The structure of a shock wave in a radiation-dominated plasma

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The structure of a shock wave (SW) propagating through a plasma in which the electromagnetic radiation density exceeds the nuclear and electron density, i.e., $\rho_r > \rho_m$, is considered, and relations between the plasma parameters ahead of and behind the SW front are found. The radiation, which has a nonequilibrium spectrum directly behind the SW front, interacts with the rarefied matter through the Compton mechanism, and after multiple scattering (at $kT \ll m_e c^2$) by electrons, its spectrum becomes a Bose–Einstein equilibrium spectrum characterized by a photon-density deficiency compared with the blackbody radiation density. The missing number of photons is produced by bremsstrahlung emission and absorption processes. The thermodynamic-equilibrium emission spectrum is the result of the combined action of the bremsstrahlung and Compton processes at considerable distances from the SW front.

A radiation-dominated plasma (RDP) is one in which the electromagnetic radiation density is higher than the nuclear and electron density:

$$\rho_r = \varepsilon_r/c^2 = 8.4\cdot 10^{-36}\, T_r^4 > \rho_m.$$

Such a plasma presumably existed in nature over a long segment of the evolutionary path of the hot universe at a temperature (in degrees Kelvin) of $10^9 > T > 10^4$. When ordinary matter is sufficiently strongly heated, it also goes over into the RDP state.

Certain variants of the modern theory of cosmology consider the production and propagation of shock waves (SW) in RDP. Thus, Peebles[1] has considered the hydrodynamics of shock waves in the universe during the era before the recombination of hydrogen (T > 4000), and has obtained limitations on the amplitudes of the matter-density and matter-velocity perturbations. In this paper we shall be interested in the structure of the shock waves and in the distortions of the black-body spectrum of the relict radiation, distortions which are connected with energy dissipation in the shock waves.

In the theory of the homogeneous and isotropic universe (Friedmann's theory) there are no shock waves, but this theory does not explain the present inhomogeneity in the distribution of matter and the existence of clusters of galaxies. If we perturb the homogeneous universe, then the development of shock waves is possible. Shock waves are apparently inevitable at the late stage of the evolution. At the early stage, when RDP exists, the appearance of shock waves is possible, although not inevitable. Under laboratory conditions, RDP can be realized only by means of powerful actions that are inevitably accompanied by shock waves. For these reasons it is of interest to consider the laws governing the propagation of shock waves in RDP. In the asymptotic limit, when $\rho_r \gg \rho_m$, we can neglect ρ_m in the investigation of the thermodynamics of the shock waves. It is sufficient to give the equation of state of the RDP: $p = \epsilon/3 = \rho c^2/3$. The presence, however, of nuclei and electrons in the composition of the RDP is important for the structure of the shock waves. It is precisely the scattering of the photons by electrons that maintains the spatial isotropy of the photon distribution. Because of this isotropy, the RDP can be treated as a Pascal fluid with an isotropic stress tensor. Without the electrons, we would have to deal with a collisionless gas of neutral particles. The energy exchange between the electrons and photons and the bremsstrahlung processes are necessary for the relaxation of the photon spectrum into the equilibrium Planckian spectrum in the compressed plasma located sufficiently far away from the shock-wave front. The evolution of the spectrum of the radiation in RDP compressed by a shock wave is the main subject of the present paper.

1. THE THERMODYNAMICS OF THE SHOCK WAVES

The general theory of relativistic shock waves has been considered earlier.[2-4] Here, for convenience, we shall give the calculation for the case $p = \epsilon/3$, together with the general formulas.

