Study of gravitational waves emitted by a rapidly rotating drop of a homogeneous gravitating liquid

M. N. Fedosov and V. P. Tsvetkov

Kalinin State University

(Submitted March 30, 1973)
Zh. Eksp. Teor. Fiz. **65**, 1289–1293 (October 1973)

The intensity of gravitational wave emission is calculated for a rapidly rotating drop of a homogeneous gravitating liquid that assumes the shape of a triaxial ellipsoid.

In connection with Weber's experiments on the registration of gravitational waves from cosmic sources, interest attaches to the possible sources of gravitational radiation of extraterrestrial origin [2-4]. We shall show that a uniformly rotating drop of a homogeneous gravitating liquid, which assumes the equilibrium shape of a triaxial ellipsoid, can be a source of gravitational waves of high intensity. The possibility of pulsed gravitational radiation from a rapidly rotating tesseral-shape drop as it changes from one tesseral figure to another was considered in [5].

Thus, assume that we have a drop of a gravitating homogeneous liquid, rotating as a unit with a constant angular velocity Ω. We assume its mass m, density ρ, and angular momentum M given. It is well known [6] that ellipsoids with three unequal axes (Jacobi ellipsoids), rotating about the minor axis, can be equilibrium figures. Poincare and Darvin have shown that the shape of a Jacobi ellipsoid is stable against small perturbations if

$$0.239G^{1/2}m^{5/3}\rho^{-1/6}<M<0.309G^{1/2}m^{5/3}\rho^{-1/6},$$

where G is the gravitational constant. Thus, the drop in the indicated region of values of the angular momentum assumes the stable form of a triaxial ellipsoid with semiaxes $a>b>c$.

The intensity of the gravitational radiation will be calculated from the well known Landau-Lifshitz formula for the quadrupole gravitational radiation [7]:

$$\frac{dI}{d\Omega}=\frac{G}{36\pi c^5}\left[\frac{1}{4}(\dddot{Q}_{\alpha\beta}n_\alpha n_\beta)^2+\frac{1}{2}\dddot{Q}_{\alpha\beta}{}^2-\dddot{Q}_{\alpha\beta}\dddot{Q}_{\alpha\gamma}n_\beta n_\gamma\right], \tag{1}$$

where

$$Q_{\alpha\beta}=\int\limits_V \rho(3x_\alpha x_\beta-r^2\delta_{\alpha\beta})\,dV$$

is the quadrupole mass tensor, n_α is a unit vector in the observation direction, and c is the speed of light, while the points denote differentiation with respect to time.

In our case, if the drop rotates about the z axis, the following independent components will differ from zero

$$\dddot{Q}_{xx}=-\dddot{Q}_{yy}=-\dddot{Q}_{xy}\,\mathrm{tg}\,2\Omega t={}^{12}/_{5}m(a^2-b^2)\Omega^3\sin 2\Omega t. \tag{2}$$

We recalculate, a, b, c, and Ω in (2) in terms of the specified m, ρ, and M. To this end, we introduce new symbols $k^2=1-b^2/a^2$ and $n=1-c^2/a^2$. As follows from [6], n and k are connected by the known relations for the Jacobi ellipsoids. Representing

$$n(k^2)=\sum_{m=0}^{\infty}c_m k^{2m}, \tag{3}$$

we find that

$$c_0=\xi,\quad c_1=\frac{1}{4\xi}(1-\xi^2),$$

and the remaining coefficients $c_{m>1}$ can be determined with the aid of a recurrence relation

$$c_m=\frac{-1}{\pi(\xi)}\left[\sum{}'\frac{(c_1)^{i'}(c_2)^{j'}\dots(c_{l'})^{t'}}{i'!j'!\dots t'!}\left(\frac{\partial^{r'-1}\pi(\xi)}{\partial\xi^{r'-1}}\right)\right.$$
$$\left.+\sum_{s=0}^{m-1}\sum\frac{(c_1)^i(c_2)^j\dots(c_l)^t}{(m-s)!\,i!\,j!\dots t!}\left(\frac{\partial^{m+r-s}\Psi(\xi,x)}{\partial\xi^r\,\partial x^{m-s}}\right)_{x=0}\right], \tag{3a}$$

where the summations Σ' and Σ in (3a) should be extended respectively to all the solutions in integer positive numbers of the equations

$$i'+2j'+\dots+l't'=m\,(l'<m),\quad i+2j+\dots+lt=s;$$

where $r'=i'+j'+\dots+t'$, $r=i+j+\dots+t$. Here

$$\pi(\xi)=\frac{2(1-4\xi^4)}{(1-\xi^2)^{1/2}(3+8\xi^2-8\xi^4)},\quad x^4\Psi(\xi,x)=F(\varphi,\lambda)-E(\varphi,\lambda)+K(\lambda)$$
$$-F(\chi,\lambda)+\frac{1}{1-\lambda^2}\left[E(\chi,\lambda)-E(\lambda)-x^2\lambda^2\frac{1-\xi^2}{1-x^2}F(\varphi,\lambda)+x^2\lambda\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right],$$
$$\varphi=\arcsin\xi,\quad \chi=\arcsin\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2},\quad \lambda=\frac{x}{\xi}, \tag{3b}$$

ξ is the root of the equation

$$\arcsin x=\frac{x(1-x^2)^{1/2}(3+10x^2)}{3+8x^2-8x^4},$$

$E(\varphi,\lambda)$, $F(\varphi,\lambda)$, $E(\lambda)$, and $K(\lambda)$ are elliptic integrals.

