Nonlinear Fraunhofer diffraction

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The problem of diffraction of an intense electromagnetic wave by an opaque screen with slits in front of a nonlinear medium is solved exactly within the framework of the nonlinear parabolic equation. Expressions are derived for the angular distribution of the diffracted radiation intensity. The waveguide channel production thresholds are found.

The nonlinear dependence of the index of refraction of a medium on the field of a wave propagating in it leads, as is well known, to a number of remarkable effects. Large numbers of both experimental and theoretical studies have been devoted to such effects as the self-focusing of radiation entering the medium and the waveguide propagation of intense light beams. These effects are specific for the nonlinear medium. There is also interest, however, in the problem of the effect of the nonlinearity of the medium on such classical linear effects as, for example, Fraunhofer diffraction. A complete theoretical investigation of these questions has not been possible until recently because of the absence of appropriate analytical techniques. The situation was changed by the paper of Zakharov and Shabat,[1] in which the method of the inverse scattering problem for a nonlinear parabolic equation was proposed, within the framework of which all the enumerated phenomena were considered.

In the present paper, the method of Zakharov and Shabat[1] is applied to the problem of the diffraction of a wave by a screen with slits backed by a nonlinear medium. Exact expressions are obtained for the intensity of the diffracted radiation as a function of direction, which are identical, in the limit of small intensity of the incident wave, with well-known formulas (see[2]). From these expressions, it is easy to find the positions of diffraction minima and maxima, which depend in simple fashion on the nonlinear characteristics of the medium and the intensity of the incident wave. In principle, this allows us to make reliable measurements of the parameters which characterize the nonlinearity of the medium.

The results that have been obtained are valid up to the wave intensities which lead to the formation of a waveguide channel. Here it turns out that the intensity of the radiation diffracted at zero angle behaves as $\ln[I_{cr}/(I_{cr}-I)]$ as $I \to I_{cr}$, where I is the intensity of the incident wave and I_{cr} represents the threshold of waveguide channel production.

From the mathematical viewpoint, the problem under consideration reduces to study of the asymptotic behavior of the solution of the Cauchy problem for a nonlinear Schrodinger equation. The results previously obtained along these lines reduce to the statement that the solution approaches zero asymptotically almost everywhere. In the present study, some integral relations are established which do not give the actual asymptotes of the solution, but which are quite satisfactory for consideration of the diffraction problem. Analytic relations can also be established for the well-known Kortweg-de Vries equation, in which we also employ the method of the inverse scattering problem.[3]

1. ASYMPTOTIC STATES

Two-dimensional stationary self-focusing of an electromagnetic wave is described by the equation[4,5]

$$2ik\frac{\partial E}{\partial z}+\frac{\partial^2 E}{\partial x^2}=-k^2\frac{\delta n_{nl}}{n_0}|E|^2E \tag{1}$$

for the complex envelope E. It is assumed that the index of refraction n varies as $n = n_0 + \delta n_{nl}|E|^2$ (cubic nonlinearity).

We introduce the new variable $t = z/2k$ and denote $k^2\delta n_{nl}/n_0$ by κ. In what follows, we limit ourselves to the case of focusing media, i.e., we shall assume that $\kappa > 0$. Equation (1) is written in the standard form

$$i\frac{\partial E}{\partial t}+\frac{\partial^2 E}{\partial x^2}+\kappa|E|^2E=0. \tag{2}$$

We shall call the variable t the time (which is very convenient, since Eq. (2) also describes the longitudinal self-modulation of a quasimonochromatic wave (see[6]), where t is the time).

The method of Zakharov and Shabat[1] of solving the Cauchy problem for Eq. (2) consists in the following. We consider the set of linear equations

$$\frac{\partial u_1}{\partial x}+i\xi u_1=q(x)u_2 \qquad \frac{\partial u_2}{\partial x}-i\xi u_2=-q^*(x)u_1, \tag{3}$$

where $q(x) = i(\kappa/2)^{1/2}E(x, 0)$, and ξ is an arbitrary real parameter. If the initial condition for Eq. (2) is that $E(x, 0)$ fall off sufficiently rapidly as $|x| \to \infty$, then each solution of (3) with the asymptote $u_1 = e^{-i\xi x}$, $u_2 = 0$ as $x \to -\infty$, has a definite asymptote as $x \to +\infty$, which we shall write down in the form $a(\xi)e^{-i\xi x} = u_1$, $u_2 = b(\xi)e^{-i\xi x}$. It turns out here (see[1]) that $a(\xi)$ is analytic in the upper half-plane of the complex variable ξ and $|a(\xi)|^2 + |b(\xi)|^2 = 1$ for real ξ.

