The nonlinear phase of monochromatic-oscillation excitation by a charged-particle beam in a plasma located in a magnetic field

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The excitation of the monochromatic longitudinal oscillations of a plasma located in a magnetic field by a stream of oscillators—charged particles—that move through the plasma along the magnetic field with constant velocity, have the same Larmor radius, and are uniformly distributed over the azimuthal angle is considered. For "oblique" waves, the primary mechanism leading to oscillation saturation is the tuning out of the resonance between a wave and a particle as a result of the oscillation-field-induced variation of the beam-particle velocity component along the magnetic field. The saturation of oscillations propagating in a direction perpendicular to the magnetic field is due to the variation of the transverse velocity and the azimuthal angle of the beam particles. Estimates for the oscillation amplitude in the nonlinear regime are given for these cases.

As is well known, various types of slow waves are excited during the passage through a plasma of a monoenergetic charged-particle beam of low density[1]. In the presence of a magnetic field, the oscillation excitation is most intense under the conditions of Cerenkov or cyclotron resonance for the beam particles. The nonresonant oscillations have a lower increment, and therefore at the initial stage of the development of the instability the monoenergetic particle beam excites a narrow packet of waves whose increment γ is close to the maximum value: $\gamma \sim \gamma_{max} \sim \omega(n_b/n_p)^{1/3}$ (n_b and n_p are the beam- and plasma-particle densities respectively). The investigation of the nonlinear phase of the development of the instability of such a narrow wave packet requires a multimode treatment. However, if the initial plasma-density perturbation is chosen in the form of a near-monochromatic wave ($\tilde{n}(\mathbf{r}, t)|_{t=0} \sim \cos \mathbf{k}\cdot\mathbf{r}$), and the width of the initial wave packet is significantly narrower than that of the wave packet whose waves have increments $\gamma \sim \gamma_{max}$, then the evolution of the plasma instability can be considered in the single-mode approximation.

The nonlinear phase of the single-mode regime of amplification of Langmuir oscillations by an electron beam has been investigated in recent years by a number of authors[2-7]. It is shown in these papers that the oscillations, after the linear phase of exponential growth, go over into a regime of nonlinear saturation, when the beam particles acquire in the field of the wave an oscillator velocity $\tilde{v} \sim \gamma_{max}/k$. In this case the beam particles are captured by the wave field.

The saturation mechanism for Langmuir oscillations amplified by a monoenergetic electron beam is identical with the well-studied[8-11] saturation mechanism for slow waves amplified by an electron beam in a traveling-wave tube (TWT). (Notice only that the picture of the nonlinear beam particle-wave interaction in a TWT is complicated by the Coulomb repulsion of the uncompensated space charge of the beam particles.)

In the present paper we investigate the nonlinear phase of the amplification by a monoenergetic charged-particle beam of the monochromatic longitudinal oscillations of a plasma located in a magnetic field[1)]. A system of equations is obtained which describes the nonlinear evolution of the oscillations under the conditions of Cerenkov or cyclotron resonances for the beam particles. For oscillations propagating in a direction inclined to the magnetic field at an angle not close to $\pi/2$, the primary mechanism responsible for the saturation of the oscillations is, as in the absence of a magnetic field, the oscillation-induced variation of the longitudinal component of the beam-particle velocity. As a result of this variation, the phase resonance between the wave and the beam particles gets destroyed. The maximum amplitude for this case is estimated.

For oscillations propagating in a direction perpendicular to the magnetic field, the oscillation saturation mechanism is substantially different. It is connected with a change in the transverse velocity of the beam particles by a value of the order of the phase velocity of the wave.

