"Kirchhoff" form of fluctuation-dissipation theorem for distributed systems

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It is shown that the "Kirchhoff" form of the fluctuation-dissipation theorem, derived and used earlier in electrodynamics, can be extended to any fields which satisfy the reciprocity theorem. A derivation is given of the conditions which must be satisfied by the spatial operators which occur in the spectral equations of such fields. A theorem analogous to the complex Lorentz lemma in electrodynamics is proved subject to these conditions. This makes it possible to extend the other (equilibrium) form of the fluctuation-dissipation theorem to any fields which are in equilibrium throughout the space occupied by them.

1. INTRODUCTION

The fluctuation-dissipation theorem[1-3] expresses the spectral densities of equilibrium thermal fluctuations in any (generally nonlinear) dissipation system in terms of the susceptibility matrix which describes the macroscopic dynamics or kinetics of this system in the linear approximation. We shall be interested in the general case of distributed systems whose states are described by a set of fluctuation fields $\xi_\alpha(t, \mathbf{r})$, which correspond to fields of the Langevin random forces $f_\alpha(t, \mathbf{r})$. The correspondence is manifested by the fact that the average power required by the system is

$$\langle W\rangle=\int\limits_V \langle f_\alpha(t, \mathbf{r})\dot{\xi}_\alpha(t, \mathbf{r})\rangle d^3r \quad (1)$$

(the angular brackets represent averaging over an ensemble and the integral applies to the whole volume V occupied by the total field).

The spectral amplitudes of the fields

$$\xi_\alpha(\omega, \mathbf{r})=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\xi_\alpha(t, \mathbf{r})e^{-i\omega t}\,dt$$

and of the forces $f_\alpha(\omega, \mathbf{r})$ obey the linearized equations of motion

$$B_{\alpha\beta}\xi_\beta(\omega, \mathbf{r})=f_\alpha(\omega, \mathbf{r}) \quad (2)$$

or in the resolved form

$$\xi_\alpha(\omega, \mathbf{r})=A_{\alpha\beta}f_\beta(\omega, \mathbf{r}), \quad (3)$$

where $A_{\alpha\beta}$ are spatial operators (elements of the susceptibility operator matrix $A = B^{-1}$).

In the case of complex amplitudes, ξ_α and f_α, the fluctuation-dissipation theorem has the form[4,3]

$$\langle\xi_\alpha(\omega, \mathbf{r})\xi_\beta^*(\omega, \mathbf{r}')\rangle=\frac{i\Theta}{2\pi\omega}(A_{\alpha\beta}-A_{\beta\alpha}^{*\prime})\delta(\mathbf{r}-\mathbf{r}'), \quad (4)$$

$$\langle f_\alpha(\omega, \mathbf{r})f_\beta^*(\omega, \mathbf{r}')\rangle=\frac{i\Theta}{2\pi\omega}(B_{\beta\alpha}^{*\prime}-B_{\alpha\beta})\delta(\mathbf{r}-\mathbf{r}'), \quad (5)$$

where

$$\Theta=\frac{\hbar\omega}{2}\mathrm{cth}\frac{\hbar\omega}{2kT_c},$$

kT_c is the average temperature in energy units, and the expressions on the left-hand sides represent the spectral densities, i.e., the factors in front of $\delta(\omega-\omega')$ in the correlation functions of the spectral amplitudes

$$\langle\xi_\alpha(\omega, \mathbf{r})\xi_\beta^*(\omega', \mathbf{r}')\rangle=\langle\xi_\alpha(\omega, \mathbf{r})\xi_\beta^*(\omega, \mathbf{r}')\rangle\delta(\omega-\omega')$$

and similarly for f_α. The primes of the conjugate operators indicate that they act on $\mathbf{r}'$.

We must stress that only the real parts of the spectral densities (4) and (5), corresponding to the symmetrized time functions of the correlation have physical meaning.

Usually, the equations (2) are given, i.e., we know the matrix B, so that the use of the fluctuation-dissipation theorem in the form of Eqs. (4) and (5) is a simple matter only in the case of the fluctuation forces f_α. However, calculation of the spectral densities ξ_α requires the determination of the susceptibility matrix A, i.e., it requires solution of the boundary-value problem for the equations of motion (2) containing distributed random forces f_α. If we find in some way (in terms of the Green's functions or by expanding in terms of eigenfunctions) the expressions for ξ_α in terms of f_β, we must then average the pair products ξ_α and use Eq. (5) in this procedure. It is quite difficult to carry out such calculations for each specific case and, what is most important, the range of the problems which can be solved in this way is limited to those for which eigenfunctions are known or for which the Green's function can be calculated.