Let the shock wave be at rest. The RDP flows in from the left with velocity (in units of the velocity of light) u_a, electron density n_a, and pressure p_a, and flows out with u_b, n_b, and p_b respectively. Let us emphasize that n_a and p_a are measured by an observer moving with velocity u_a (similarly, for n_b and p_b the velocity of the observer is u_b). The velocities are relativistic: $u_a > 1/\sqrt{3}$, since the velocity A of sound in the RDP is $1/\sqrt{3}$. Therefore, in constructing the conservation equations, we should "Lorentz-transform" all the quantities into the rest frame of the shock wave. The quantities in this frame will be written with a superior bar. Thus,

$$\bar{n}_a = n_a/(1-u_a^2)^{1/2}, \quad \bar{n}_b = n_b/(1-u_b^2)^{1/2}.$$

The conservation of the number of electrons yields

$$n_a u_a/(1-u_a^2)^{1/2} = n_b u_b/(1-u_b^2)^{1/2}.$$

Further, the components of the energy-momentum tensor can be written in the form

$$(\bar{T}_{00})_a = \bar{\varepsilon}_a = \frac{\varepsilon_a + p_a u_a^2}{1-u_a^2} = p_a \frac{3+u_a^2}{1-u_a^2},$$

$$(\bar{T}_{01})_a = \frac{(\varepsilon_a + p_a) u_a}{1-u_a^2} = 4p_a \frac{u_a}{1-u_a^2},$$

$$(\bar{T}_{11})_a = \frac{\varepsilon_a u_a^2 + p_a}{1-u_a^2} = p_a \frac{1+3u_a^2}{1-u_a^2}.$$

and similarly for the quantities with the subscript b.

The nonrelativistic equations of conservation of momentum

$$p_a + \rho_a u_a^2 c^2 = p_b + \rho_b u_b^2 c^2$$

and energy

$$u_a(\varepsilon_a+p_a+{}^1/{}_2\rho_a u_a{}^2c^2)=u_b(\varepsilon_b+p_b+{}^1/{}_2\rho_b u_b{}^2c^2)$$

become

$$(\bar{T}_{11})_a=(\bar{T}_{11})_b,\quad (\bar{T}_{01})_a=(\bar{T}_{01})_b,$$

from which, by dividing one equation by the other, we obtain

$$u_a/(1+3u_a{}^2)=u_b/(1+3u_b{}^2).$$

The function $u/(1+3u^2)$ has a maximum at $u=1/\sqrt{3}$, whence it can be seen that a nontrivial solution for u_b, with $u_b<1/\sqrt{3}$ and $u_b\neq u_a$, exists only for $u_a>1/\sqrt{3}$. It is evident that the nontrivial solution is $u_b=1/3u_a$. We shall, for brevity, henceforth set $u_a=u$ and $u_b=1/3u$. Then $u_a u_b=A^2$ identically. We find the relation

$$\frac{n_b}{n_a}=u\left(\frac{9u^2-1}{1-u^2}\right)^{1/2},\quad \frac{p_b}{p_a}=\frac{9u^2-1}{3(1-u^2)}. \tag{1}$$

It is worth noting that $\bar{n}_b/\bar{n}_a=3u^2$ in the rest frame of the shock wave. In the system of coordinates in which the matter compressed in the shock wave is at rest the free-stream velocity is $u_a'=(3u^2-1)/2u$, while the density ratio is

$$\left(\frac{n_b}{n_a}\right)'=\frac{n_b}{n_a}[1-(u_a')^2]^{1/2}=\frac{9u^2-1}{2}; \tag{2}$$

similarly, in the rest frame of the matter still uncompressed by the shock wave the velocity is $u_b''=(3u^2-1)/2u$, and the density ratio is

$$\left(\frac{n_b}{n_a}\right)''=\frac{n_b}{n_a}\frac{1}{[1-(u_b'')^2]^{1/2}}=\frac{2u^2}{1-u^2}$$

