Macroscopic bodies with zero rest mass

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A solution of the general relativity equations is found describing a macroscopic body of zero rest mass. Such a body moves as a whole with the speed of light and can emit electromagnetic waves only if it interacts with outside particles.

According to the theory of relativity, particles of finite mass cannot move with the speed of light. However, this assertion is valid only for free particles. An interacting particle can move with the speed of light. Such a possibility is realized, for instance, in the case when a particle of finite mass is part of a body which as a whole has zero rest mass, and therefore always moves with the speed of light. In this case the rest mass of the particle must be compensated by the negative mutual attraction energy of the particles making up the body. Motion with the speed of light of massive particles is in fact realized in the following realistically observable process. Assume that at some point in space a positron and an electron annihilated. If the photon produced in this way again produces an electron and a positron, then, since nothing prevents one from considering the particles produced to be the same as the ones before the annihilation, the positron and the electron have moved with the speed of light as a result of the process under consideration.*

In a preceding paper[1] the author has shown the possibility, in principle, of existence of a macroscopic body of zero rest mass, where the finite rest mass of the component particles is compensated by the gravitational attraction energy. Owing to the presence of the gravitational field, probe particles of finite mass which are situated at a finite distance of such a body can, not only stay behind it, but also remain at rest or even catch up with it, in spite of the fact that relative to a distant observer the body is moving with the speed of light. If the distant observer would interpret the motion of the particle which catches up with the zero-mass body as a motion in flat space under the action of some forces, he would be forced to attribute to the particle a speed exceeding the speed of light.

It is quite possible that there exist in nature astronomical objects which are bodies of zero rest mass, moving with the speed of light. The proper luminosity of such objects would be equal to zero, since a zero-mass object cannot by itself emit electromagnetic radiation, as a consequence of the conservation of energy and momentum. In other words, one can say that such a body does radiate, but that owing to the Doppler effect the frequency of the radiation is always shifted to zero, since the source moves with the speed of light. Radiation at a finite frequency is possible only as a result of interaction with outside particles. We shall use the term "tachar" for the description of such a body, which seems reasonable in view of the properties mentioned above.

The present paper is devoted to a quantitative clarification of the properties of a tachar consisting of dustlike matter, i.e., of matter with vanishing pressure. In this case it turns out to be possible to obtain an exact solution of the equations of the general theory of relativity. In its mathematical aspects, this solution is completely analogous to the well-known Tolman solution (cf.[2], problem 4 in Sec. 100) for spherically symmetric dustlike matter. The physical meaning of these two solutions is of course quite different.

1. We shall look for a solution of the Einstein equations corresponding to a tachar exhibiting the maximally possible symmetry. We first consider a body of finite mass, situated in a space which is Galilean at infinity. By means of an appropriate Lorentz transformation one can always make the total momentum **P** of the body as a whole vanish. Then the energy will be equal to the total mass m of the body. In this reference frame the most symmetric object of finite mass is spherically symmetric. The corresponding solution of the field equations is invariant with respect to the rotation group, which is the little group (stationary subgroup) of the 4-momentum (m, 0, 0, 0) of the body.

The three-momentum of a body of zero mass cannot be made to vanish by means of a Lorentz transformation. One can however reduce the 4-momentum to the form $(E, 0, 0, P_z)$ with $P_z = E$. A zero-mass body will exhibit maximal symmetry if the corresponding solution of the field equations is symmetric with respect to those Lorentz transformations which leave invariant the 4-vector (E, 0, 0, E). As is well known, these transformations form a group (little group) isomorphic to the group E(2) of motions (translations and rotations) of the Euclidean 2-plane[3]. Thus, we must search for an E(2)-symmetric solution of the equations of the gravitational field. It should be stressed that, in the same manner in which the spherical symmetry of a solution guarantees the vanishing of the 3-momentum of the body, the presence of the E(2)-symmetry guarantees the equality of the energy and momentum, i.e., the light-like character of the motion of the body as a whole.

