

$$k^2=k_{1,2}^2=k_\perp^2(p\pm\sqrt{p^2-1}),$$

$$\delta k=\frac{i}{2}k_\perp\left\{\frac{k'}{k}+\frac{(k^2-k_\perp^2p)'}{k^2-k_\perp^2p}+\frac{\varepsilon_1k^2+k_\perp^2(p'-\varepsilon_2)}{k^2-k_\perp^2p}\right\}$$

Since $k^2 \neq 0$ for any values of x, it is clear that the geometric-optics approximation is violated only near the branching points of the function k^2, i.e., at the points

$$x_*^2=4\frac{k^2}{k_\perp^2}|K|^2L^2-\left(\frac{2}{i}\frac{\delta\omega}{\omega_{Le}(0)}+\frac{\nu_{ei}}{\omega_{Le}(0)}\right)^2L^2. \qquad (9)$$

At these points, the wave vectors $k_1(x)$ and $k_2(x)$ which correspond to different plasmons are identical, i.e., mutual transformation of the plasmons is possible. In fact, the transparency regions for both types of plasmon coincide and lie near values of x defined by $x_* > x > -x_*$. Outside this region, neither plasmon exists since they are damped exponentially when the absolute magnitude of x increases without limit. On the other hand, it is readily shown that the group velocities of the plasmons in the direction of the plasma inhomogeneity are antiparallel. Moreover, it is clear from the following representation for the wave vector inside the transparency region

$$k_{1,2}=k_\perp\{\sqrt{p+1}\pm\sqrt{p-1}\}/\sqrt{2},$$

that at a branching point there can be a change in only the sign in front of the second term in this expression, i.e., we have transitions of the form $k_1 \rightleftarrows k_2$, and therefore it may be concluded that the incident plasma wave approaching the branching point is completely transformed into a plasma wave of another type which is then backward scattered. Hence, plasmons are trapped between the branching points x_* and $-x_*$ which, for purely imaginary $\delta\omega$, lie on the real axis. Therefore, we may conclude that the pump wave decays into two plasmons which remain in the plasma.

The quantization condition which describes the growth rate for the $\omega \to 2\omega_{Le}$ decay can be obtained by matching the solutions on either side of the branching points. This condition is

$$\int_{-x_*}^{x_*} dx\,(k_1-k_2)\pm\frac{i}{2}k_\perp\int_{-x_*}^{x_*} dx\,\frac{\varepsilon_1p+p'-\varepsilon_2}{(p^2-1)^{1/2}}=\pi(2n+1). \qquad (10)$$

For $\epsilon_1 = 0$, this quantization condition becomes identical with that given in[8].

Integration shows that the second term in Eq. (10) is real and small [$\sim (k_\perp L)^{-1}$ as compared with the first]. Finally, we find that the growth rate for the $\omega_0 \to 2\omega_{Le}$ decay is given by

$$4(\alpha/2b-1)\varphi(q)\,b=\pi(2n+1), \qquad (11)$$

where

$$\varphi(q)=K(q)+\frac{1}{q}[K(q)-E(q)],$$

$$q=\frac{\alpha-2b}{\alpha+2b}=\frac{\mathscr{K}_-}{\mathscr{K}_+}, \quad \mathscr{K}_\pm=2\frac{|k_y|}{k_\perp}|K|\pm\left(\frac{\nu_{ei}}{\omega_{Le}(0)}+\frac{2\gamma}{\omega_{Le}(0)}\right),$$

$\gamma = \mathrm{Im}\,\delta\omega$ (it is assumed that $\mathrm{Re}\,\delta\omega = 0$), and K(q) and E(q) are the complete elliptic integrals of the first and second kind, respectively. At the threshold, the growth rate is zero and the criterion for the $\omega_0 \to 2\omega_{Le}$ decay instability in the inhomogeneous plasma is

$$\frac{|k_y|}{k_\perp}|K|=\frac{1}{2}\frac{\nu_{ei}}{\omega_{Le}(0)}+\frac{\pi(2n+1)}{8k_\perp L\varphi(q)}. \qquad (12)$$