It is shown in[5] (see also[3]) that in the case of thermal fluctuations of an electromagnetic field we can use the electrodynamic reciprocity theorem to obtain the spectral density of the field intensities **E** and **H** in a form different from Eq. (4) and this generalizes the classical Kirchhoff law, namely:

$$\langle A(\omega, \mathbf{r}_1)B^*(\omega, \mathbf{r}_2)\rangle=\pm\frac{2}{\pi}\Theta Q_{0AB^*}(\mathbf{r}_1, \mathbf{r}_2). \quad (6)$$

Here, A and B are any two out of six components of **E** and **H** (the plus sign corresponds to the case when both components are either electric or magnetic, and the minus sign corresponds to the case when one component is electric and the other magnetic) and Q_{0AB^*} is the power of the "mixed" Joule losses of some auxiliary fields excited in a given system of bodies and media by specified point sources which are a dipole of type and orientation A located at a point $\mathbf{r}_1$ and another dipole of type and orientation B located at point $\mathbf{r}_2$.

If we denote the field intensities created by these dipoles at a point **r** (i.e., the Green's functions) by $\mathbf{E}_0^A(\mathbf{r}, \mathbf{r}_1)$, $\mathbf{H}_0^A(\mathbf{r}, \mathbf{r}_1)$ and $\mathbf{E}_0^B(\mathbf{r}, \mathbf{r}_2)$, $\mathbf{H}_0^B(\mathbf{r}, \mathbf{r}_2)$, respectively, we can write the volume density of the losses of the combined field $\mathbf{E}_0^A + \mathbf{E}_0^B$, $\mathbf{H}_0^A + \mathbf{H}_0^B$ in the form

$$q_0=q_{0AA^*}+q_{0AB^*}+q_{0A^*B}+q_{0BB^*},$$

which clarifies the meaning of each of the terms and, particularly, the meaning of

$$Q_{0AB^*}=\int\limits_V q_{0AB^*}\,d^3r.$$

Formula (6) generalizes the Kirchhoff law in several ways[3] but the important point here is that this formula is easier to understand physically than Eq. (4) because it gives a direct relationship between the spectral densities and the Joule losses; moreover, it has important methodological advantages. It applies to the problems for which the exact solutions are either known or can be obtained and to the more numerous and more varied problems in which the losses Q_{OAB^*} can be found by any of the available approximate methods. Therefore, the fluctuation-dissipation theorem in the "Kirchhoff" form of Eq. (6) is much more flexible and widely applicable than in the standard form (4).

If the gradients of the average temperature are sufficiently weak, Eq. (6) can be extended in a natural manner to inhomogeneously heated bodies by the simple introduction of Θ in the integral:

$$\langle A(\omega, \mathbf{r}_1) B^*(\omega, \mathbf{r}_2)\rangle = \pm\frac{2}{\pi}\int\limits_V \Theta q_{0AB^*}\, d^3r. \tag{7}$$

If the whole volume V occupied by the total field is in equilibrium (Θ = const), we can use the complex Lorentz lemma to show that the total losses Q_{OAB^*} can be expressed linearly in terms of the real parts of the Green's functions.[5,3] Consequently, we find that in the equilibrium case

$$\langle E_\alpha(\omega, \mathbf{r}_1) E_\beta^*(\omega, \mathbf{r}_2)\rangle = -\frac{\Theta}{\pi}\operatorname{Re} E_{0\alpha}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{\Theta}{\pi}\operatorname{Re} E_{0\beta}(\mathbf{r}_2, \mathbf{r}_1) \tag{8}$$

with corresponding equations for the fields H and H_0. The values of E and H are not correlated and it follows, in particular, that the energy flux is everywhere zero. The formulas (8) allow us to find the integrated (over the spectrum) correlation functions of the equilibrium field by the use of the standard theory of analytic functions.

The advantages of the fluctuation-dissipation theorem in the form given by Eqs. (6)–(8), compared with the standard formulation (4), are so great that it is natural to enquire whether the formulas (6)–(8) can be extended to thermal fluctuations of any fields. We shall show that such an extension is indeed possible. The fluctuation-dissipation theorem formulated by Eqs. (6)–(8) applies to any field $\xi_\alpha(\omega, \mathbf{r})$ which satisfies the equations of motion (2) if the operator matrix B ensures that the reciprocity theorem is satisfied subject to suitable boundary conditions.