The most convenient coordinate system for the discussion of spherically symmetric solutions are spherical coordinates, where there are two coordinates (the radius and the time) which are invariant with respect to the rotation group. Quantities which define a spherically symmetric field can depend only on these two coordinates. A similar role for the E(2)-symmetric solutions is played by the so-called horospheric coordinate system (cf.[4]), which for our purposes is most conveniently defined in the following manner. We introduce the coordinates ζ, η, u, v in terms of the Cartesian coordinates t, x, y, z by means of the relations

$$x=\eta u, \quad y=\eta v, \quad \eta=t-z, \quad \zeta=\xi-(x^2+y^2)/\eta, \qquad (1)$$

where $\xi = t + z$.

The transformations of the group E(2) are those Lorentz transformations which reduce to the translations and rotations of the (u, v) plane for constant η. Since the product $\zeta\eta = t^2 - x^2 - y^2 - z^2$ is invariant under general Lorentz transformations, the coordinate ζ is automatically invariant with respect to the group E(2). Quantities which define an E(2)-symmetric field can depend only on the coordinates η and ζ.

Using Eq. (1) it is easy to determine the metric of flat space in horospheric coordinates:

$$ds^2=d\eta d\zeta-\eta^2(du^2+dv^2). \tag{2}$$

It is interesting to note that such a metric arises near the front of a plane gravitational wave (cf.[2], Sec. 103). The transformation (1) was used to reduce it to Galilean form.

The most general metric exhibiting E(2)-invariance can be written in the form

$$ds^2=A(\tau,w)d\tau^2-B(\tau,w)dw^2+D(\tau,w)d\tau\,dw-C(\tau,w)(du^2+dv^2), \tag{3}$$

where the coordinates τ and w can be subjected to the transformations

$$w'=w'(\tau,w),\qquad \tau'=\tau'(\tau,w),$$

which do not violate the E(2)-symmetry. We make use of this arbitrariness in order to reduce to zero the quantity $D(\tau, w)$ and the w-component of the velocity of matter. The components of the velocity along u and v vanish on account of the E(2)-symmetry. The energy-momentum tensor of matter has consequently, in the general case, the following nonvanishing components

$$T_0^0=\varepsilon,\quad T_1^1=T_2^2=T_3^3=-p,$$

where ϵ is the energy density, p is the pressure. The coordinates τ, w, u, v have been denoted here by the indices 0, 1, 2, 3, respectively.

From the conservation equations $T^k_{i,k} = 0$ we obtain two equations

$$\dot\varepsilon+\left(\frac{\dot B}{2B}+\frac{\dot C}{C}\right)(\varepsilon+p)=0,\quad p'+\frac{A'}{2A}(\varepsilon+p)=0; \tag{4}$$

where the dot denotes differentiation with respect to τ and the dash denotes differentiation with respect to w. These equations coincide exactly with the analogous equations in the spherically symmetric case (cf.[2], problem 3 of Sec. 100). For vanishing pressure it can be seen from the second equation of (4) that the function A can be selected equal to one, i.e., the metric is simultaneously comoving and synchronous. Setting $C = \eta^2$ we find the following field equations:

$$\begin{gathered}8\pi k\varepsilon=-2\frac{\eta''}{B\eta}+\frac{\dot B\dot\eta}{B\eta}+\frac{B'\eta'}{B^2\eta}+\frac{\dot\eta^2}{\eta^2}-\frac{\eta'^2}{B\eta^2},\\ 2\eta\ddot\eta+\dot\eta^2-\frac{\eta'^2}{B}=0,\qquad \dot B\eta'-2B\dot\eta'=0,\\ \frac{\ddot B}{2B}+\frac{\ddot\eta}{\eta}-\frac{1}{4}\frac{\dot B^2}{B^2}-\frac{\eta''}{B\eta}+\frac{1}{2}\frac{\dot B\dot\eta}{B\eta}+\frac{1}{2}\frac{B'\eta'}{B^2\eta}=0.\end{gathered} \tag{5}$$

Here k is the Newtonian gravitational constant.