The consideration of the single-mode regime is sometimes linked with the discreteness of the wave-number spectrum in a plasma of finite dimensions, when $k_{zn} = 2\pi n/L$, n = 1, 2, ... (k_z is the component of the wave vector and L is the length of the system in the direction of motion of the beam). However, the condition that the neighboring harmonics should have an increment significantly less than γ_{max}, i.e., that

$$|\omega - k_{z\,n\pm1}V_0| = 2\pi V_0/L \gg \gamma_{max} \quad (k_{zn}V_0 \approx \omega),$$

leads to a situation in which the beam particles traverse the plasma in a time $\Delta t = L/V_0$ shorter than the time required by the oscillations to reach the nonlinear phase ($\gamma_{max}\Delta t \ll 1$). In this case the amplification of the oscillations of the confined plasma is due to particles which are successively injected into the plasma and which amplify the oscillations during the time of interaction by a small amount. The nonlinear saturation of the oscillations occurs when the field increases to such a value that in the time taken by a particle to traverse the distance L the hf field induces a phase shift of the order of π between the wave and the particle. A nonlinear theory for this problem has been developed by Kurilko[13].

1. THE MOTION OF A CHARGED PARTICLE IN THE FIELD OF A PLANE WAVE AND IN A CONSTANT MAGNETIC FIELD

In order to derive the basic equations that describe the nonlinear phase of the interaction of the monoenergetic beam with the longitudinal plasma oscillations, we must determine the flux's volume charge density perturbation, which can be computed by integrating the kinetic equation over the trajectories. For this purpose it is necessary to solve the problem of the motion of a charged particle in the electric field of a plane wave

$$\mathbf{E}=-\nabla\varphi=-\mathrm{Re}[ik\varphi_0 e^{i(\mathbf{kr}-\omega t+\alpha)}] \qquad (1)$$

and in a constant magnetic field $\mathbf{B}_0$. In (1), φ_0 is the amplitude, α is the initial phase, and ω and $\mathbf{k}$ are the frequency and wave vector of the longitudinal oscillations.

The equation of motion of the particle

$$m\frac{d\mathbf{v}}{dt}=e\mathbf{E}+\frac{e}{c}[\mathbf{v}\times\mathbf{B}_0] \qquad (2)$$

can be integrated approximately in the case of a weak electric field, when the characteristic drift velocity $v_d = ck\varphi_0/B_0$ of the particle motion is considerably less than the wave's phase velocity $v_{ph} = \omega/k$ and the particle transverse velocity $v_\perp$. In this case the method of averages[14,15] can be used to solve Eq. (2).

Let us choose a system of coordinates with the z axis directed parallel to $\mathbf{B}_0$ and the x axis lying in the plane of the vectors $\mathbf{k}$ and $\mathbf{B}_0$. Then, introducing the variables $v_\perp$, θ, ξ, and η:

$$v_x=v_\perp\cos\theta, \quad v_y=v_\perp\sin\theta, \qquad x=\xi-\frac{v_\perp}{\omega_B}\sin\theta, \quad y=\eta+\frac{v_\perp}{\omega_B}\cos\theta, \qquad (3)$$

where $\omega_B = eB_0/mc$, we rewrite (2) in the form

$$\frac{dv_z}{dt}=\frac{ek_z\varphi_0}{m}\sum_{s=-\infty}^{\infty}J_s(a)\sin(\Phi_s+\alpha),$$

$$\frac{dv_\perp}{dt}=\frac{ek_x\varphi_0}{m}\sum_{s=-\infty}^{\infty}\frac{s}{a}J_s(a)\sin(\Phi_s+\alpha), \qquad (4)$$

$$\frac{d\theta}{dt}=-\omega_B-\frac{ek_x\varphi_0}{mv_\perp}\sum_{s=-\infty}^{\infty}J_s'(a)\cos(\Phi_s+\alpha),$$

where

$$\Phi_s=k_z z+k_x\xi-s\theta-\omega t,$$

$J_s(a)$ is a Bessel function, and $a = k_x v_\perp/\omega_B$. (Notice that the coordinate ξ is an integral of the motion.)