In the nonrelativistic theory, the compression ratio in the shock wave is finite. This result should be compared with the expression $\bar{n}_b/\bar{n}_a=3u^2$, which varies from 1 to 3 as u varies from $1/\sqrt{3}$ to 1. The ratio n_b/n_a includes the Lorentz contraction and therefore tends to infinity as $u\to 1$. The specific entropy for the equilibrium RDP is $S=\alpha p^{3/4}n^{-1}$. We obtain

$$S_b/S_a=\varphi^{1/4},\quad \varphi=\frac{9u^2-1}{27u^4(1-u^2)}.$$

As was to be expected, the entropy increases: when $u>1/\sqrt{3}$ the quantity $\varphi>1$. For $u=1/\sqrt{3}+\varkappa$, where $\varkappa\ll 1$, we have

$$\varphi=1+12\sqrt{3}\,\varkappa^3,\quad S_b/S_a=1+3\sqrt{3}\,\varkappa^3 \tag{3}$$

This is in accord with the general shock-wave theory, in which it is shown that $\Delta S/S\propto(\Delta u)^3\propto\varkappa^3$.

2. THE STRUCTURE OF THE SHOCK WAVES

The general structure and the division into regions are shown in Fig. 1. The region $[-\infty, A]$ contains an unperturbed equilibrium RDP having (in its rest frame) the quantities n_a and p_a, and flowing with the velocity $u_a=u$. The changeover to the quantities n_b, p_b, and u_b occurs in the layer AB. For this purpose, a few collisions of the photons with the electrons are sufficient. Therefore, the thickness of the layer is of the order of $[AB]\propto(n\sigma_T)^{-1}$, where $n\sim n_a\sim n_b$ and σ_T is the Thompson scattering cross section. For a weak wave, for which $u=1/\sqrt{3}+\varkappa$, $\varkappa\ll 1$, the thickness increases: $[AB]\propto\varkappa^{-1}(n\sigma_T)^{-1}$, in accord with the general theory of

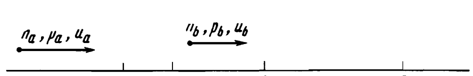

FIG. 1

shock waves. In this case, however, there arises around B a spatially isotropic, but substantially nonequilibrium, spectrum which is a linear superposition of Planckian spectra and which satisfies the two conditions $N_b=N_a n_b/n_a$ for the photon density and $\epsilon_b=3p_b=\epsilon_a p_b/p_a$ for the energy density. The spectral energy density of the radiation can then be expressed by the formula

$$\varepsilon_\nu=\int P(\nu,T)R(T)\,dT, \tag{4}$$

where R(T) is the temperature distribution function obeyeing, as shown by theory[5], the conditions

$$R>0,\quad \int R(T)\,dT=1. \tag{5}$$

Here

$$P(\nu,T)=\frac{8\pi h\nu^3}{c^3}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}$$

is the spectral energy density in a black-body radiation. The conditions on the photon density and the energy can be written with the aid of (4) and (5) in the form

$$\int T^3R\,dT=T_a{}^3\frac{N_b}{N_a}=T_a{}^3\frac{n_b}{n_a},\quad \int T^4R\,dT=T_a{}^4\frac{p_b}{p_a}.$$

That R(T) does not reduce to a δ-function is clear from the fact that the entropy exceeds the initial entropy. It is obvious that the above two conditions cannot determine completely the function R(T).

In the limit of weak shock waves, however, the distribution function R, being close to a δ-function, can be specified by two parameters: the Rayleigh-Jeans temperature Θ and the dimensionless quantity b characterizing the width:

$$\Theta=\int TR\,dT,\quad b^2=\frac{1}{2\Theta^2}\int(T-\Theta)^2R\,dT.$$

We neglect the influence of the remaining moments of the type $\int(T-\Theta)^nR\,dT$, $n>2$, which are of higher order in smallness with respect to $\varkappa=u-1/\sqrt{3}$. Let us expand the Planck function $P(\nu, T)$ in a series about the temperature Θ up to terms of second order in $T-\Theta$. Omitting the intermediate calculations, which can be found in [5], we obtain a spectrum of the type