From the third equation of (5) it follows that

$$B=\eta'^2/f(w), \tag{6}$$

where $f(w) > 0$ is an arbitrary function. The second and fourth equations in (5) are equivalent to the single equation

$$\dot\eta^2=f-F(w)/\eta, \tag{7}$$

where F(w) is yet another arbitrary function. Integrating (7) we obtain

$$\pm(\tau_0(w)-\tau)=\frac{(f\eta^2-F\eta)^{1/2}}{f}+\frac{F}{f^{3/2}}\ln\left\{\left(\frac{f\eta}{F}\right)^{1/2}+\left(\frac{f\eta}{F}-1\right)^{1/2}\right\}, \tag{8}$$

where $\tau_0(w)$ is also an arbitrary function of the variable w, the quantity η being assumed to be positive. By substituting these formulas into the first equation of (5) we find for the density of matter

$$\varepsilon=-F'/8\pi k\eta^2\dot\eta'. \tag{9}$$

Equations (6), (8), and (9) determine the required exact solution of the Einstein equations. If one selects in place of w the variable f = f(w) one obtains

$$ds^2=d\tau^2-\eta'^2df^2/f-\eta^2dl^2, \tag{10}$$

where $dl^2 = du^2 + dv^2$; here and in the sequel the dash denotes differentiation with respect to f.

2. In the preceding paper of the author[1] and earlier in the paper of Aichelburg and Sexl[5] the gravitational field of a point particle of zero mass and energy E was determined in the linear approximation. The corresponding metric has the form

$$ds^2=d\xi d\eta-dr^2-r^2d\varphi^2+8kE\delta(\eta)\ln r\,d\eta^2, \tag{11}$$

where

$$r=(x^2+y^2)^{1/2},\qquad \varphi=\operatorname{arc\,tg}(y/x),$$

where it was assumed that the particle has only a z-component of the momentum $P_z = E$ and moves along the line x = y = 0.

We show that one can impose such conditions on the arbitrary functions F(f) and $\tau_0(f)$ that the solution (8)–(10) has the same structure at large distances as the solution (11). This allows one to relate the energy and momentum of the tachar with parameters which determine its internal structure.

By means of a transition from the variable f to the variable η one can rewrite the interval (10) in the form

$$ds^2=-\frac{d\eta^2}{f}\mp\frac{2}{f}\left(f-\frac{F}{\eta}\right)^{1/2}d\eta\,d\tau+\frac{F}{f\eta}d\tau^2-\eta^2\,dl^2, \tag{12}$$

where the two signs correspond to those in Eq. (8). If $F \equiv 0$ matter and the gravitational field are absent. In this case the metric (12) reduces to the form (2) by means of the transformation

$$\zeta=\frac{\eta}{f}\mp2\int\frac{\tau_0'\,df}{f^{1/2}}.$$

For $F/f\eta \ll 1$ one can linearize all the relations with respect to F. Introducing in this case the variable

$$\zeta=\frac{\eta}{f}\mp2\int\frac{\tau_0'\,df}{f^{1/2}}+\frac{3}{4}\frac{F}{f^2}\ln\frac{f\eta}{F}+\chi(\eta), \tag{13}$$

where $\chi(\eta)$ is some function for which the concrete form will be selected later, we obtain from (12), taking into account (8)

$$ds^2=d\zeta\,d\eta-\eta^2\,dl^2+\frac{F'}{4}\frac{\ln(f\eta/F)}{\eta\pm2f^{3/2}\tau_0'}\left(\frac{d\eta^2}{f}-d\eta\,d\zeta\right)-\frac{d\chi}{d\eta}d\eta^2+\frac{F}{4\eta}d\zeta^2. \tag{14}$$

The last terms here are small corrections to the first two terms.