The first stage in the solution of the Cauchy problem consists in the determination of $a(\xi)$, $b(\xi)$ (the scattering matrix). Knowledge of the scattering matrix for the set (3) has fundamental value, inasmuch as it turns out that: first, if $E(x, t)$ changes in time in accord with eq. (2), then $a(\xi)$ does not depend on the time and $b(\xi, t) = b(\xi, 0)\exp(4i\xi^2t)$; second, the "potential" q(x) is uniquely established by the scattering matrix, for which it suffices to solve the following set of linear integral equations for the functions $K_{1,2}(x, y)$:

$$K_1(x,y)=F^*(x+y)+\int_x^\infty K_2^*(x,s)F^*(s+y)\,ds,$$
$$K_2^*(x,y)=-\int_x^\infty K_1(x,s)F(s+y)\,ds, \tag{4}$$

where[1)]

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\xi)}{a(\xi)}e^{i\xi x}d\xi, \qquad (5)$$

q(x) is expressed in terms of the solution of the set (4):

$$q(x)=-2K_1(x,x), \qquad \int_x^\infty |q(s)|^2 ds=-2K_2(x,x). \qquad (6)$$

It is obviously not possible to obtain the solution of the set (4) in the general case. However, effective study of the asymptotic behavior of the solution of (4) as $t\to\infty$ turns out to be possible. We multiply the first of Eqs. (4) by $e^{i\xi y}$, the second by $e^{-i\xi y}$, and integrate them over y from x to ∞. We denote the integrals

$$\int_x^\infty K_{1,2}(x,y)e^{i\xi y}dy$$

by $K_{1,2}(x,\xi)$. We obtain the following equations for these quantities:

$$K_1(x,\xi)=-\frac{1}{2\pi i}\int d\xi'\frac{b^*(\xi')}{a^*(\xi')}\frac{e^{i(\xi-2\xi')x}}{\xi-\xi'+i0}-\frac{1}{2\pi i}\int d\xi' K_2^*(x,\xi')\frac{b^*(\xi')}{a^*(\xi')}\frac{e^{i(\xi-\xi')x}}{\xi-\xi'+i0},$$

$$K_2^*(x,\xi)=-\frac{1}{2\pi i}\int K_1(x,\xi')\frac{b(\xi')}{a(\xi')}\frac{e^{-i(\xi-\xi')x}}{\xi-\xi'-i0}d\xi'. \qquad (7)$$

For their derivation, we made use of the fact that

$$\int_x^\infty F(s+y)e^{-i\xi y}dy=-\frac{1}{2\pi i}\int d\xi'\frac{b(\xi')}{a(\xi')}e^{i\xi' s}\frac{e^{i(\xi'-\xi)x}}{\xi'-\xi+i0}.$$

We consider the asymptotic form of the solutions of the set (7) on the lines $x-vt=\text{const}$, $t\to\infty$. We note in advance that the zeroth order of the asymptotic expansion of the integral

$$I=\int f(\xi')\frac{e^{i\Phi(\xi')t}}{\xi'-\xi-i0}d\xi'$$

in powers of $1/t$ has the form

$$I=\begin{cases}2\pi i f(\xi)e^{i\Phi(\xi)t}, & \Phi'(\xi)>0\\ 0, & \Phi'(\xi)<0\end{cases}. \qquad (8)$$

Therefore, as $t\to\infty$, one can write formulas (7) in the form

$$K_1(x,\xi)=c_1^*(\xi)\exp[-4i\xi^2t-i\xi x]+$$
$$+\frac{1}{2\pi i}\int d\xi' K_2^*(x,\xi')c^*(\xi')\frac{\exp(4i\xi'^2t+i(\xi-\xi')x)}{\xi'-\xi-i0},$$
$$K_2^*(x,\xi)=\frac{1}{2\pi i}\int d\xi' K_1(x,\xi')c(\xi')\frac{\exp(4i\xi'^2t-i(\xi-\xi')x)}{\xi'-\xi+i0}.$$