$$\varepsilon_\nu=\int P(\nu,T)R\,dT\approx P(\nu,\Theta)+b^2\Theta^2\left.\frac{\partial^2P(\nu,T)}{\partial T^2}\right|_{T=\Theta},$$

a radiation energy density

$$\varepsilon=\varepsilon(\Theta)(1+12b^2), \tag{6}$$

and a photon density

$$N=N(\Theta)(1+6b^2), \tag{7}$$

where $\epsilon(\Theta)$ and $N(\Theta)$ are the photon energy and density in black-body radiation of temperature Θ. Comparing (6) and (7) with the formulas (1) and (3), and taking account of the fact that

$$\frac{\varepsilon(\Theta)}{\varepsilon_a}=\left(\frac{N(\Theta)}{N_a}\right)^{4/3}=\left(\frac{\Theta}{T_a}\right)^4,$$

we find the Rayleigh-Jeans temperature Θ and the parameter b for small $\varkappa\ll 1$:

$$\Theta=T_a u\sqrt{3}=T_a(1+\varkappa\sqrt{3}),\quad b^2=\varkappa^3\sqrt{3}.$$

Thus, the width of the distribution function R, characterized by b, is proportional to $\varkappa^{3/2}$. On the face of it, the Doppler effect should, when the velocity of the RDP changes by a value of the order of $\varkappa$, yield $b\sim\varkappa$. The decrease of the width to $\varkappa^{3/2}$ can clearly be understood from the following arguments. The radiation, on crossing the shock-wave front, is scattered a large number of times $\kappa^{-1} \gg 1$ by the free electrons. The temperature spread of the function R over one Thompson mean free path is proportional to the velocity difference across this path: $(\Delta u)_1 \propto \kappa \Delta u \propto \kappa^2$. In the subsequent scatterings this temperature spread grows according to the diffusion law in proportion to the square root of the total number of collisions, i.e., $b \sim \kappa^2\sqrt{\kappa^{-1}} = \kappa^{3/2}$. In the case of a strong shock wave $(u \sim 1)$ the shape of the superposition function R(T) and, consequently, the spectrum of the radiation behind the front are approximately determined in the Appendix. The general conclusion is that the photon spectrum is non-Planckian, being of the Rayleigh-Jeans type in the low-frequency region, and falling off exponentially in the high-frequency region. The Rayleigh-Jeans temperature is, however, relatively small, and does not correspond to the total energy density. How does the nonequilibrium spectrum further evolve and approach equilibrium? This question was first precisely formulated for the tenuous plasma by Kompaneets.[6] Let us return to Fig. 1 showing the division of the shock wave into zones.

In the zone BC, as a result of the Compton exchange of energy between the photons and electrons, the spectrum is reconstructed into an "equilibrium spectrum at the given photon density," i.e., into the Bose-Einstein (indicated by the index BE) form

$$\varepsilon_\nu(\mu, T_{BE}) = \frac{8\pi h\nu^3}{c^3}\left[\exp\left(\frac{h\nu}{kT_{BE}}+\mu\right)-1\right]^{-1},$$

in which the temperature T_{BE} and the chemical potential μ are determined from the relations

$$\int \varepsilon_\nu(\mu, T_{BE})\,d\nu = \varepsilon_b, \qquad \int \frac{\varepsilon_\nu(\mu, T_{BE})}{h\nu}\,d\nu = N_b. \tag{8}$$

Thus, in the case $u = 1/\sqrt{3} + \kappa$, $\kappa \ll 1$, using (6) and (7), and expanding the formula (8) for small $\mu \ll 1$, we obtain the relations

$$T_{BE}^4(1-d_1\mu) = \Theta^4(1+12b^2),$$
$$T_{BE}^3(1-d_2\mu) = \Theta^3(1+6b^2),$$

where

$$d_1 = \frac{3\int x^2(e^x-1)^{-1}dx}{\int x^3(e^x-1)^{-1}dx} \approx 1.11,$$
$$d_2 = \frac{2\int x(e^x-1)^{-1}dx}{\int x^2(e^x-1)^{-1}dx} \approx 1.37.$$