In Eqs. (4), $\theta_1 = \theta$ and $\theta_2 = \omega t - k_z z$ are rapidly rotating phases. Further, let us consider only the resonant particles whose velocity v_z along the constant magnetic field $\mathbf{B}_0$ is close to the resonance velocity, so that the condition

$$\omega-k_z v_z\approx n\omega_B \quad (n=0, \pm1, \pm2, \ldots). \qquad (5)$$

is fulfilled. Retaining in (4) the resonance terms, we obtain for the averaged quantities $\bar{v}_z$, $\bar{v}_\perp$, and $\bar{\theta}$, upon the fulfillment of the condition (5), the following equations:

$$\frac{d\bar{v}_z}{dt}=\frac{ek_z\varphi_0}{m}J_n(\bar{a})\sin(\bar{\Phi}_n+\alpha), \qquad (6)$$

$$\frac{d\bar{v}_\perp}{dt}=\frac{ek_x\varphi_0}{m}\frac{n}{\bar{a}}J_n(\bar{a})\sin(\bar{\Phi}_n+\alpha), \qquad (7)$$

$$\frac{d\bar{\theta}}{dt}=-\omega_B-\frac{ek_x\varphi_0}{m\bar{v}_\perp}J_n'(\bar{a})\cos(\bar{\Phi}_n+\alpha). \qquad (8)$$

It is convenient to introduce in place of Eq. (8) an equation for the "spiral" phase $\bar{\Phi}_n \equiv k_z\bar{z} - \omega t - n\bar{\theta} + k_x\bar{\xi}$:

$$\frac{d\bar{\Phi}_n}{dt}=n\omega_B-\omega+k_z v_z+\frac{ek_x\varphi_0}{m\bar{v}_\perp}nJ_n'(\bar{a})\cos(\bar{\Phi}_n+\alpha). \qquad (9)$$

Further, to simplify the notation, we shall drop the bar over the averages.

For a constant value of the wave amplitude φ_0, Eqs. (6)–(9) have the solutions:

$$\frac{k_z v_z}{\omega_B}-\frac{k_z^2a^2}{2k_x^2n}=C_1, \qquad (10)$$

$$J_n(a)\cos(\Phi_n+\alpha)+c_1\frac{a^2}{2}+Q\frac{a^4}{4}=C_2, \qquad (11)$$

where

$$c_1=\left(C_1-\frac{\omega-n\omega_B}{\omega_B}\right)\frac{m\omega_B^2}{ne\varphi_0k_x^2}, \quad Q=\frac{k_z^2m\omega_B}{2k_x^2n^2e\varphi_0}$$

Equations (10) and (11) determine the particle trajectories in the phase plane (a, Φ_n). Analysis of these solutions shows that the motion of the particles in the phase plane can be bounded or unbounded. Then for $n \neq 0$ (and for a given c_1), several centers of motion can exist in the region $0 \le \Phi_n \le 2\pi$ in the phase plane.

From (7), (9), and (10) we obtain the following equation:

$$\frac{d\Phi_n}{da}=\frac{c_1a+Qa^3+J_n'(a)\cos(\Phi_n+\alpha)}{J_n(a)\sin(\Phi_n+\alpha)}, \qquad (12)$$

whence the coordinates of the singular points a_c and Φ_{nc} in the a-Φ_n phase plane can be determined. We shall not consider the special types of singular points for which $J_n(a_c) = 0$. Then we obtain for the "spiral" phase of the singular point the following values:

$$\Phi_{nc}=l\pi-\alpha \quad (l=0, \pm1, \pm2, \ldots) \qquad (13)$$

and a_c is found from the equation

$$c_1a_c+Qa_c^3+(-1)^lJ_n'(a_c)=0. \qquad (14)$$

For a particle moving along a trajectory, the velocity changes $\Delta v_\perp$ and Δv_z are of the order of, or less than, $(e\varphi_0/m)^{1/2}$ if $k_z \sim k_x$. Taking into account the fact that $\Delta v_\perp \ll v_\perp$, we obtain from (6)–(9) for the trajectories lying near the singular point the equation

$$\frac{d^2(\Phi_n-\Phi_{nc})}{dt^2}-\frac{ek_z^2\varphi_0}{m}J_n(a_c)(-1)^l(\Phi_n-\Phi_{nc})=0. \qquad (15)$$