For numerical estimates it is important to approximate the function $n(k^2)$ by a sufficiently simple expression. Calculating with the aid of (3a) and (3b) the value of c_2:

$$c_2=\frac{1-\xi^2}{\xi^3(4\xi^4-1)}\left(\frac{117}{384}\xi^6+\frac{119}{768}\xi^4-\frac{219}{1536}\xi^2-\frac{79}{3072}\right) \tag{4}$$

and using the asymptotic form of $n(k^2)$ as $k^2\to 1$, we find that the function $n(k^2)$ can be approximated with a high degree of accuracy (as shown by a comparison with numerical calculations of Darwin [6], the discrepancy is not more than 0.001) by the expression

$$n(k^2)\approx\xi+(c_1-1)k^2+(c_2+{}^3/_2)k^4+({}^7/_2-4\xi-3c_1-2c_2)k^6$$
$$+(3\xi-3+2c_1+c_2)k^8-(1-k^2)^2\ln(1-k^2)+{}^1/_6k^4(1-\nu k^2)^2$$
$$\times\ln(1-\nu k^2)-{}^1/_6k^6(1-\nu)[(1-k^2)\nu+(2-k^2)\ln(1-\nu)], \tag{5}$$

where ν is the root of the equation

$$x+{}^1/_2x^2+\ln(1-x)=6(\ln 4+{}^5/_2-6\xi-3c_1-c_2).$$

We denote by M_0 the drop angular momentum starting with which the Jacobi ellipsoids become stable. Its value is

$$M_0{}^2=\frac{3}{25}\left(\frac{3}{4\pi}\right)^{1/3}f(\xi)Gm^{10/3}\rho^{-1/3},\quad f(\xi)=\frac{\xi^2(1-\xi^2)^{1/3}}{{}^3/_8+\xi^2-\xi^4}. \tag{6}$$

It is convenient to introduce the parameter $u=M/M_0$, which characterizes the deviation of the angular momentum of the drop from the critical value M_0. In the region of values $1<u<1.293$, the Jacobi ellipsoids are stable.

Putting $k = \tanh \eta$ and a second relation for the Jacobi ellipsoid [6], we obtain

$$\frac{(2\,\mathrm{cth}\,\eta-\mathrm{th}\,\eta)^2\,\mathrm{ch}^{1/3}\,\eta}{n(1-n^2)^{1/2}}\left[F(\varphi',\lambda')-\left(1+\frac{1-n^2}{1-\mathrm{cth}^2\,\eta n^2}\right)E(\varphi',\lambda')+\frac{n(1-n^2)^{1/2}}{\mathrm{ch}\,\eta(1-\mathrm{cth}^2\,\eta n^2)}\right]=(1-\xi^2)^{1/6}f(\xi)u^2, \tag{7}$$

where $\varphi' = \sin^{-1} n$ and $\lambda' = (\tanh \eta)/n$.

The angular velocity Ω of the drop is then equal to

$$\Omega=(4\pi\rho G)^{1/2}f^{1/2}(\xi)u\,\mathrm{ch}^{4/3}\,\eta\,\frac{(1-n^2)^{1/6}}{1+\mathrm{ch}^2\,\eta}. \tag{8}$$

Formulas (3a), (3b), (5), (7), and (8) give a parametric representation of the dependence of n, k, and Ω on u, and consequently of the components $Q_{\alpha\beta}$ on m, ρ, and M.

Choosing a spherical coordinate system in which $n_x = \sin\theta\cos\varphi$, $n_y = \sin\theta\sin\varphi$, $n_z = \cos\theta$, we obtain from (1) the instantaneous distribution of the radiation:

$$dI(t)/d\Omega=A[\cos^2\theta+{}^1/_4\sin^4\theta\sin^2(\omega t+2\varphi)],$$
$$A=\frac{12\pi}{25c^5}\left(\frac{3}{4\pi}\right)^{1/3}G^4m^{10/3}\rho^{5/3}f^3(\xi)u^6\,\mathrm{sh}^4\,2\eta\,\mathrm{ch}^{20/3}\,\eta\,\frac{(1-n^2)^{4/3}}{(1+\mathrm{ch}^2\,\eta)^6}, \tag{9}$$

$\omega = 2\Omega$ is the frequency of the gravitational radiation.

Averaging (9) over the period of the revolution of the drop, we obtain

$$d\bar{I}/d\Omega={}^1/_8A(1+6\cos^2\theta+\cos^4\theta). \tag{10}$$

Consequently, the maximum intensity of the gravitational radiation is directed along the rotation axis, and the minimum is in a perpendicular direction. For the time-averaged radiation intensity, their ratio is equal to 8.