2. RECIPROCITY THEOREM

We shall take the equations of motion (2) for the fields $\xi_\alpha^{(1)}, f_\alpha^{(1)}$ and $\xi_\alpha^{(2)}, f_\alpha^{(2)}$, multiply the former by $\xi_\alpha^{(2)}$ and the latter by $\xi_\alpha^{(1)}$ and subtract the results. Making the substitution $\alpha \rightleftarrows \beta$ in the second term of the difference, we obtain

$$f_\alpha^{(1)}\xi_\alpha^{(2)} - f_\alpha^{(2)}\xi_\alpha^{(1)} = \xi_\alpha^{(2)} B_{\alpha\beta}\xi_\beta^{(1)} - \xi_\beta^{(1)} B_{\beta\alpha}\xi_\alpha^{(2)}.$$

Integration of this equation over the volume V occupied by the total field leads to the reciprocity theorem

$$\int\limits_V (f_\alpha^{(1)}\xi_\alpha^{(2)} - f_\alpha^{(2)}\xi_\alpha^{(1)})\, d^3r = 0 \tag{9}$$

subject to the condition

$$\int\limits_V \{\xi_\alpha^{(2)} B_{\alpha\beta}\xi_\beta^{(1)} - \xi_\beta^{(1)} B_{\beta\alpha}\xi_\alpha^{(2)}\}\, d^3r = 0. \tag{10}$$

If we represent all the operators $B_{\alpha\beta}$ in the integral form

$$B_{\alpha\beta}\varphi(\mathbf{r}) = \int\limits_V b_{\alpha\beta}(\mathbf{r}, \mathbf{r}')\varphi(\mathbf{r}')\, d^3r', \tag{11}$$

we find that the substitution $\mathbf{r} \rightleftarrows \mathbf{r}'$ in the second term of the integrand modifies the condition (10) to

$$\iint\limits_V \xi_\alpha^{(2)}(\mathbf{r})\xi_\beta^{(1)}(\mathbf{r}')\{b_{\alpha\beta}(\mathbf{r}, \mathbf{r}') - b_{\beta\alpha}(\mathbf{r}', \mathbf{r})\}\, d^3r\, d^3r' = 0. \tag{12}$$

We shall now separate the operators $B_{\alpha\beta}$ and the corresponding kernels $b_{\alpha\beta}(\mathbf{r}, \mathbf{r}')$ into two categories. The operators and kernels depending on the parameters which govern the energy dissipation in a system will be called dissipative and denoted by $D_{\alpha\beta}$ and $d_{\alpha\beta}(\mathbf{r}, \mathbf{r}')$. The other operators and kernels will be denoted by $N_{\alpha\beta}$ and $n_{\alpha\beta}(\mathbf{r}, \mathbf{r}')$ and will be called nondissipative.

We note that the dissipative kernels have the symmetry[1)]

$$d_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = d_{\beta\alpha}(\mathbf{r}', \mathbf{r}), \tag{13}$$

which generalizes the Onsager condition of symmetry of the transport coefficients. It follows from Eq. (13) that the dissipative kernels drop out of the condition (12) so that it becomes

$$\iint\limits_V \xi_\alpha^{(2)}(\mathbf{r})\xi_\beta^{(1)}(\mathbf{r}')\{n_{\alpha\beta}(\mathbf{r}, \mathbf{r}') - n_{\beta\alpha}(\mathbf{r}', \mathbf{r})\}\, d^3r\, d^3r' = 0.$$

The assumption which we shall now make about the operators under consideration is that the same symmetry also applies to the nondissipative kernels (which, generally speaking, may be independent of the parameters of the medium), i.e.,

$$n_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = n_{\beta\alpha}(\mathbf{r}', \mathbf{r}). \tag{14}$$

This is equivalent to the assumption that the nondissipative terms of the integrand in Eq. (10) represent the divergence of a difference flux:

$$\begin{aligned}\xi_\alpha^{(2)} N_{\alpha\beta}\xi_\beta^{(1)} - \xi_\beta^{(1)} N_{\beta\alpha}\xi_\alpha^{(2)} = \int\limits_V \{\xi_\alpha^{(2)}(\mathbf{r})\, n_{\alpha\beta}(\mathbf{r}, \mathbf{r}')\,\xi_\beta^{(1)}(\mathbf{r}')\\ -\xi_\beta^{(1)}(\mathbf{r})\, n_{\beta\alpha}(\mathbf{r}, \mathbf{r}')\,\xi_\alpha^{(2)}(\mathbf{r}')\}\, d^3r' = \operatorname{div}\mathbf{\Pi}(\mathbf{r}), \quad \mathbf{\Pi} = \mathbf{\Pi}_{12} - \mathbf{\Pi}_{21},\end{aligned} \tag{15}$$

where the normal component Π_n vanishes on the closed surface S surrounding the volume V. However, if the total field occupies all space (although the sources are concentrated in the finite region), the value of Π_n at infinity tends to zero faster than does $1/R^2$, i.e., the terms of the order of $1/R^2$ drop out from the difference flux.