As follows from (15), the entrapped particles execute about the center oscillatory motion with frequency

$$\Omega=\left|\frac{ek_z^2\varphi_0}{m}J_n(a_c)\right|^{1/2}. \qquad (16)$$

The particle motion is of a different nature when $k_z = 0$. Restricting ourselves to the case when $\omega = n\omega_B$, we obtain for the motion near the singular point

$$\Phi_{nc}=l\pi-\alpha, \quad a_c=\frac{k_xv_{\perp c}}{\omega_B}=p_{n\nu}',$$

where $p'_{n\nu}$ is a root of the function J'_n, the following equation:

$$\frac{d^2(\Phi_n-\Phi_{nc})}{dt^2}+\Omega_\perp^2(\Phi_n-\Phi_{nc})=0. \qquad (17)$$

Here $\Omega_\perp$, the frequency of oscillation of the entrapped particle, is equal to

$$\Omega_\perp=\frac{ek_x^2\varphi_0}{m\omega_B}\left|\frac{n^2J_n(a)J_n''(a)}{a^2}\right|^{1/2}_{a=p'_{n\nu}}. \qquad (18)$$

In contrast to the case $k_z \approx k_x$, when $\Omega \propto \sqrt{\varphi_0}$, the frequency $\Omega_\perp$ is proportional to φ_0 when $k_z = 0$.

2. THE BASIC EQUATIONS

Let us derive the equation for the time variation of the amplitude $\varphi_0(t)$ and phase $\alpha(t)$ of the wave being amplified, taking into account the nonlinear character of the motion of the beam particles under the resonance conditions (5). The plasma oscillations in the case under consideration can be described in the linear approximation[2-7]. Let the wave being amplified have a frequency ω and a growth rate γ. Averaging the Poisson equation over the time interval T ($\gamma^{-1} \gg T \gg \omega^{-1}$) and over space ($kL \gg 1$), we obtain

$$k^2\varepsilon_l(\omega,\mathbf{k})\varphi(t)+k^2\frac{\partial\varepsilon_l}{\partial\omega}i\frac{d\varphi(t)}{dt} = \frac{8\pi e}{L^3T}\int_{-L/2}^{+L/2}dx\,dy\,dz\int_{-T/2}^{+T/2}dt\exp(i\omega t-i\mathbf{kr})\left(\int f_b\,d\mathbf{v}-n_b\right), \tag{19}$$

where $\epsilon_l(\omega, \mathbf{k})$ is the permittivity of the plasma, $\varphi(t) = \varphi_0(t)e^{i\alpha(t)}$, f_b is the beam-particle distribution function, and n_b is the mean value of the beam-particle density. The space charge of the beam particles is assumed to be compensated in the absence of oscillations.

We shall assume that in the absence of oscillations (at t = 0) the beam particles move with constant velocity $v_z = v_{z0}$, have the same Larmor radius, and are uniformly distributed over the azimuthal angle θ, i.e., their distribution function at t = 0 has the form

$$f_b(\mathbf{r}_0,\mathbf{v}_0,0)=\frac{n_b}{2\pi V_{\perp 0}}\delta(v_{\perp 0}-V_{\perp 0})\delta(v_{z0}-V_{z0}). \tag{20}$$

Such particles are sometimes called oscillators. A linear theory of longitudinal plasma oscillation excitation by an oscillator stream has been developed in[16,17].

We shall assume that the frequency ω is close to the frequency $\omega(\mathbf{k})$ of the natural oscillations of the plasma. Using the Liouville theorem ($d\mathbf{r}d\mathbf{v} = d\mathbf{r}_0 d\mathbf{v}_0$ and $f_b(\mathbf{r}, \mathbf{v}, t) = f_b(\mathbf{r}_0, \mathbf{v}_0, 0)$), and going over in (19) to the variables $\bar{\xi}$, $\bar{\eta}$, $\bar{z}$, $\bar{v}_\perp$, $\bar{v}_z$, and $\bar{\theta}$, we obtain

$$k^2\varepsilon_l(\omega,\mathbf{k})\varphi(t)+k^2\frac{\partial\varepsilon_l}{\partial\omega}i\frac{d\varphi}{dt}=4en_b\int_{-\pi}^{+\pi}d\Phi_{n0}J_n(a)e^{-i\Phi_n}, \tag{21}$$

where the integration is over the values of the spiral phase at t = 0. We are considering the excitation of only the resonance frequencies for which the condition (5) is fulfilled. The excitation of oscillations with frequencies that are integral multiples of the fundamental frequency of the wave being amplified can be disregarded, since their amplitude is ω/γ times smaller than that of the fundamental harmonic.