Here $c(\xi)=b(\xi)/a(\xi)$, and $c_1(\xi)$, in view of (8), is

$$c_1(\xi)=\begin{cases}c(\xi), & \xi<-v/4\\ 0, & \xi>-v/4\end{cases}. \qquad (9)$$

We seek the solution of the set (7) in the form

$$K_1(x,\xi)=A(\xi)\exp(-4i\xi^2t-i\xi x),\quad K_2(x,\xi)=B(\xi)e^{i\xi x},$$

($x=vt+x_0$). Using (8), we obtain for A and B:

$$A(\xi)=c_1^*(\xi)(1+B(\xi)),$$
$$B(\xi)=\frac{1}{2\pi i}\int\frac{A(\xi')c(\xi')}{\xi'-\xi+i0}d\xi'. \qquad (10)$$

The solution of the set (10) can now be found in general form. Specifically, we consider a function $a_1(\xi)$, which is analytic in the upper half-plane, and which does not have zeros there, such that $|a_1(\xi)|=|a(\xi)|$ if $\xi<-v/4$, and $|a_1(\xi)|=1$ for $\xi>-v/4$. Further, let $b_1=c_1a_1$. Then, as is not difficult to establish, $B(\xi)=a_1^*-1$, $A(\xi)=b_1^*(\xi)$. It is obvious that this solution satisfies the first of Eqs. (10). We now verify the satisfaction of the second. Its right-hand side is

$$\frac{1}{2\pi i}\int\frac{b_1^*(\xi')c(\xi')}{\xi'-\xi+i0}d\xi'=\frac{1}{2\pi i}\int\frac{|b_1|^2}{a_1}\frac{d\xi'}{\xi'-\xi+i0}$$
$$=\frac{1}{2\pi i}\int\left(\frac{1}{a_1(\xi')}-1-a_1^*(\xi')+1\right)\frac{d\xi'}{\xi'-\xi+i0}. \qquad (11)$$

Inasmuch as $a_1(\xi)$ has no zeros in the upper half-plane and $a_1\to1$ as $\xi\to\infty$, we have

$$\int\left(\frac{1}{a_1(\xi')}-1\right)\frac{d\xi'}{\xi'-\xi+i0}=0.$$

The contour of integration for the remaining integral can be closed in the lower half-plane, since $a_1^*(\xi^*)$ is an analytic function of ξ there. In this case,

$$-\frac{1}{2\pi i}\int\frac{a_1^*(\xi')-1}{\xi'-\xi+i0}d\xi'=a_1^*(\xi)-1,$$

i.e., the second of Eqs. (10) is also satisfied.

It remains to find $a_1(\xi)$, knowing the modulus of this function on the real axis:

$$|a_1(\xi)|=\begin{cases}|a(\xi)|, & \xi<-v/4\\ 1, & \xi>-v/4\end{cases}. \qquad (12)$$

The function $a_1(\xi)$ has no zeros in the upper halfplane; therefore $\ln a_1(\xi)$ is analytic for $\operatorname{Im}\xi>0$. On the real axis, $\ln a_1(\xi)=\ln|a_1(\xi)|+i\arg a_1(\xi)$ and $\arg a_1(\xi)$ is easily found:

$$\arg a_1(\xi)=-\frac{1}{\pi}\int_{-\infty}^{-v/4}\frac{\ln|a(\xi')|d\xi'}{\xi'-\xi}. \qquad (13)$$

We have thus found $K_{1,2}(x,\xi)$. In particular,

$$K_2(x,\xi)=(a_1(\xi)-1)e^{i\xi x}.$$

Carrying out the Fourier transformation with respect to y, we find

$$K_2(x,x)=\lim_{y\to x+0}\frac{1}{2\pi}\int(a_1(\xi)-1)e^{i\xi(x-y)}d\xi.$$

this last integral is determined by the residue of the function $a_1(\xi)$ at infinity, which is easily found from (13). As a result, we have that as $t\to\infty$, $x-vt=\text{const}$,

$$\lim K_2(x,x)=-\frac{1}{\pi}\int_{-\infty}^{-v/4}\ln\frac{1}{|a(\xi)|}d\xi.$$

Using Eq. (6), we find that

$$\lim_{t\to\infty}\int_{x_0+vt}^\infty |q(s,t)|^2ds=\frac{2}{\pi}\int_{-\infty}^{-v/4}\ln\frac{1}{|a(\xi)|}d\xi. \qquad (14)$$