When the inhomogeneity has very little effect on the threshold (in this case, $q \approx 0$), the last term in Eq. (12) is a small correction. The threshold is given by the following expression:

$$\frac{|k_y|}{k_\perp}|K|=\frac{1}{2}\frac{\nu_{ei}}{\omega_{Le}(0)}+\frac{2n+1}{4k_\perp L}. \qquad (13)$$

Conversely, when the argument q is close to unity, the inhomogeneity has a substantial effect on the decay threshold which is then given by

$$\frac{|k_y|}{k_\perp}|K|=\pi(2n+1)/8k_\perp L\ln\left[\frac{1}{8}\frac{k_\perp}{|k_y|}\frac{1}{|K|}\frac{\nu_{ei}}{\omega_{Le}(0)}\right]. \qquad (14)$$

If

$$\frac{1}{\pi}\frac{\nu_{ei}}{\omega_{Le}(0)}\frac{k_\perp L}{2n+1}<e,$$

where e is the base of natural logarithms, the solution of Eq. (14) may be written with logarithmic accuracy in the form

$$\frac{|k_y|}{k_\perp}|K|=\frac{\pi(2n+1)}{8k_\perp L}\left\{\ln\left[\frac{1}{\pi}\frac{\nu_{ei}}{\omega_{Le}(0)}\frac{k_\perp L}{2n+1}\right]\right\}^{-1}. \qquad (15)$$

Comparison of Eqs. (13) and (15) shows that the inhomogeneity has a substantial effect on the threshold.

We can generalize the above results to the case where the dissipation of plasma waves is determined by their Cerenkov interaction with electrons. This occurs when the wave vector for the growing oscillations turns out to be sufficiently large: $k > k_{st}$, where k_{st} is the value of the wave vector for which Landau damping is comparable with damping due to electron-ion collisions. We shall suppose, for the sake of simplicity, that $k_y \gg k_z$. In that case, if

$$k_{st}<k_y<k_d=\frac{1}{2r_{De}}\left\{\ln\left[\frac{1}{8}\sqrt{\frac{\pi}{8}}\frac{1}{|K|}\right]\right\}^{-1/2}, \quad (16)$$

where $r_{De} = v_{Te}/\omega_{Le}(0)$ is the Debye length of the electrons, then k_x^2 is not very different from k_y^2 in the region of plasma transparency. We can therefore replace k_x^2 by k_y^2 in the expression for the Landau damping, and the corresponding generalization of Eq. (12) takes the form

$$|K|=\frac{1}{2}\frac{\nu_{ei}}{\omega_{Le}(0)}+\frac{1}{8}\left(\frac{\pi}{8}\right)^{1/2}\frac{1}{k_y^3r_{De}^3}\exp\left\{-\frac{1}{4}\frac{1}{k_y^2r_{De}^2}\right\}+\frac{\pi(2n+1)}{8k_yL\varphi(q)}. \quad (17)$$

At the same time, in the expression for x_* and q we must substitute $\nu_{ei} \to 2\tilde{\gamma}$, where

$$\tilde{\gamma}=\frac{1}{2}\nu_{ei}+\frac{1}{8}\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_{Le}(0)}{k_y^3r_{De}^3}\exp\left\{-\frac{1}{4}\frac{1}{k_y^2r_{De}^2}\right\}.$$

It is clear from Eq. (17) that the threshold reaches its minimum value given by

$$|K|=\frac{\pi(2n+1)}{32\varphi(q)}\frac{r_{De}}{L}\left\{\ln\left[\frac{16\varphi(q)}{\sqrt{2\pi}(2n+1)}\frac{L}{r_{De}}\right]\right\}^{1/2} \quad (18)$$

for values of the wave vector given by

$$k_{min}=\frac{1}{2r_{De}}\left\{\ln\left[\frac{16\varphi(q)}{\sqrt{2\pi}(2n+1)}\frac{L}{r_{De}}\right]\right\}^{-1/2}.$$

