On the theory of self-similar degeneracy of homogeneous isotropic turbulence

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An analysis of recent experiments of Ling and co-workers on the degeneracy of homogeneous isotropic turbulence in flow behind a grid shows that the Kármán complete self-similarity hypothesis is not confirmed at the intermediate stage of the turbulence degeneracy. Another more general self-similarity hypothesis is proposed. All solutions of the Kármán–Howarth equation of propagation of correlations consistent with this hypothesis are obtained and it is found that the experimental data are in agreement with this more general incomplete self-similarity hypothesis.

1. The study of homogeneous isotropic turbulence was begun by Taylor in a series of pioneering papers,[1] but this field acquired its present importance after A. N. Kolmogorov's investigations[2] (see also [3,4]) in which the homogeneity and isotropy of the local structure of arbitrary turbulent flows at sufficiently large Reynolds numbers were predicted. The experimental confirmation of the microstructural laws obtained on the basis of these predictions made local isotropy and homogeneity a fundamental property of the microstructure of developed turbulent flows.

Of the utmost importance in the theory of homogeneous isotropic turbulence is the study of the intermediate phase of the decay (the decay of the homogeneous isotropic turbulent field created at the initial moment of time). At this stage nonlinearity effects are still important, and viscous dissipation already exerts some influence on the flow. At the same time, the details of the initial conditions cease to be important at the intermediate stage: universal laws of decay come into play.

The study of isotropic homogeneous turbulence is based on the joint-moment, or correlation-function, technique[3-5]: moment (or correlation) tensors of the flow characteristics are introduced into consideration, and from the Navier-Stokes equations of motion and the equation of continuity, equations which are valid for each realization of the random hydrodynamic field, is obtained an infinite chain of equations for the moments. The first equation in this chain is the Kármán-Howarth "equation of propagation of correlations"[5,4]:

$$\partial_t B_{LL} = 2\nu r^{-4}\partial_r r^4 \partial_r B_{LL} - 2r^{-4}\partial_r r^4 B_{NN,L}, \tag{1.1}$$

connecting the functions $B_{LL}(r, t) = \langle u_L(\mathbf{x}) u_L(\mathbf{x}+\mathbf{r})\rangle$ and $B_{NN,L}(r, t) = \langle u_N^2(\mathbf{x}) u_L(\mathbf{x}+\mathbf{r})\rangle$, which are respectively components of the second- and third-order moment tensors; on account of the isotropy, homogeneity, and incompressibility of the flow field, these components completely determine the tensors. Here ν is the kinematic viscosity of the fluid, $r = |\mathbf{r}|$, t is the time, u_L is the velocity component along the vector $\mathbf{r}$, u_N is the velocity component in the direction perpendicular to the vector $\mathbf{r}$, and the sign $\langle\ \rangle$ denotes mathematical expectation.

For the final stage of the decay, at which the contribution of the third-order moments is negligibly small, so that the relation (1.1) goes over into an equation for the function $B_{LL}(r, t)$, Millionshchikov[6] (see also [3,4]) has established the decay law for the second-order moments:

$$B_{LL}(r,t) = \frac{\Lambda_0}{48\sqrt{2\pi}[\nu(t-t_0)]^{5/2}} \exp\left[-\frac{r^2}{8\nu(t-t_0)}\right],$$
$$\Lambda_0 = \int_0^\infty r^4 B_{LL}(r,0)\,dr. \tag{1.2}$$

For this law of decay of the second-order moment distribution to be valid, Λ_0 should be finite, i.e., $0 < \Lambda_0 < \infty$, at the beginning of the final phase of the decay.

At the intermediate stage of the decay, the contribution of the third-order moments (the nonlinearity) is important, so that the relation (1.1) connects two functions, i.e., it is not a closed relation. Nevertheless, Loĭtsyanskiĭ[7] (see also [3,4]) established, on the basis of the fact that the term with the third-order moments in (1.1) has the form of a divergence, an integral law of conservation of the second-order moments that is applicable to the intermediate phase of the decay: if, as $r \to \infty$, $B_{LL}(r, t)$ decreases faster than r^{-5} and $B_{NN,L}(r, t)$ faster than r^{-4}, then the integral

$$\int_0^\infty r^4 B_{LL}(r,t)\,dr = \Lambda \tag{1.3}$$

is finite and is conserved in time. A connection between this conservation law and the law of conservation of angular momentum was shown by L. D. Landau and E. M. Lifshitz.[3]

To make (1.1) a closed relation, Kármán[5,4] put forward the hypothesis of self-preservation (self-similarity), according to which there exist functions of the time, $v(t)$ and $l(t)$, such that the moments B_{LL} and $B_{NN,L}$ can be represented in the form

$$B_{LL}(r,t) = v^2(t) f\left(\frac{r}{l(t)}\right), \quad B_{NN,L}(r,t) = v^3(t) h\left(\frac{r}{l(t)}\right). \tag{1.4}$$

The relation (1.4) is valid in the case of complete self-preservation of the velocity field, when the velocity field eventually varies in self-preserving fashion during the decay, i.e., only the spatial $l(t)$ and velocity $v(t)$ scales vary. Kármán used the hypothesis (1.4) to establish a family of possible decay laws valid at the initial stage of the decay of homogeneous isotropic turbulence, when the Reynolds number is large and the term with the viscosity in (1.1) can be neglected. Kolmogorov[8] derived from this family a decay law corresponding to the preservation of Loĭtsyanskiĭ's finite integral. Sedov[9] (see also Dryden's paper[10]) found all the solutions to the Kármán-Howarth equation (1.1) that are allowed by Kármán's complete self-preservation hypothesis (1.4) when the viscous terms in (1.1) are retained.