The total radiation intensity is obtained by integrating (9) with respect to $d\Omega$:

$$I(t)=\bar{I}={}^8/_5\pi A. \tag{11}$$

We are particularly interested in the case when $(u-1) \ll 1$. Then the Jacobi ellipsoid differs insignificantly from the ellipsoid of revolution. Using the method of expanding the corresponding expressions in powers of η, we have at $(u-1) \ll 1$

$$I=\bar{I}=\frac{8\pi}{5}A=\frac{3\pi}{25c^5}\left(\frac{3}{4\pi}\right)^{1/3}G^4m^{10/3}\rho^{5/3}f^3(\xi)\beta(\xi)(1-\xi^2)^{4/3}(u-1)\approx1.403G^4c^{-5}m^{10/3}\rho^{5/3}(u-1), \tag{12}$$

where

$$\beta(\xi)=9216\xi^4(4\xi^4-1)\Gamma(\xi),\quad \Gamma^{-1}(\xi)=2880\xi^8+616\xi^6-1412\xi^4+754\xi^2-339.$$

The frequency of the gravitational radiation is determined in this case by the formula

$$\omega=(4\pi\rho Gf(\xi))^{1/2}(1-\xi^2)^{1/4}[1-\gamma(\xi)(u-1)],$$
$$\gamma(\xi)=3\Gamma(\xi)[432\xi^8+368(\xi^4-1)\xi^2-8\xi^4+113]. \tag{13}$$

The emission of gravitational waves from the drop leads to a decrease in the energy E and in the angular momentum M of the drop with time, and consequently also a decrease of the parameter η. Using the obvious relation

$$-dE/dt=I={}^8/_5\pi A,$$

we can easily obtain

$$\int_{\eta_0}^{\eta}\left(\frac{dE(\eta)}{d\eta}\right)A^{-1}(\eta)d\eta=-\frac{8\pi}{5}t, \tag{14}$$

where

$$E=\frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3}Gm^{5/3}\rho^{1/3}\left[\frac{f(\xi)u^2(1-n^2)^{1/3}\mathrm{ch}^{4/3}\,n}{1+\mathrm{ch}^2\,\eta}-\frac{2(1-n^2)^{1/6}}{n\,\mathrm{ch}^{1/3}\,\eta}F(\varphi',\lambda')\right],$$

In the case $(u-1) \ll 1$, the integration in (14) can be carried out in terms of elementary functions

$$u=1+\frac{\eta^4}{\beta(\xi)}=1+(u_0-1)e^{-pt},$$
$$p=\frac{24}{25}\left(\frac{\pi}{6}\right)^{1/3}f(\xi)\beta(\xi)(1-\xi^2)G^3m^{5/3}\rho^{4/3}\approx5{,}449G^3m^{5/3}\rho^{4/3}c^{-5},\quad u_0=u(t=0). \tag{15}$$

Substituting (15) in (12), we obtain the dependence of the intensity of the gravitational radiation of the drop on the time at $(u-1) \ll 1$.

We can regard white dwarfs and neutron stars, which are at present identified with pulsars, as rapidly rotating drops of a gravitating liquid. At the characteristic neutron-star parameters $m = m_\odot$ and $\rho = 4\times10^{14}$ g/cm^3, assuming $M = 1.0296M_0$, we have $I = 10^{53}$ erg/sec. At the given value of the angular momentum, the shape of the triaxial ellipsoid is stable. The frequency of the gravitational radiation is equal to 1764 Hz, which corresponds to a drop-revolution period $T = 0.001$ sec, which is smaller by a factor 30 than the periods of the presently known pulsars. Choosing the distance from the earth to be 2×10^{22} cm, we obtain a gravitational-energy flux on earth 4×10^7 erg/sec-cm^2. Using (15), we find that after approximately 2.3 sec the radiation intensity drops to 10^{30} erg/sec.

In the case of a white dwarf, choosing $\rho = 10^8$ g/cm^3, $m = m_\odot$, and $M = 1.0296$, we obtain $I \approx 10^{42}$ erg/sec, a radiation frequency 0.88 Hz, and a radiation intensity that decreases by a factor 10 after 10 years.

It appears that one cannot exclude the possibility that some of the presently known pulsars have an average density of less than 10^{14} g/cm^3 ($\sim10^{11}-12^{12}$ g/cm^3). They can in this case perfectly well assume a stable form of a triaxial ellipsoid.

[1] J. Weber, Phys. Rev. Lett. **22**, 1320 (1969); **25**, 180 (1970).
[2] R. L. Forward and D. Berman, Phys. Rev. Lett. **18**, 1071 (1968).
[3] K. S. Thorne, Ap. J., **158**, 1, 997 (1969).
[4] J. R. Ipser, Ap. J. **166**, 175 (1971).
[5] A. I. Tsygan, Zh ETF Pis. Red. **14**, 465 (1971) [JETP Lett. **14**, 317 (1971)].
[6] P. E. Appell, Equilibrium Figures of a Rotating Homogeneous Liquid (Russian translation), ONTI, 1936.
[7] L. D. Landau and E. M. Lifshitz, Teoriya polya (Field theory), Nauka, 1967 [Pergamon, 1971].

Translated by J. G. Adashko
130