In cylindrical coordinates

$$r=\eta(u^2+v^2)^{1/2},\qquad \varphi=\operatorname{arctg}(v/u),\qquad \xi=\zeta+r^2/\eta$$

one can rewrite Eq. (14) in the following form

$$\begin{aligned}ds^2=d\xi\,d\eta-dr^2-r^2\,d\varphi^2+\frac{F'}{4}\frac{\ln(f\eta/F)}{f^2\psi'+\eta}\left[\frac{d\eta^2}{f}-d\eta\left(d\xi+\frac{r^2}{\eta^2}d\eta-\frac{2r}{\eta}dr\right)\right]\\ -\frac{d\chi}{d\eta}d\eta^2+\frac{F}{4\eta}\left(d\xi+\frac{r^2}{\eta^2}d\eta-\frac{2r}{\eta}dr\right)^2,\end{aligned} \tag{15}$$

where $\psi(f)$ is an arbitrary function, related to $\tau_0(f)$ through the relation

$$\psi(f)=\pm 2\int \frac{df}{f^{1/2}}\tau_0'.$$

The density of matter in the approximation under consideration is

$$\varepsilon=-\frac{fF'}{4\pi k\eta^2(f^2\psi'+\eta)}. \quad (16)$$

We assume that for $f \to 0$ the function $F(f)$ tends to zero like f^2, and the function $\psi(f)$ being positive, tends to infinity like $h(f)/f$, where the function $h(f)$ tends to infinity, but slower than the square root of $\ln(1/f)$. Setting for definiteness $h(f) \propto \ln^{1/n}(1/f)$, with $n > 2$, we have

$$\psi(f)\approx af^{-1}\ln^{1/n}(1/f), \quad F(f)\approx bf^2.$$

Here $a > 0$ and $b > 0$ are constants having the dimension of length.

Let us determine the asymptotic behavior of the metric (15) for constant ξ and η and as $r \to \infty$. Recognizing that according to (13) we have

$$f=\frac{a\eta}{r^2}\ln^{1/n}\left(\frac{r^2}{a\eta}\right),$$

we obtain

$$ds^2=d\xi\, d\eta-dr^2-r^2d\varphi^2+\frac{b}{2}\ln\frac{r^2}{ab}\frac{d\eta^2}{\eta}. \quad (17)$$

The matter density tends to zero according to the law

$$\varepsilon=\frac{ab}{2\pi kr^4}\ln^{1/n}\left(\frac{r^2}{a\eta}\right).$$

Let now $\eta \to \infty$ for constant r and t, i.e., we consider the region far behind the body. In this case

$$f=\exp[-(\eta/a)^n],$$

and if one selects

$$\chi(\eta)=-{}^1/_4 b(\eta/a)^{2n},$$

the metric becomes

$$ds^2=d\xi\, d\eta-dr^2-r^2\, d\varphi^2+\frac{nb}{2a}\left(\frac{\eta}{a}\right)^{2n-1}\exp\left[-\left(\frac{\eta}{a}\right)^n\right]$$
$$\times\left\{d\eta\, d\xi-\frac{n\xi}{a}\left(\frac{\eta}{a}\right)^{n-1}d\eta^2\right\}+\frac{b}{4\eta}\exp\left[-2\left(\frac{\eta}{a}\right)^n\right]d\xi^2. \quad (18)$$

The matter density is

$$\varepsilon=\frac{bn}{2\pi ka^3}\left(\frac{\eta}{a}\right)^{n-3}\exp\left[-2\left(\frac{\eta}{a}\right)^n\right].$$

The last two equations show that the constant a has the meaning of the size of the tachar in the z-direction.

Thus, the gravitational field in the region $\eta > 0$ decreases exponentially for $\eta \to \infty$, increases logarithmically for $r \to \infty$ and has a singularity on the plane[1)] $\eta = 0$. For this reason we must assume that for $\eta < 0$, i.e., ahead of the body, there is no gravitational field.