Equations (6), (7), (9), and (21) determine the nonlinear evolution of the monochromatic plasma wave being amplified by the spiral beam. Let us write the basic system of equations in dimensionless variables:

$$\frac{d\varepsilon}{d\tau}=\int_{-1/2}^{1/2}J_n(a)\sin(2\pi\zeta+\alpha)d\zeta_0, \tag{22}$$

$$\varepsilon\left(\frac{d\alpha}{d\tau}-\Delta\right)=\int_{-1/2}^{1/2}J_n(a)\cos(2\pi\zeta+\alpha)d\zeta_0, \tag{23}$$

$$\frac{d\nu}{d\tau}=-\varepsilon J_n(a)\sin(2\pi\zeta+\alpha), \tag{24}$$

$$\frac{da}{d\tau}=-\varepsilon\frac{k_x^2}{k_z^2}\frac{\gamma_0}{\omega_B}\frac{nJ_n(a)}{a}\sin(2\pi\zeta+\alpha), \tag{25}$$

$$\frac{d\zeta}{d\tau}=\frac{\nu}{2\pi}-\frac{\varepsilon}{2\pi}\frac{k_x^2}{k_z^2}\frac{\gamma_0}{\omega_B}\frac{nJ_n'(a)}{a}\cos(2\pi\zeta+\alpha), \tag{26}$$

where

$$\zeta=\frac{\Phi_n}{2\pi},\quad \varepsilon=-\frac{ek_z^2\varphi_0}{m\gamma_0^2},\quad \tau=\gamma_0 t,\quad \nu=\frac{k_z}{\gamma_0}\left(v_z-\frac{\omega-n\omega_B}{k_z}\right),$$
$$\gamma_0=\left(\frac{2k_z^2}{k^2}\frac{\omega_b^2}{\partial\varepsilon_l/\partial\omega}\right)^{1/3},\quad \Delta=\frac{\omega-\omega(\mathbf{k})}{\gamma_0},$$

$\omega(\mathbf{k})$ is the frequency of the natural plasma oscillations, and ω_b is the Langmuir frequency of the beam particles.

3. THE DEVELOPMENT OF THE OSCILLATIONS

In the linear phase of the development of the oscillations, we shall seek the solution to the system of equations (22)–(26) in the form $\epsilon(\tau) = \epsilon_0 e^{\Gamma\tau}$ and $\alpha(\tau) = \alpha_0 + \sigma\tau$. Considering the terms linear in the field, we obtain from (24)–(26) the solutions

$$a=a_0+\Delta a,\quad \zeta=\frac{\nu_0}{2\pi}\tau+\zeta_0+\Delta\zeta, \tag{27}$$

where $a_0 = k_x v_{\perp 0}/\omega_B$,

$$\Delta a=-\varepsilon_0 e^{\Gamma\tau}\frac{k_x^2}{k_z^2}\frac{\gamma_0}{\omega_B}\frac{nJ_n(a_0)}{a_0}\frac{1}{2i}\left[\frac{\exp\{i(2\pi\zeta_0+\nu_0\tau+\alpha_0+\sigma\tau)\}}{\Gamma+i(\nu_0+\sigma)}-\text{c.c.}\right],$$
$$\Delta\zeta=-\frac{\varepsilon_0}{2\pi}e^{\Gamma\tau}J_n(a_0)\frac{1}{2i}\left[\frac{\exp\{i(2\pi\zeta_0+\nu_0\tau+\alpha_0+\sigma\tau)\}}{(\Gamma+i(\nu_0+\sigma))^2}-\text{c.c.}\right] \tag{28}$$
$$-\frac{\varepsilon_0}{2\pi}e^{\Gamma\tau}\frac{k_x^2}{k_z^2}\frac{\gamma_0}{\omega_B}\frac{nJ_n'(a_0)}{2a_0}\left[\frac{\exp\{i(2\pi\zeta_0+\nu_0\tau+\alpha_0+\sigma\tau)\}}{\Gamma+i(\nu_0+\sigma)}-\text{c.c.}\right].$$