3 CONSERVATION OF ENERGY AND MIXED LOSSES

In the time representation, the equations of motion (2) are

$$\tilde{B}_{\alpha\beta}\xi_\beta(t, \mathbf{r}) = f_\alpha(t, \mathbf{r}), \tag{16}$$

where $\tilde{B}_{\alpha\beta}$ are the space-time real operators, so that

$$\tilde{B}_{\alpha\beta}e^{i\omega t} = e^{i\omega t}B_{\alpha\beta}, \quad \tilde{B}_{\alpha\beta}e^{-i\omega t} = e^{-i\omega t}B_{\alpha\beta}^*. \tag{17}$$

We shall write the law of conservation of energy in the form which follows from the system (16). Multiplying this system by $\dot{\xi}_\alpha(t, \mathbf{r})$, we obtain on the right the volume density of the instantaneous power $w = f_\alpha(t, \mathbf{r})\dot{\xi}_\alpha(t, \mathbf{r})$ transferred to the system by the forces f_α. The consumption of this instantaneous power has three components, one of which changes the energy density E in the system, another is dissipated per unit volume q, and the third is carried away by the flux of density $\mathbf{P}$:

$$w = \frac{\partial E}{\partial t} + q + \operatorname{div}\mathbf{P} = \dot{\xi}_\alpha(t,\mathbf{r}) B_{\alpha\beta} \xi_\beta(t,\mathbf{r}). \qquad (18)$$

We shall consider harmonic fields

$$\xi_\alpha(t,\mathbf{r}) = {}^1/_2\{\xi_\alpha(\omega,\mathbf{r})e^{i\omega t} + \text{c.c.}\},$$

average Eq. (18) over the period $T = 2\pi/\omega$, and denote this operation by a bar above a quantity. Since $\partial E/\partial t = 0$, we find, subject to Eq. (17), that

$$\overline{w} = \overline{q} + \operatorname{div}\overline{\mathbf{P}} = {}^1/_4 i\omega\{\xi_\beta B_{\beta\alpha}{}^* \xi_\alpha{}^* - \xi_\alpha{}^* B_{\alpha\beta}\xi_\beta\}.$$

If ξ_α represents the superposition of fields 1 and 2,

$$\xi_\alpha = \xi_\alpha^{(1)} + \xi_\alpha^{(2)},$$

we must divide $\overline{q}$ and $\overline{\mathbf{P}}$ into three components:

$$\overline{q} = q_{11} + q_{12} + q_{22}, \quad \overline{\mathbf{P}} = \mathbf{P}_{11} + \mathbf{P}_{12} + \mathbf{P}_{22},$$

where q_{11}, $\mathbf{P}_{11}$ and q_{22}, $\mathbf{P}_{22}$ correspond to fields 1 and 2 separately and the mixed ("interference") term is

$$q_{12} + \operatorname{div}\mathbf{P}_{12} = {}^1/_4 i\omega\{\xi_\beta^{(1)} \dot{B}_{\beta\alpha} \xi_\alpha^{(2)*} + \xi_\alpha^{(2)} \dot{B}_{\alpha\beta} \xi_\beta^{(1)*}\} + \text{c.c.}$$

We shall find it convenient to divide this expression into two parts and use them in the complex-conjugate rather than the real form:

$$q_{12} = q_{12^*} + q_{1^*2}, \quad \mathbf{P}_{12} = \mathbf{P}_{12^*} + \mathbf{P}_{1^*2},$$

separating in $q_{12}{}^*$ and $\mathbf{P}_{12}{}^*$ the terms containing the fields $\xi_\alpha^{(1)}$ and the complex conjugate fields $\xi_\alpha^{(2)}$:

$$q_{12^*} + \operatorname{div}\mathbf{P}_{12^*} = {}^1/_4 i\omega\{\xi_\beta^{(1)} \dot{B}_{\beta\alpha} \xi_\alpha^{(2)*} - \xi_\alpha^{(2)*} B_{\alpha\beta} \xi_\beta^{(1)}\}. \qquad (19)$$

Obviously, $q_{12}{}^*$ should include only the dissipative operators $D_{\alpha\beta}$ and $\operatorname{div}\mathbf{P}_{12}{}^*$ should include only the nondissipative operators $N_{\alpha\beta}$. Using the integral representation (11) and the conditions (13) and (14), we thus obtain

$$q_{12^*} = \frac{i\omega}{4} \int\limits_V \{\xi_\beta^{(1)}(\mathbf{r}) \dot{d}_{\beta\alpha}(\mathbf{r}',\mathbf{r}) \xi_\alpha^{(2)*}(\mathbf{r}') - \xi_\alpha^{(2)*}(\mathbf{r}) d_{\alpha\beta}(\mathbf{r},\mathbf{r}') \xi_\beta^{(1)}(\mathbf{r}')\} d^3r' \qquad (20)$$

and exactly the same expression for $\operatorname{div}\mathbf{P}_{12}{}^*$ but with the $n_{\alpha\beta}$ kernels.