Comparing (17) and (11) we determine the total energy and the momentum of the system:

$$E=P_z=\frac{b}{8k}\int\frac{d\eta}{\eta} \quad (19)$$

In order that the last term of (17) should indeed represent a small correction to the Galilean metric it is necessary that the inequality $\eta \gg b$ be verified. On the other hand, we have assumed in the derivation of (17) that the first term in the right-hand side of (13) is much smaller than the second term which is equal to $\psi(f)$. This means that the equation (17) is valid only under the condition $\eta \gg a \ln^{1/n}(r^2/a\eta)$. Therefore, if one neglects the double logarithms, one must use a quantity of the order a as an upper limit in the integral (19) and a quantity of the order b as the lower limit. Moreover, it is in any case necessary that $b \ll a$. Taking all this into account, we obtain

$$E=P_z=\frac{b}{8k}\ln\frac{a}{b}. \quad (20)$$

This formula is valid only in the case when the dimension a of the object is considerably larger than its gravitational radius kE. In the opposite case the whole matching procedure to the linear approximation and the linear approximation itself loses all meaning. It is important to note here that, as can be seen from the results of the linear approximation, the gravitational field of the zero-mass body does not vanish at infinity. Therefore the usual method for the computation of the energy and momentum by means of the energy-momentum pseudotensor (cf.[2], Sec. 101) is in general not applicable, and the only possibility is matching up the solution to the linear approximation.

From what was said one might have drawn the conclusion that the existence of zero-mass bodies is impossible. However, the fact that the photon and the neutrino exist means that in reality general relativity experiences difficulties in describing such bodies. This difficulty becomes essential for energies of the order a/k. For a photon a characteristic dimension is the wavelength $\hbar/E$, and the linear approximation becomes incorrect for energies of the order of $(\hbar/k)^{1/2} \sim 10^{28}$ eV $\sim 10^{-5}$ g. For photons of lower energy, as well as for macroscopic bodies which satisfy the condition $a \gg kE$ the indicated difficulty does not play an important role.

If $a \lesssim kE$ it is impossible to relate the energy to the parameters which determine the solution. In this case one can find the most general form of the metric far behind the body (for large η) where the matter density is small and can be neglected. If $\epsilon = 0$ it can be seen from (9) that the function F equals a constant. In this case the metric (12) reduces to the form[1]

$$ds^2=d\eta\, d\zeta-\eta^2\, dl^2+\frac{\eta_0}{\eta}d\zeta^2, \quad (21)$$

where $\eta_0 = F/4 > 0$ is a constant. The case considered above corresponds to $\eta_0 = 0$. In the general case the exponential decrease for $\eta \to \infty$ may be replaced by the considerably slower decrease according to the $1/\eta$ law, as can be seen from (21).

In the metric (18) the boundaries of the light cone ds = 0 for du = dv = 0, i.e., in particular, for motion along the axis along which the tachar moves, are determined by the equations

$$d\eta=-\frac{b}{4\eta}\exp\left[-2\left(\frac{\eta}{a}\right)^n\right]d\xi, \quad d\xi=\frac{n^2b}{2a^2}\xi\left(\frac{\eta}{a}\right)^{3n-2}\exp\left[-\left(\frac{\eta}{a}\right)^n\right]d\eta.$$

Similar formulas for the metric (21) have the form $d\eta = -(\eta_0/\eta)d\xi$, $d\xi = 0$. In both cases the interior of the future cone will contain directions corresponding to a decrease of η. It follows that probe particles of finite mass which are at a finite distance from the tachar can catch up with it. For the metric (21) this is true for $\eta_0 > 0$. Subjecting the metric (21) to the transformation $\eta \to -\eta$, $\zeta \to -\zeta$, the quantity η_0 changes sign. At the same time the energy of the system changes sign. Therefore the case $\eta_0 > 0$ corresponds to a physically meaningless object with negative energy, which gravitationally repels particles. For negative η_0 even light is compelled to stay behind the body.

Since the variables f, u, and v are the space coordinates of a comoving frame, their values for each given particle of matter remain constant. Therefore Eq. (13) determines the law of motion of the matter situated on the exterior part of the tachar. The "velocity" of matter along the z axis for $\eta \to \infty$ (in this case $\eta \approx |z|$) is

$$\frac{dz}{dt} = 1 - 2\exp\left[-\left(\frac{\eta}{a}\right)^n\right],$$

i.e., it is smaller than the speed of light but by a quantity which decreases exponentially for $|z| \to \infty$. Matter lags behind the "center" of the tachar, but in an extremely slow way: $\eta(t) = a \ln^{1/n} t$.