Hence we obtain

$$\mu = \frac{4b^2}{{}^4/{}_3 d_2 - d_1} = 9.6\varkappa^3 = 5.6b^2,$$
$$T_{BE} = \Theta(1+3b^2+{}^1/{}_4 d_1\mu) = \Theta(1+7{,}9\varkappa^3) = \Theta(1+4.5b^2).$$

In the limiting case of a strong shock wave when $u \approx 1$ and the chemical potential is large, i.e., $\mu > 1$, the radiation spectrum has the Wien form: $\epsilon_\nu \propto \nu^3 \exp(-h\nu/kT_{BE})$ with the temperature

$$T_{BE} = \frac{\varepsilon_b}{3kN_b} \approx \frac{0.3T_a}{u}\left(\frac{9u^2-1}{1-u^2}\right)^{1/2}$$

and the value for the chemical potential

$$\mu = 4\ln\frac{(16\pi)^{1/4}}{N}\left(\frac{\varepsilon}{3hc}\right)^{3/4} = 4\ln 0.9\frac{S_b}{S_a}.$$

The thickness of the zone BC (in the rest frame of the matter compressed in the shock wave) is approximately m_ec^2/kT times the photon mean free path. This corresponds to an interval of time sufficient for the Compton diffusion of the photons in frequency space to bring about the formation of a Bose-Einstein distribution.

Finally, in the zone CD, owing to the still slower bremsstrahlung process, the radiation spectrum is converted into a black-body radiation spectrum (or a Planckian spectrum, indicated by the index Pl) with temperature

$$T_{Pl} = T_a\left[\frac{9u^2-1}{3(1-u^2)}\right]^{1/4}$$

and the corresponding photon density

$$N_{Pl} = N_a\left[\frac{9u^2-1}{3(1-u^2)}\right]^{3/4} = N_b\left[\frac{9u^2-1}{27u^4(1-u^2)}\right]^{1/4} = N_b\frac{S_b}{S_a},$$

exceeding the photon density in the radiation in the zone BC by a factor of S_b/S_a. The Planckian spectrum is realized in the layer D and further does not change up to $+\infty$.

Notice especially that the reconstruction of the spectrum in the regions BC and CD is accomplished at constant energy ϵ, i.e., the relation $p = \epsilon/3$ does not depend on the type of spectrum. Therefore, the entire process proceeds at constant velocity u_b and constant matter density n_b. These mechanical quantities vary rapidly in the layer AB and at once assume constant equilibrium values—the same as at $+\infty$. The formation of the black-body radiation spectrum takes place via a series of Bose-Einstein spectra with a gradually decreasing chemical potential. The production of the deficient number of photons in the radiation occurs during the emission of bremsstrahlung photons at frequencies $h\nu \gtrsim x_0kT$, $x_0 \ll 1$, which are picked up by the Compton diffusion toward the region of higher energies $h\nu \sim 3kT$ (cf. the picture developed in [6,7]).

The frequency $x_0 \ll 1$ characterizes the weakness of the bremsstrahlung absorption and emission as compared to the gaining of energy by the low-frequency photons when they undergo Compton scattering by hotter electrons:

$$\frac{x_0^2}{g(x_0)} = \frac{w_{ff}}{\pi ch^2}\left(\frac{ch}{2kT}\right)^3\frac{m_e}{\sigma_T n_e} \approx 10^{-16} n_e\left(\frac{T}{10^6}\right)^{-4,5},$$

where

$$g(x) = \frac{\sqrt{3}}{\pi}\ln\frac{2.35}{x}$$

is the Gaunt factor (see, for example, [8]) and $w_{ff} = 1.43 \times 10^{-27} n_e^2 T^{1/2}$ is the rate of volume energy loss by a plasma of density n_e and temperature T during the emission of bremsstrahlung. Since each regenerated quantum increases its energy on the average, the already existing quanta must lose this energy in order for the total energy ϵ to remain unchanged. In consequence, the radiation temperature $T = \epsilon/3fkN$ (where $3fkT = \epsilon/N$ is the mean photon energy) should fall, the factor f weakly varying in the process from 1 for the Wien spectrum to 0.9 for the black-body radiation spectrum.