The right side of this formula represents the part of the wave packet moving with velocity greater than v. The integrand on the left side is identical, with accuracy to within a factor, with the canonical actions for the Hamiltonian system (2) (see[7]). Relation (14) shows that these canonical operations are simply connected with the asymptotic state of the system.

The expressions (12), (13) for $a_1(\xi)$ and $b_1(\xi)$, given by $b_1(\xi)=c_1(\xi)a_1(\xi)$, give us in essence the matrix of scattering by the part of the potential moving with velocity greater than v, which allows us to establish a collection of relations of the type (14) for the integrals of certain polynomials in q(x) and its derivatives. However, this is already beyond the scope of the present paper.

Relation (14) is sufficient for the solution of the problem of the diffraction of a nonlinear wave on a screen with a slit (or any set of slits). Inasmuch as $t=z/2k$, Eq. (14) essentially gives us the integrated intensity of the diffracted radiation at angles larger than $\arctan(v/2k)$. Since Eq. (1) is applicable only for the consideration of diffraction at small angles, then, by setting the angle $\theta \ll 1$, we obtain a simple expression for the intensity of the radiation diffracted in the angular range θ to $\theta + d\theta$:

$$dI(\theta)=\frac{k}{\pi\varkappa}\ln\left|a\left(-\frac{k\theta}{2}\right)\right|^{-2}d\theta. \qquad (15)$$

2. SLIT DIFFRACTION

We apply the general relations obtained above to the problem of diffraction from a slit. Let a screen be placed in the plane $z = 0$, and let the slit be a strip $0 < x < l$. We shall assume the field in the plane of the screen to be given:

$$E(x)=\begin{cases}0, & x<0,\ x>l\\ E_0, & 0<x<l\end{cases}. \qquad (16)$$

The scattering matrix for the system (3) with potential $q(x)$, which is of the form (16), can easily be found. Simple calculations give

$$a(\xi)=\frac{1}{2\chi(\xi)}[(\chi+\xi)e^{-i\chi l}+(\chi-\xi)e^{i\chi l}]e^{i\xi l},$$
$$\chi(\xi)=\operatorname{sign}\xi(\xi^2+|q_0|^2)^{1/2}. \qquad (17)$$

Then,

$$\frac{1}{|a(\xi)|^2}=\frac{\xi^2+|q_0|^2}{\xi^2+|q_0|^2\cos^2\chi l}$$

Substituting the resultant expression in (15), we obtain the angular intensity distribution of the diffracted radiation:

$$\frac{dI(\theta)}{d\theta}=\frac{k}{\pi\varkappa}\ln\left\{\frac{k^2\theta^2/4+|q_0|^2}{k^2\theta^2/4+|q_0|^2\cos^2[l(k^2\theta^2/4+|q_0|^2)^{1/2}]}\right\}, \qquad (18)$$

where $|q_0|^2 = \kappa I_0/2l$, and I_0 is the integrated intensity of the wave incident on the slit:

$$I_0=\int|E|^2dx=|E_0|^2l.$$

In the limit $\kappa \to 0$, we have from (18)

$$dI(\theta)=\frac{2I_0}{\pi lk}\sin^2\frac{kl\theta}{2}\frac{d\theta}{\theta^2}, \qquad (19)$$

which is identical with the well-known expression for Fraunhofer diffraction (see, for example,[2]). The zeros of Eq. (18) occur at points θ where

$$\cos^2[l((k\theta)^2/4+|q_0|^2)^{1/2}]=1,$$

i.e.,

$$\theta^2_{min}=4\pi^2n^2/k^2l^2-2\delta n_{nl}I_0/l. \qquad (20)$$

In linear theory $(\theta^2_{min})_l = 4\pi^2n^2/k^2l^2$. Hence,

$$\theta^2_{min}=(\theta^2_{min})_l-2\delta n_{nl}I_0/l. \qquad (21)$$

It is not difficult to establish the fact that exactly the same relation holds for the positions of the diffraction maxima, i.e.,

$$\theta^2_{max}=(\theta^2_{max})_l-2\delta n_{nl}I_0/l.$$

It is important to note that the value of the shift of the maximum or minimum does not depend on the integrated intensity. Everything is determined by the value of $|E_0|^2$ in the plane of the screen.