4. DERIVATION OF KIRCHOFF FORM OF FLUCTUATION-DISSIPATION THEOREM

We shall apply the reciprocity theorem (9) to the case when a field 2 is of thermal (fluctuation) origin: $\xi_\alpha^{(2)} = \xi_\alpha$, $f_\alpha^{(2)} = f_\alpha$, and a field 1 is an auxiliary field excited by a point source located at $\mathbf{r}_1$ and having only the component with $\alpha = m$:

$$f_\alpha^{(1)} = \frac{\delta_{\alpha m}}{i\omega} \delta(\mathbf{r} - \mathbf{r}_1). \qquad (21)$$

The factor $1/i\omega$ is introduced here in order to obtain more compact forms of the final formulas. The field excited by the force (21), i.e., the Green's function, will be denoted by

$$\xi_\alpha^{(1)} = \xi_{0\alpha}^m(\mathbf{r},\mathbf{r}_1).$$

The reciprocity theorem (9) then yields Eq. (3) in the form

$$\xi_m(\mathbf{r}_1) = i\omega \int\limits_V f_\alpha(\mathbf{r}) \xi_{0\alpha}^m(\mathbf{r},\mathbf{r}_1) d^3r \equiv A_{m\alpha} f_\alpha(\mathbf{r}_1). \qquad (22)$$

Using Eq. (22), we shall obtain the mutual spectral density of the m-th and n-th components of the thermal field taken at the points $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively:

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = \omega^2 \iint\limits_V \langle f_\alpha(\mathbf{r}) f_\beta{}^*(\mathbf{r}') \rangle \xi_{0\alpha}{}^m(\mathbf{r},\mathbf{r}_1) \xi_{0\beta}^{n*}(\mathbf{r}',\mathbf{r}_2) d^3r\, d^3r'. \qquad (23)$$

It follows from the fluctuation-dissipation theorem of Eq. (5) and from the integral representation (11) that the spectral density of the fluctuation forces is

$$\langle f_\alpha(\mathbf{r}) f_\beta{}^*(\mathbf{r}') \rangle = \frac{i\Theta}{2\pi\omega} \{b_{\beta\alpha}{}^*(\mathbf{r}',\mathbf{r}) - b_{\alpha\beta}(\mathbf{r},\mathbf{r}')\}. \qquad (24)$$

It follows from the meaning of the fluctuation-dissipation theorem that the right-hand side of Eq. (24) can include only the kernels of the dissipative operators $d_{\alpha\beta}(\mathbf{r},\mathbf{r}')$. Consequently, the nondissipative kernels $n_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ should obey an additional condition

$$n_{\alpha\beta}(\mathbf{r},\mathbf{r}') = n_{\beta\alpha}{}^*(\mathbf{r}',\mathbf{r}), \qquad (25)$$

which yields—after the substitution of Eq. (24) into Eq. (23)—

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = \frac{i\omega}{2\pi} \iint\limits_V \Theta\{d_{\beta\alpha}{}^*(\mathbf{r}',\mathbf{r}) - d_{\alpha\beta}(\mathbf{r},\mathbf{r}')\} \xi_{0\alpha}{}^m(\mathbf{r},\mathbf{r}_1) \xi_{0\beta}{}^{n*}(\mathbf{r}',\mathbf{r}_2) d^3r\, d^3r'. \qquad (26)$$

The conditions (14) and (25) mean that the kernels should be real. If we separate the real and imaginary parts of the nuclei ($b_{\alpha\beta} = b_{\alpha\beta}^r + ib_{\alpha\beta}^i$), we can rewrite the conditions (14) and (25) in the form

$$n_{\alpha\beta}{}^r(\mathbf{r},\mathbf{r}') = n_{\beta\alpha}{}^r(\mathbf{r}',\mathbf{r}), \quad n_{\alpha\beta}{}^i(\mathbf{r},\mathbf{r}') = 0. \qquad (27)$$

Thus, the nondissipative kernels should have the Onsager symmetry and should be real.

We must make the following point. If we assume that Θ is a point function or even if it is a two-point function depending on $\mathbf{r}$ and $\mathbf{r}'$ (i.e., if it is a correlation function), we must still assume (in the phenomenological theory being developed here) that the nonlocalization radius or the characteristic scale of variation of Θ is much greater than the nonlocalization radius of the dissipative kernels $d_{\alpha\beta}$. Therefore, in the case of interchange $\mathbf{r} \rightleftarrows \mathbf{r}'$ in the integral (26), we need not distinguish $\mathbf{r}$ and $\mathbf{r}'$ in Θ. Bearing this point in mind, we can easily show that the interchange $\alpha \rightleftarrows \beta$ and, moreover, the interchange $\mathbf{r} \rightleftarrows \mathbf{r}'$ in the second term of the integrand in Eq. (26) [subject to the symmetry of Eq. (13)] reduce the expression (26) to

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = \frac{2}{\pi} \int\limits_V \Theta(\mathbf{r}) q_{0mn^*}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) d^3r, \qquad (28)$$

where $q_{0mn}{}^*$ is the volume density of the mixed losses of the auxiliary fields $\xi_{0\beta}^m$ and $\xi_{0\alpha}^{n*}$, as given by Eq. (20).