The equations for the energy and the rate of photon production during bremsstrahlung emission can be written in the form:

$$3fkTN = \varepsilon,$$
$$3kT\frac{dN}{dt} = w_{ff}(T)F(\mu, x_0).$$

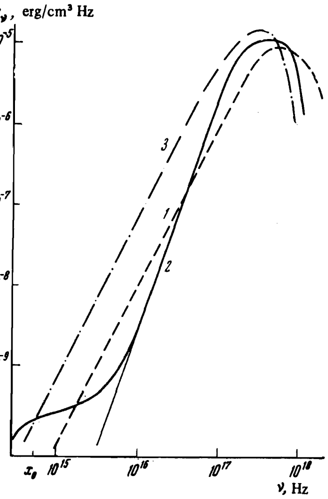

FIG. 2. The shape of the radiation spectrum in the various zones behind the shock-wave front. The radiation energy density is $\epsilon_b = 6 \times 10^{12}$ erg/cm^{-3}, the particle concentration in the plasma is $n_b = 10^{14}$ cm^{-3}, and $u = 0.99$. The curve 1 is a superposition of black-body radiation spectra (the region B), and the curve 2 is the plot of a Bose-Einstein spectrum with a chemical potential $\mu \approx 2$ (the region C). Because of the bremsstrahlung processes, the radiation spectrum near $\nu \leqslant 10^{15}$ Hz obeys the Rayleigh-Jeans law; the curve 3 represents the final blackbody radiation spectrum with a temperature of $T_{Pl} = 5 \times 10^6$ °K (the region D).

The function $F(\mu, x_0)$ takes account of both the $2.5g^2(x_0)$-fold increase in the rate of energy loss by the plasma due to Comptonization when $\mu > x_0$ [6] and the decrease in the energy loss as a result of the bremstrahlung self-absorption when $\mu < x_0$ [9,10]; this function is of the form

$$F(\mu, x_0) = \begin{cases} 2{,}5g^2(x_0), & \mu>1 \\ 2.5[g^2(x_0)-g^2(\mu)], & x_0<\mu<1. \\ 1.4\mu g(x_0)/x_0, & \mu<x_0 \end{cases} \qquad (9)$$

In the case of a small chemical potential ($\mu < 1$) the number of photons in the radiation

$$N=N_{Pl}[1+\mu(^3/_4 d_1-d_2)] = N_{Pl}(1-0{,}53\mu),$$

and the equation for the rate of production of photons can be written in the form

$$0.53\varepsilon d\mu/dt=-w_{ff}F(\mu, x_0).$$

It follows from (9) that for $x_0 < \mu < 1$ the chemical potential decreases almost linearly with the characteristic time $\tau = 0.2\epsilon\mu_0/w_{ff}g^2(x_0)$, while in the case when $\mu < x_0$ it decreases exponentially with the characteristic time $\tau = 0.4\epsilon x_0/w_{ff}g(x_0)$. To this time corresponds the zone width $[CD] = \tau c/3u$. Figure 2 shows for $u = 0.99$ a few spectra characteristic of the different zones of the shock wave (the conditions roughly correspond to the hot Universe with $z = 2\times10^6$ and $\Omega = 1$ when $\epsilon_b = 6 \times 10^{12}$ erg/cm^3 and $n_b = 10^{14}$ cm^{-3}).

We have presented above the available results pertaining to the observable (in principle) quantities. The justification of the picture and the methods of the calculation in their most general form are contained in Kompaneets's[6] and Waymann's[11] papers, and, in greater detail, in our previous papers on spectrum relaxation. Owing to the constancy of n_b after the layer AB, the solution of the cosmological problem can be directly carried over to the shock-wave problem.