Using Eqs. (22), (23), and (27), we obtain

$$\left.\frac{\partial\varepsilon_l}{\partial\omega}\right|_{\omega(\mathbf{k})}(\omega'-\omega(\mathbf{k}))=\frac{k_z^2}{k^2}J_n^2(a_0)\frac{\omega_b^2}{(\omega'-k_zV_{z0}-n\omega_B)^2}+\frac{2k_x^2}{k^2}nJ_n'(a_0)J_n(a_0)\frac{\omega_b^2}{\omega_B(\omega'-k_zV_{z0}-n\omega_B)} \tag{29}$$

where $\omega' = \omega + \gamma_0(i\Gamma - \sigma)$.

Equation (29) is, in the presence of an oscillator stream with the distribution function (20), a dispersion equation for the longitudinal plasma oscillations, an equation which has been investigated in[16,17]. Since the flux density n_b is assumed to be low in comparison with the plasma density, an instability develops only under resonance conditions ($\omega' \approx k_zV_{z0} + n\omega_B$). If $k_z \sim k_x$ and a_0 is not close to a root $p_{n\nu}$ of the Bessel function, or, more exactly, if the inequality

$$\frac{\gamma_0}{\omega_B}\frac{k_x^2}{k_z^2}\frac{nJ_n'(a)}{J_n^{1/3}(a)}\ll 1, \tag{30}$$

is fulfilled, then the second term on the right-hand side of (29) can be neglected. Then, assuming that the frequency $k_zV_{z0} + n\omega_B$ is close to the frequency $\omega(\mathbf{k})$ of the natural oscillations of the plasma, we find that

$$\omega-\omega(\mathbf{k})=(i-3^{-1/2})\gamma_L, \tag{31}$$

where

$$\gamma_L=\frac{\sqrt{3}}{2}\left|\frac{k_z^2}{k^2}\frac{\omega_b^2J_n^2(a_0)}{\partial\varepsilon_l/\partial\omega}\right|^{1/3}. \tag{32}$$

If, in particular, the velocity V_{z0} is considerably higher than the thermal velocity of the plasma electrons, then there get excited those electronic oscillations of the "cold" plasma having the frequencies

$$\omega(\mathbf{k})=\pm\omega_\pm(\vartheta),$$
$$\omega_\pm^2={}^1/_2(\omega_{pe}^2+\omega_{Be}^2)\pm{}^1/_2[(\omega_{pe}^2+\omega_{Be}^2)^2-4\omega_{pe}^2\omega_{Be}^2\cos^2\vartheta]^{1/2}, \tag{33}$$

(where $\cos\vartheta = k_z/k$) and the maximum growth rate

$$\gamma_L=\frac{\sqrt{3}}{2^{4/3}}\left|\frac{\cos^2\vartheta\,\omega_b^2\omega_\pm(\omega_\pm^2-\omega_{Be}^2)}{\omega_+^2-\omega_-^2}J_n^2(a_0)\right|^{1/3}. \tag{34}$$

The increment (34) for $\omega_{pe} \sim \omega_{Be}$, $k_z \sim k_x$, and $a_0 \sim 1$ has the same order of magnitude as the increment of the Langmuir oscillations excited by a beam when $\mathbf{B}_0 = 0$.

In analyzing the nonlinear oscillation regime, let us first consider the case when the quantity k_z is large enough for the inequality (30) to be fulfilled and the quantity a is not close to $p_{n\nu}$. Then in the linear phase the oscillations grow with the increment (34). If the inequality

$$\frac{k_x^2}{k_z^2}\left|\frac{\gamma_0}{\omega_B}\frac{n}{a}J_n'(a)\right| \ll 1, \qquad (35)$$

is valid, then the second term on the right-hand side of Eq. (26) can be neglected. Furthermore, we can disregard in Eqs. (22)—(24) the time variation of the quantity a, and set $a = a_0$.