However, $q_{0mn}{}^*$ can be represented in a simpler form if we apply the condition (13) to the first term in the integrand (26). We then obtain

$$q_{0mn^*}(\mathbf{r};\mathbf{r}_1,\mathbf{r}_2) = \frac{\omega}{2} \int\limits_V d_{\alpha\beta}{}^i(\mathbf{r},\mathbf{r}') \xi_{0\alpha}{}^m(\mathbf{r},\mathbf{r}_1) \xi_{0\beta}{}^{n*}(\mathbf{r}',\mathbf{r}_2) d^3r'$$

$$\equiv \xi_{0\alpha}{}^m(\mathbf{r},\mathbf{r}_1) F_\alpha{}^{n*}(\mathbf{r},\mathbf{r}_2),$$

so that $F_\alpha^n(\mathbf{r},\mathbf{r}_2)$ can be called the "active" part of the force acting on the coordinate $\xi_{0\alpha}^m(\mathbf{r},\mathbf{r}_1)$.

Thus, under the conditions of Eq. (27), the fluctuation-dissipation theorem reduces to the form (28) which generalizes the result (7) obtained earlier for an electromagnetic field. In the special case when Θ is independent of $\mathbf{r}$, we obtain the following generalization of Eq. (6):

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = \frac{2}{\pi} \Theta Q_{0mn^*}(\mathbf{r}_1,\mathbf{r}_2). \qquad (29)$$

[We shall return later to the problem of two signs in Eqs. (6) and (7)].

5. FLUCTUATION-DISSIPATION THEOREM UNDER THERMAL EQUILIBRIUM CONDITIONS

We shall now demonstrate that in the case of thermal equilibrium throughout the volume V occupied by the total field we can express the fluctuation-distribution theorem in the form of Eq. (8) if Eq. (29) is valid, i.e., we can prove that the total mixed losses $Q_{0mn}{}^*$ obey a theorem which is an analog of the complex Lorentz lemma in electrodynamics.

Multiplying the equations of motion (2) for the fields and forces $\xi_\alpha^{(1)}$, $f_\alpha^{(1)}$ by $\xi_\alpha^{(2)}$ and multiplying the complex conjugate equations $\xi_\alpha^{(2)}$, $f_\alpha^{(2)}$ by $\xi_\alpha^{(1)}$ and subtracting the results, we obtain

$$\xi_\beta^{(1)} B_{\beta\alpha}{}^* \xi_\alpha^{(2)*} - \xi_\alpha^{(2)*} B_{\alpha\beta} \xi_\beta^{(1)} = f_\alpha^{(2)*} \xi_\alpha^{(1)} - f_\alpha^{(1)} \xi_\alpha^{(2)*},$$

which—according to Eq. (19)—can be written in the form

$$\frac{4}{i\omega}(q_{12^*} + \operatorname{div} \mathbf{P}_{12^*}) = f_\alpha^{(2)*} \xi_\alpha^{(1)} - f_\alpha^{(1)} \xi_\alpha^{(2)*}.$$

If we integrate this equation over the volume V, the flux of the vector $\mathbf{P}_{12}{}^*$ vanishes because of the conditions (27) and we obtain the following expression for the total mixed losses of the fields 1 and 2:

$$\frac{4}{i\omega} Q_{12^*} = \int_V \{f_\alpha^{(2)*} \xi_\alpha^{(1)} - f_\alpha^{(1)} \xi_\alpha^{(2)*}\} d^3r, \tag{30}$$

which is the analog of the complex Lorentz lemma in electrodynamics.

We shall now apply Eq. (30) to fields 1 and 2 excited by point sources located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, and we shall assume that the first source has only the component $\alpha = m$ and the second only the component $\alpha = n$:

$$f_\alpha^{(1)} = \frac{\delta_{\alpha m}}{i\omega} \delta(\mathbf{r}-\mathbf{r}_1), \quad f_\alpha^{(2)} = \frac{\delta_{\alpha n}}{i\omega} \delta(\mathbf{r}-\mathbf{r}_2). \tag{31}$$

The fields excited by these forces will be denoted by

$$\xi_\alpha^{(1)} = \xi_{0\alpha}{}^m(\mathbf{r}, \mathbf{r}_1), \quad \xi_\alpha^{(2)} = \xi_{0\alpha}{}^n(\mathbf{r}, \mathbf{r}_2). \tag{32}$$