We take this opportunity to thank G. S. Bisnovatiĭ-Kogan and R. A. Syunyaev for a discussion of the paper.

APPENDIX

Let us find for the case of a strong ($u \sim 1$) shock wave the approximate form of the radiation spectrum near the point B directly behind the shock-wave front and the superposition function $R^*(T)$. Let us imagine an observer moving with velocity u_b relative to the shock-wave front, but located in the zone $[-\infty, A]$ (the observer moves with velocity $u_a' = (3u^2-1)/2u$ relative to the matter before the shock-wave front). From the point of view of this observer, the radiation in the zone $[-\infty, A]$ is anisotropic; remaining in each element of solid angle a black-body radiation, it has the angle-dependent temperature:

$$T(\psi)=T_a[1-(u_a')^2]^{1/2}/(1-u_a'\cos\psi). \qquad (A.1)$$

After being scattered several times by electrons in the zone AB of the shock-wave front, the radiation becomes isotropic, the photon density increasing, according to (2), by a factor of $r^3 = (n_b/n_a)' = (9u^2-1)/2$, and the energy of each photon increasing by roughly a factor of r in the process. It follows from this that the photon density

$$\frac{N_b^*}{N_b}=r^3\int T^3(\psi)\frac{d\cos\psi}{2}=1$$

coincides with the exact value ($N_b^* = N_b$).

The radiation energy density differs slightly from ϵ_b, which characterizes the accuracy of the approximation:

$$\frac{\varepsilon_b^*}{\varepsilon_b}=r^4\int T^4(\psi)\frac{d\cos\psi}{2}=\frac{(3u^2+1)^2}{(36u^2-4)^{2/3}}\to1.6 \quad \text{as} \quad u\to1.$$

The superposition function $R^*(T)$ is easily found (the angle ψ is connected with the temperature through the formula (A.1)):

$$R^*(T)=\frac{r}{2}\frac{d\cos\psi}{dT}=\begin{cases}\dfrac{T_1T_2}{T_2-T_1}\dfrac{1}{T^2}, & T_1<T<T_2,\\ 0, & T<T_1, \quad T>T_2\end{cases}$$

where

$$T_2=T_a\frac{1+u}{(1-u^2)^{1/2}}\frac{(9u^2-1)^{5/6}}{(3u+1)2^{1/3}}\to T_a\frac{2{,}25}{\sqrt{1-u^2}} \quad \text{as} \quad u\to1,$$

$$T_1=T_a\frac{\sqrt{1-u^2}}{1+u}\frac{(9u^2-1)^{5/6}}{(3u-1)2^{1/3}}\to1{.}1T_a\sqrt{1-u^2} \quad \text{as} \quad u\to1.$$

Using the approximate function $R^*(T)$, we obtain the Rayleigh-Jeans temperature

$$\Theta^*=\int TR\,dT=T_a\frac{\sqrt{1-u^2}(9u^2-1)^{5/6}}{2^{4/3}(3u^2-1)}\ln\frac{1+u}{1-u}\frac{3u-1}{3u+1}\to T_1\ln\frac{1}{1-u} \quad \text{as} \quad u\to1$$

and the form of the spectrum

$$\varepsilon_\nu^*=\int P(\nu, T)R^*\,dT=B\nu^2\ln\frac{1-\exp(-h\nu/kT_2)}{1-\exp(-h\nu/kT_1)} \approx\begin{cases}B\nu^2\ln(T_2/T_1), & h\nu<kT_1\\ B\nu^2\ln(kT_2/h\nu), & kT_1<h\nu<kT_2,\\ B\nu^2\exp(-h\nu/kT_2), & kT_2<h\nu\end{cases}$$

where

$$B=\frac{8\pi k}{c^3}\frac{T_1T_2}{T_2-T_1}.$$

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