When the value of the wave vector k_{min} is close to the right-hand end of the interval defined by Eq. (16), i.e., $k_{min} \approx k_d$, the argument of the elliptic functions is close to zero and the threshold for the excitation of the $\omega_0 \to 2\omega_{Le}$ parametric instability is given by the following expression:

$$|K|=\frac{2n+1}{16}\frac{r_{De}}{L}\left\{\ln\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{8}{2n+1}\frac{L}{r_{De}}\right]\right\}^{3/2}. \quad (19)$$

When the wave vector k_{min} lies on the left hand part of the interval defined by Eq. (16), the argument q is of the order of unity and the threshold for the development of the $\omega_0 \to 2\omega_{Le}$ decay instability is given by

$$|K|=\frac{(2n+1)\pi}{32}\frac{\nu_{ei}}{\omega_{Le}(0)}\frac{l}{L\ln 4}\left[\ln\left(\frac{16}{\sqrt{2\pi}}\frac{1}{2n+1}\frac{L}{r_{De}}\right)\right]^{3/2}, \quad (20)$$

where $l = v_{Te}/\nu_{ei}$ is the mean free path. Comparison of Eqs. (19) and (20) shows that they are close to one another, but their ratio is determined by the particular values of the plasma parameters.

The ratio of the threshold (19) to the threshold for the development of the $\omega_0 \to 2\omega_{Le}$ instability in homogeneous plasma is given by the formula

$$\frac{|K|}{|K|_{hom}}=\frac{2n+1}{16}\frac{l}{L}\left[\ln\left(\left(\frac{\pi}{2}\right)^{1/2}\frac{8}{2n+1}\frac{L}{r_{De}}\right)\right]^{3/2}. \quad (21)$$

The square of this ratio gives the increase in the threshold value of the light flux due to the influence of the plasma inhomogeneity relative to the threshold flux necessary for the excitation of the instability in homogeneous plasma. We shall estimate the threshold radiation flux in the particular case which is of practical importance for producing controlled thermonuclear reactions with the aid of a laser. In this case, the plasma electron temperature is 1 keV and the size of the plasma inhomogeneity varies from 0.01 cm to 0.001 cm. The threshold values of the light flux necessary for the development of the $\omega_0 \to 2\omega_{Le}$ decay instability in homogeneous plasma is 10^{12} W/cm^2 if the plasma is produced by a neodymium laser beam of 1.06 μ wavelength, and 10^{10} W/cm^2 if the plasma is exposed to CO_2 laser radiation of 10.6 μ wavelength. The effect of the inhomogeneity of the plasma amounts to an increase in the threshold flux to 4×10^{12} W/cm^2 for L = 0.01, and to 2×10^{14} W/cm^2 for L = 0.001 in the case of the neodymium laser; the increase is to 2×10^{12} W/cm^2 for L = 0.01 cm and to 10^{14} W/cm^2 for L = 0.001 cm in the case of the CO_2 laser.

In the region where Eq. (12) is valid, there is a substantial increase in the threshold for the $\omega_0 \to 2\omega_{Le}$ decay. It may therefore be concluded that when plasma is exposed to radiation with an amplitude exceeding the value given by Eq. (19), we have excitation of plasmons with wavelengths of the order of k_{min}.

It is important to note the following fact which is concerned with the effect of thermal corrections to the plasmon spectrum: with increasing distance from the branching points, this influence can be neglected since the wave number $k_\perp$ is not very different from k_x throughout the transparency region. This differs very substantially from the situation occurring in inhomogeneous plasma without the pump wave, when departure from the branching point along the plasma profile leads to an increase in k_x, i.e., to an exponential increase in the damping.

It is clear from the foregoing that allowance for spatial inhomogeneity of plasma leads to two effects: the inhomogeneity leads to a substantial increase in the threshold for the decay of the pump wave into two high-frequency electron waves and these plasma waves, which grow in time, turn out to be trapped in the plasma. This trapping corresponds to absolute instability.

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