The substitution of the forces (31) and (32) in Eq. (30) yields

$$Q_{0mn^*} = -{}^1/_4 \{\xi_{0n}{}^m(\mathbf{r}_2, \mathbf{r}_1) + \xi_{0m}{}^{n*}(\mathbf{r}_1, \mathbf{r}_2)\}. \tag{33}$$

However, the application of the reciprocity theorem (9) to the same forces and fields leads to

$$\xi_{0n}{}^m(\mathbf{r}_2, \mathbf{r}_1) = \xi_{0m}{}^n(\mathbf{r}_1, \mathbf{r}_2). \tag{34}$$

Using Eq. (34), we can write the total mixed losses (33) in one of the following two forms:

$$Q_{0mn^*} = -{}^1/_2 \operatorname{Re} \xi_{0n}{}^m(\mathbf{r}_2, \mathbf{r}_1) = -{}^1/_2 \operatorname{Re} \xi_{0m}{}^n(\mathbf{r}_1, \mathbf{r}_2).$$

Consequently, the fluctuation-dissipation theorem (29) becomes

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = -\frac{\Theta}{\pi} \operatorname{Re} \xi_{0n}{}^m(\mathbf{r}_2, \mathbf{r}_1) = -\frac{\Theta}{\pi} \operatorname{Re} \xi_{0m}{}^n(\mathbf{r}_1, \mathbf{r}_2), \tag{35}$$

which is a generalization of the electrodynamic formulas (8).

Thus, subject to the conditions (27), the fluctuation-dissipation theorem expressed in the forms (28), (29), and (35) applies to correlation functions in thermal fluctuation fields and retains all the conditions mentioned in Sec. 1.

6. CONCLUSIONS

A very important comment must be made to complement the results given above. It does not follow at all that the conditions (27) are necessarily satisfied for arbitrarily selected fields ξ_α and forces f_α compatible with the general requirement of their energy conjugacy (1). The conditions (27) are satisfied, for example, by an acoustic field described by a displacement vector, temperature, and a set of relaxing and/or diffusing scalar parameters but they are not satisfied by the Maxwell equations in their usual form. However, the Kirchhoff form of the fluctuation-dissipation theorem is valid in electrodynamics. The essence of this point is as follows.

Within the framework of the condition (1) we have a wide choice in the selection of variables that describe the state of the system. Apart from any original variables ξ_α and f_α used to describe thermal fluctuations, we can equally well use variables η_α and g_α, obtained by linear transformations of the type

$$\xi_\alpha = a_{\alpha\beta} \eta_\beta, \quad f_\alpha = a_{\beta\alpha}^{-1*} g_\beta \tag{36}$$

with constant coefficients. Such transformations ensure the invariance of the power transmitted to the system by fluctuation forces

$$\overline{w} = {}^1/_4 i\omega(\xi_\alpha f_\alpha{}^* - \xi_\alpha{}^* f_\alpha) = {}^1/_4 i\omega(\eta_\alpha g_\alpha{}^* - \eta_\alpha{}^* g_\alpha),$$

of the equations of motion (2)

$$g_\alpha = \hat{B}_{\alpha\beta} \eta_\beta = \int_V \hat{b}_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \eta_\beta(\mathbf{r}') d^3r',$$

and of the fluctuation-dissipation theorem of Eqs. (5) and (24):

$$\langle g_\alpha(\omega, \mathbf{r}) g_\beta{}^*(\omega, \mathbf{r}') \rangle = \frac{i\Theta}{2\pi\omega}(\hat{B}_{\beta\alpha}^{*\prime} - \hat{B}_{\alpha\beta}) \delta(\mathbf{r}-\mathbf{r}')$$
$$= \frac{i\Theta}{2\pi\omega}\{\hat{b}_{\beta\alpha}{}^*(\mathbf{r}', \mathbf{r}) - \hat{b}_{\alpha\beta}(\mathbf{r}, \mathbf{r}')\},$$

where the new kernels are related to the old by the transformations

$$\hat{b}_{\alpha\beta}(\mathbf{r}, \mathbf{r}') = a_{\nu\alpha}{}^* b_{\nu\mu}(\mathbf{r}, \mathbf{r}') a_{\mu\beta}.$$

If the transformed kernels $\hat{b}_{\alpha\beta}$ satisfy the conditions (13) and (27), all the results obtained above—including the reciprocity theorem (9) and the fluctuation-dissipation theorem in the forms (28), (29), and (35)—apply to the new variables η_α and g_α. What does it mean if we return to the original variables ξ_α and f_α?