The radial "velocity" of matter for $\eta \to \infty$ is determined by

$$\frac{dr}{dt} = \frac{2r}{\eta}\exp\left[-\left(\frac{\eta}{a}\right)^n\right].$$

Along the z axis, i.e. for $r \to \infty$, the analogous formulas have the form:

$$\frac{dz}{dt} = 1 - \frac{2\eta^2}{r^2} \quad \frac{dr}{dt} = \frac{2\eta}{r}$$

An interesting feature is exhibited by the motion of matter in the region $\eta \to \infty$ for finite η_0. If F (and consequently η_0) is constant but small, it can be seen from (14) that the metric (12) reduces to the form (21) by means of the transformation (13) with $\chi(\eta) = 0$. This yields for the speed of matter

$$\frac{dz}{dt} = \frac{1-f}{1+f}$$

which, since f is positive, is smaller than the speed of light. The metric (21) is however invariant with respect to the transformation $\xi' = -\xi - \eta^2/2\eta_0$. Therefore, in the coordinates (ξ', η) the space is also Gahlean for $\eta \to \infty$. The "velocity" of matter in the new coordinate system equals

$$\frac{dz}{dt} = 1 + \frac{2\eta_0}{\eta}$$

i.e., it is larger than the speed of light.

3. If the arbitrary function $\psi(f)$ in the solution under discussion remains the same, but one assumes that as $f \to 0$ the function F(f) tends to zero sufficiently rapidly, one can obtain a metric that approaches the Galilean metric arbitrarily rapidly at infinity. The computation of the energy and momentum by means of the pseudotensor leads in this case obviously to a vanishing result. Such a solution corresponds to a body moving with the speed of light, but as a whole not having any energy or momentum.

A completely analogous procedure can be carried out in the spherically symmetric solution of Tolman. As a result there arises a solution corresponding to a resting body (or in general to a body moving with a speed below the speed of light), with nonvanishing energy and momentum.

Here it is important to make the following remark. The solution (8)—(10) has a singularity for $\eta \to 0$ and finite u and v, i.e., for $\eta \to 0$ and $r \to 0$. In view of the speed of light with which it moves the singularity appears also in the whole $\eta = 0$ plane. Assume that the matter is distributed in such a manner that the body has vanishing energy and momentum and that the body collides with a small outside particle. As a result of the collision at the "periphery" of the body there appears a perturbation, but the speed of light with which the singularity moves does not change, obviously. After a certain "relaxation time" part of the matter will be emitted into the surrounding space and the body will move, like the singularity, with the speed of light and will now have energy and momentum equal to each other, but generally nonzero. In a similar manner a body moving below the speed of light, but not having energy and momentum will be transformed by a collision with an arbitrary small particle into a usual body with finite rest mass. Thus, the equations of the general theory of relativity admit solutions describing bodies without energy and momentum, but such solutions are unstable.

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*)This reasoning is obviously false, since one-photon annihilation (and the subsequent pair production) cannot occur without the presence of a spectator particle which takes up the necessary momentum balance. If the annihilation is into two photons, it becomes necessary to bring the two photons together again in order to produce the "same" electron-positron pair at a different point. The reader will find other instances of loose reasoning in the body of the article. [Translator's note].

1)In order to avoid misunderstandings we make the following remarks. As can be seen from Eq. (12) on the line $u = v = 0, \eta = \text{const.}$ the metric is defined by $ds^2 = (F/f\eta)d\tau^2 > 0$, i.e., is timelike. It might seem that one could conclude from this that the singularity $\eta = 0$ moves with a speed below the speed of light, as does the body as a whole. It is clear, however, that such an analysis of the "world line" described by a singularity has nothing to do with the speed of motion of the body as a whole. It suffices to remember that in the famous Lemaître metric (cf. [2], Sec. 100) for a centrally symmetric field the singularity $r = 0$ describes a spacelike "world line," which does not however imply that it moves with a speed exceeding that of light.

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