Then setting

$$\varepsilon'=g\varepsilon,\quad \tau'=g^{-2}\tau,\quad \nu'=g^2\nu,\quad \Delta'=g^{-2}\Delta, \qquad (36)$$

where $g = J_n^{-1/3}(a_0)$, we obtain for the quantities ϵ', α, ν', and ζ the "universal" system of equations

$$\frac{d\varepsilon'}{d\tau'}=\int_{-1/2}^{1/2}\sin(2\pi\zeta+\alpha)\,d\zeta_0,$$

$$\varepsilon'\left(\frac{d\alpha}{d\tau'}-\Delta'\right)=\int_{-1/2}^{1/2}\cos(2\pi\zeta+\alpha)\,d\zeta_0, \qquad (37)$$

$$\frac{d\nu'}{d\tau'}=-\varepsilon'\sin(2\pi\zeta+\alpha),\quad \frac{d\zeta}{d\tau'}=\frac{\nu'}{2\pi}.$$

These equations coincide with the equations that determine the development of the Langmuir oscillations excitable by a monoenergetic electron beam when $\mathbf{B}_0 = 0$, equations which have been derived and thoroughly investigated before[2,3-6]. As in the isotropic plasma, when the conditions (35) are fulfilled, the dominant nonlinear effect limiting the growth of the amplitude of the longitudinal oscillations excited in a magnetoactive plasma by an oscillator stream is the phase shift between the wave and the stream particles that results from the oscillation-field induced change in the longitudinal component of the stream-particle velocity. As follows from the results of[2,3-6], the saturation of the oscillations occurs at $\tau' \sim 5$—10, when the quantity ϵ' oscillates about the value $\epsilon' \sim 1$. It follows from this that the amplitude of the electric field of the oscillations in the nonlinear phase is equal to

$$E=k\varphi_0\sim[4\pi n_b m V_{z0}^2]^{1/2}\frac{k}{k_z}\frac{\gamma_L^2}{\omega_b|\omega(\mathbf{k})-n\omega_B|J_n(a_0)}. \qquad (38)$$

This estimate for the quantity E is easy to obtain if we take into account the fact that the saturation of the oscillations sets in when the frequency of oscillation of the entrapped particles of the stream becomes equal in order of magnitude to the linear increment (i.e., when $\Omega \sim \gamma_L$).

When $\omega_{pe} \sim \omega_{Be}$ and $k_z \sim k_x$, we have

$$E\sim[4\pi n_b m V_{z0}^2]^{1/2}\left(\frac{n_b}{n_p}\right)^{1/4}J_n^{1/3}(a_0). \qquad (39)$$

It follows from this that for $\mathbf{B}_0 \neq 0$, and under the resonance conditions (5), the maximum amplitude decreases by a factor of $J_n^{-1/3}(a_0)$. This is due to the fact that when $\mathbf{B}_0 \neq 0$ the linear increment ($\gamma_L \propto J_n^{2/3}$) decreases, and although the oscillation frequency Ω of the entrapped particles is proportional to $J_n^{1/2}$, the condition $\gamma_L \sim \Omega$ is fulfilled at smaller values of the field amplitude.

If $k_z \ll k_x$, then the quantity (38) increases, in comparison with the $\mathbf{B}_0 = 0$ case, by a factor of $(k/k_z)^{2/3}$. This is connected with the fact that the disappearance of the phase resonance is due to the field component $E_z = k_z\varphi_0$, which in this case is not only significantly less than E, but is $(k/k_z)^{1/3}$ times lower than the maximum intensity of the electric field in the isotropic case (the latter is due to the fact that $\gamma_L \sim k_z^{4/3}$).

It follows from the results of[3-6] that under the action of the oscillation field the particles group themselves in the phase plane (v_z, Φ_n) into bunches, which are captured by the wave field in the nonlinear phase and made to execute a complicated spiral motion with a characteristic frequency equal to Ω.