Using the transformations inverse to Eq. (36),

$$\eta_\alpha = a_{\alpha\beta}{}^{-1} \xi_\beta, \quad g_\alpha = a_{\beta\alpha}{}^* f_\beta,$$

we can easily show that, in the case of ξ_α and f_α, the reciprocity theorem becomes

$$\int_V (\eta_\alpha^{(2)} g_\alpha^{(1)} - \eta_\alpha^{(1)} g_\alpha^{(2)}) d^3r = a_{\gamma\alpha}{}^* a_{\alpha\beta}{}^{-1} \int_V (\xi_\beta^{(2)} f_\gamma^{(1)} - \xi_\beta^{(1)} f_\gamma^{(2)}) d^3r = 0,$$

whereas the fluctuation-dissipation theorem of Eq. (29) is

$$\langle \xi_m(\mathbf{r}_1) \xi_n{}^*(\mathbf{r}_2) \rangle = a_{n\beta}{}^* a_{m\alpha}{}^{-1} \frac{2}{\pi} \int_V \Theta(\mathbf{r}) q_{0\alpha\beta^*}(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2) d^3r. \tag{37}$$

In the Maxwell equations it is sufficient to replace the usual variables

$$\xi_\alpha = \frac{1}{4\pi}\{\mathbf{E}, \mathbf{H}\}, \quad f_\alpha = \{\mathfrak{D}, \mathfrak{B}\} = \frac{4\pi}{i\omega}\{\mathbf{j}_e, \mathbf{j}_m\},$$

where $\mathfrak{D}$ and $\mathfrak{B}$ are external fluctuation inductions and $\mathbf{j}_e$ and $\mathbf{j}_m$ are the densities of external electric and magnetic currents, with new variables obtained using diagonal transformations with the coefficients

$$a_{\alpha\beta} = a_\alpha \delta_{\alpha\beta}, \quad a_1 = a_2 = a_3 = 1, \quad a_4 = a_5 = a_6 = i.$$

We then obtain the fluctuation fields $\eta_\alpha = (4\pi)^{-1}\{\mathbf{E}, -i\mathbf{H}\}$

and forces $g_\alpha = \{\mathfrak{D}, -i\mathfrak{B}\}$ so that $\tilde{E} = E$, $\tilde{H} = -iH$, $\mathfrak{D} = \mathfrak{D}$, and $\mathfrak{B} = i\mathfrak{B}$.

In terms of the new variables, the Maxwell equations

$$\tilde{\mathfrak{D}}=-\tilde{D}+\frac{c}{\omega}\operatorname{rot}\tilde{H},\quad \tilde{\mathfrak{B}}=-\tilde{B}+\frac{c}{\omega}\operatorname{rot}\tilde{E},$$

satisfy the conditions (13) and (27) and the reciprocity theorem has its standard form (9). In terms of the original variables, this yields the usual form of the electrodynamic reciprocity theorem with different signs of the electric and magnetic terms:

$$\int_V \{E^{(2)}j_e^{(1)}-H^{(2)}j_m^{(1)}\}d^3r=\int_V \{E^{(1)}j_e^{(2)}-H^{(1)}j_m^{(2)}\}d^3r.$$

Finally, in terms of the old variables E and H, the fluctuation-dissipation theorem of Eq. (29) assumes the form of Eq. (6), i.e.,

$$\langle \xi_m(r_1)\xi_n^*(r_2)\rangle=\pm\frac{2}{\pi}\int_V \Theta(r)\,q_{0mn^*}(r;r_1,r_2)\,d^3r, \tag{38}$$

where the upper sign corresponds to m, n, = 1, 2, 3 (electric components) or m, n = 4, 5, 6 (magnetic components) and the lower sign corresponds to mixed correlation functions with m = 1, 2, 3 or n = 4, 5, 6, or conversely.

Thus, if the conditions (27) are not satisfied, it does not follow that the Kirchhoff form of the fluctuation-dissipation theorem is invalid. We must also consider the possibility of such a choice of the variables—among those which can be formed by the transformations (36)—which would satisfy the conditions (27). Naturally, if the reciprocity theorem is known in some form for specific fields, there is no need to go over to unified equations simply to avoid the complexity of the form (37) of the fluctuation-dissipation theorem [for example, in order to avoid the two signs in Eq. (38) for an electromagnetic field].

[1)] We are excluding gyrotropy in an external magnetic field and rotations of the system. The later generalization of the results to the presence of such gyrotropy or to rotation presents no basic difficulties.

[1] H. B. Callen and T. A. Welton, Phys. Rev. 83, 34 (1951).
[2] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika, izd. 2—e, Nauka, M., 1964 (Statistical Physics, 2nd ed., Pergamon Press, Oxford, 1969).
[3] M. L. Levin and S. M. Rytov, Teoriya ravnovesnykh teplovykh fluktuatsiĭ v elektrodinamike (Theory of Equilibrium Thermal Fluctuations in Electrodynamics), Nauka, M., 1967.
[4] S. M. Rytov, Dokl. Akad. Nauk SSSR 110, 371 (1956) [Sov. Phys.-Doklady 1, 555 (1957)].
[5] M. L. Levin, Dokl. Akad. Nauk SSSR 102, 53 (1955).

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