As follows from (18), the intensity at the diffraction maximum $\theta = 0$ behaves as

$$\frac{k}{\pi\varkappa}\ln\cos^{-2}\left(I_0l\frac{\varkappa}{2}\right)^{-1/2}.$$

This expression becomes infinite for $I_0 l\kappa/2 = \pi^2/4$, i.e.,

$$I_{cr}=\frac{\pi^2}{2l}\frac{n_0}{\delta n_{nl}k^2}. \qquad (22)$$

As $I_0 \to I_{cr}$ the intensity of the diffraction at zero angle behaves as

$$\left(\frac{dI}{d\theta}\right)_{\theta=0}=\frac{2k}{\pi\varkappa}\ln\frac{2\pi}{(I_{cr}-I)l\varkappa}.$$

The critical value of the intensity (22) is the threshold for production of a uniform waveguide channel, which was found previously.[8] Finally, we note that, in accord with what was said above, all the results of the present section are applicable in the subcritical regime, i.e., for $0 < I < I_{cr}$.

3. DIFFRACTION BY A PLANE GRATING

We now consider diffraction from a grating, i.e., in a system of N identical slits; the distance between the centers of neighboring slits is denoted by L. As before, we shall assume the field in the plane of the screen to be specified, setting $E(x) = 0$ if x does not fall on any slit and $e(x) = E_0$ in the opposite case. The scattering matrix of the system (3) with $y(x)$ of the given type is represented in the form of the product of matrices on potentials of the type (16). This feature allows us to find $|a(\xi)|^2$:

$$|a(\xi)|^2=1-\frac{|q_0|^2}{\xi^2+|q_0|^2}\sin^2\chi l\frac{\sin^2N\lambda}{\sin^2\lambda}. \qquad (23)$$

Here l is the width of the slit, $\chi^2 = \xi^2 + |q_0|^2$, and the quantity λ is

$$\lambda=\arccos(\cos\chi l\cos\xi(L-l)-\xi\chi^{-1}\sin\chi l\sin\xi(L-l)).$$

The angular intensity distribution of the diffracted radiation is given by the general expression (15). The transition to the linear limit can be made in the following way. If $\kappa \to 0$, then $\lambda = \xi$; here it is easy to establish the fact that $\lambda = \xi L$. Further, expanding $\ln|a(\xi)|^{-2}$ in powers of κ, we obtain the well-known expression (see[3]):

$$\frac{dI}{d\theta}=\frac{2I_{tot}}{\pi Nkl}\frac{\sin^2{}^1\!/_2kl\theta\sin^2({}^1\!/_2NkL\theta)}{\theta^2\sin^2({}^1\!/_2kL\theta)}.$$

Here I_{tot} is the total light intensity incident on all the slits.

We now turn to the nonlinear problem. It is evident that the intensity of the diffraction at zero angle is given by

$$\left(\frac{dI(\theta)}{d\theta}\right)_{\theta=0}=\frac{k}{\pi\varkappa}\ln\frac{1}{\cos^2N|q|l}.$$

This expression becomes infinite for $N|q|l = \pi/2$. Here the total intensity of the radiation incident on the slit is $I_{cr} = \pi^2/2\kappa Nl$. At intensities exceeding this value, a waveguide channel is formed (to which corresponds the δ-like singularity in $dI/d\theta$). It is curious to note that both the threshold value of the intensity and the dependence $(dI/d\theta)_{\theta=0}$ on I_{tot} for a set of N slits of width l are exactly the same as the same quantities found in the previous section for a single slit, the width of which is Nl.

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1)The expression given for F(x) is valid only in the absence of zeros in $a(\xi)$ in the upper half-plane of ξ. Inasmuch as the appearance of such zeros leads to the generation of a waveguide channel, the representation (5), together with all the subsequent formulas, is valid for incident wave intensities not sufficient for the production of a waveguide channel.

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