Alternatively, there occurs a buildup of oscillations propagating in a direction perpendicular to the magnetic field. For $k_z = 0$ the linear increment is equal to

$$\gamma=\omega_b\left[-\frac{2nJ_n(a_0)J_n'(a_0)}{\omega_B\partial\varepsilon_l/\partial\omega}\right]^{1/2}, \qquad (40)$$

where $\omega = \omega(\mathbf{k}) = n\omega_B$ and $J_n(a_0)J_n'(a_0) < 0$. In particular, for the oscillations of a cold plasma, we have

$$\omega(\mathbf{k})=(\omega_{pe}^2+\omega_{Be}^2)^{1/2}=n\omega_{Be},$$
$$\gamma=(n_b/n_p)^{1/2}[-(n^2-1)J_n(a_0)J_n'(a_0)]^{1/2}|\omega_{Be}|. \qquad (41)$$

In contrast to the case $k_z \sim k_x$, for $k_z = 0$, only oscillations with sufficiently large values of a_0, i.e., with $\gamma \sim \sqrt{n_b}$, are excited.

An estimate for the maximum amplitude of the oscillations can be obtained from the condition $\gamma \sim \Omega_\perp$, where the quantities γ and $\Omega_\perp$ are given by the formulas (41) and (18). Assuming, for simplicity, that $a_0 \sim 1$ and $n \sim 1$, we obtain

$$E\sim[4\pi n_b m V_{\perp0}^2]^{1/2}. \qquad (42)$$

This estimate can be obtained from the equations which describe the evolution of the oscillations at the resonance $\omega(\mathbf{k}) = n\omega_B$ in the nonlinear phase, and which are similar to the Eqs. (37) for the "oblique" propagation:

$$\frac{d\varepsilon}{d\tau}=-\frac{1}{2\pi}\int_{-\pi}^{\pi}d\Phi_{n0}J_n(a)\sin(\Phi_n+\alpha),$$

$$\left(\Delta-\frac{d\alpha}{d\tau}\right)\varepsilon=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\Phi_{n0}J_n(a)\cos(\Phi_n+\alpha),$$

$$\frac{da}{d\tau}=\varepsilon\frac{nJ_n(a)}{a}\sin(\Phi_n+\alpha), \qquad (43)$$

$$\frac{d\Phi_n}{d\tau}=\varepsilon\frac{nJ_n'(a)}{a}\cos(\Phi_n+\alpha),$$

where

$$\varepsilon=\frac{ek_x^2\varphi_0}{m\omega_b}\left(\frac{\partial\varepsilon_l}{\partial\omega}\frac{1}{2\omega_B}\right)^{1/2},\quad \tau=\gamma_1 t,$$

$$\gamma_1=\omega_b\left(\frac{\partial\varepsilon_l}{\partial\omega}\frac{\omega_B}{2}\right)^{-1/2},\quad \Delta=\frac{n\omega_B-\omega(\mathbf{k})}{\gamma_1}.$$

It follows from (43) that the oscillation saturation begins when $\epsilon \sim 1$. The quantity a, the spiral phase Φ_n, and the azimuthal angle θ then change in value in a time $\sim\Omega_\perp^{-1}$ by the amounts $\Delta a \sim 1$, $\Delta\Phi_n \sim \pi$, and $\Delta\theta \sim \pi$. Thus, in spite of the fact that the linear increment for $k_z = 0$ is less than for $k_z \sim k_x$, because for $k_z = 0$ the Doppler effect does not lead in the nonrelativistic case to a change in the phase resonance between the wave and a particle, the nonlinear effects turn out in the $k_z = 0$ case to be important only for the oscillation amplitude (42), which can be appreciably larger than the oscillation amplitude (39) for the $k_z \sim k_x$ case. Thus, according to (42), for $k_z = 0$, the energy transferred by the electron beam to the plasma oscillations is of the same order of magnitude as the beam energy.

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[1)] We note that the damping of a monochromatic wave propagating in a direction perpendicular to a weak magnetic field has been investigated by Sagdeev and Shapiro [12].

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