In [9], it was established, in particular, that the self-preserving decay law (1.4) is incompatible with the preservation of Loĭtsyanskiĭ's finite integral: if the Loĭtsyanskiĭ integral Λ is finite and the relations (1.4) are fulfilled, then it is not preserved in time.

Attempts have been made since Taylor's work to use the homogeneous isotropic turbulence as a model for turbulence behind a grid in wind tunnels. It must however be admitted that these attempts, which continued until recently (for a review of them, see [4]), are not quite satisfactory (the absence of homogeneity and isotropy, a large variance, etc.). Only in the recent experimental investigations performed by Ling and Huang[11] and Ling and Wan[12] was the intermediate phase of the decay of homogeneous isotropic turbulence adequately simulated, apparently for the first time, in a turbulent flow behind a grid in a low-velocity water channel, and thoroughly investigated.

It is shown in the present paper that the results of the experiments [11,12] are at variance with the preservation of Loĭtsyanskiĭ's finite invariant and with the complete self-preservation of the flow (Kármán's hypothesis (1.4)) at the intermediate stage of the decay of homogeneous isotropic turbulence. A more general concept of incomplete self-preservation of the flow field is formulated in which the velocity fields at different moments of time are not similar, but the correlation coefficients decay in a self-preserving fashion. All the solutions to the Kármán-Howarth equation that are allowed by the incomplete self-preservation hypothesis are constructed, and it is shown that the experiments [11,12] confirm the incomplete self-preservation hypothesis at the intermediate stage of the decay.

2. The Ling-Huang[11] and Ling-Wan[12] experiments were performed in a carefully checked—with respect to homogeneity and isotropy—turbulent flow behind a grid in a low-velocity water channel. A special analysis showed that the flow was at the intermediate stage of decay. The authors succeeded in representing with great accuracy the law of decay of the second-order moments in the self-preserving form

$$B_{LL}(r,t)=b(t)f\left(\frac{r}{l(t)}\right),\quad b(t)=\frac{A}{(t-t_0)^n},\quad l(t)=\sqrt{\nu(t-t_0)}. \qquad (2.1)$$

Here, as usual, $t=x/U$, where x is the distance from the grid and U is the mean flux velocity, and A and t_0 are constants. The Reynolds number MV/ν (M is the dimension of a grid mesh and V is the characteristic velocity) of the grid was varied within a wide range of values—from 470 to 34 000. It is significant that not only the constants A and t_0, but the exponent n in the decay law, turned out to be dependent upon the initial conditions: by varying the grids—setting different combinations of fixed and mobile grids and varying, for the mobile grids, the velocity of oscillation of the agitating bars—they succeeded in obtaining $n=2.00$, $n=1.73$, and $n=1.35$; the exponent n decreased with increasing grid Reynolds number.

The exponent n was in all the experiments invariably less than $5/2$; substitution of (2.1) into (1.3) shows that the Loĭtsyanskiĭ integral (1.3) cannot therefore be finite and constant: if it is finite, then it grows like $(t-t_0)^{5/2-n}$. This shows that the experiments [11,12] do not confirm the assumptions underlying the derivation of Loĭtsyanskiĭ's conservation law, since the conservation law can be derived quite rigorously after making these assumptions. The verification of the validity of the Kármán similarity law (1.4) at the intermediate stage of the decay can be carried out on the basis of Sedov's results [9], which include all the possible solutions to equation (1.1) that are allowed by this law. The substitution of the expressions (1.4) into the Kármán-Howarth equation (1.1) led Sedov to the relation ($\chi=r/l$, $b(t)=v^2(t)$):

$$\begin{aligned} &h'(\chi)+4h(\chi)/\chi-{}^1\!/_2\chi f'(\chi)b^{-3/2}(dl/dt)\\ &+{}^1\!/_2 f(\chi)lb^{-3/2}(db/dt)=\nu b^{-1/2}l^{-1}[f''(\chi)+4f'(\chi)/\chi]. \end{aligned} \qquad (2.2)$$

Sedov considered three possible cases, one of which, as it turned out, was a priori physically uninteresting and was therefore discarded. In the first of the remaining cases, one necessarily obtains that $l=\sqrt{\nu(t-t_0)}$, which corresponds to the experimental dependence (2.1), and $b=A(t-t_0)^{-1}$, which does not correspond to this dependence. The second possible case leads to a system of ordinary differential equations for the functions $b(t)$ and $l(t)$:

$$\frac{1}{\sqrt{b}}\frac{dl}{dt}=\frac{1}{2}\frac{\nu}{\sqrt{b}\,l}+p,\qquad \frac{l}{b^{3/2}}\frac{db}{dt}=-n\frac{\nu}{\sqrt{b}\,l}. \qquad (2.3)$$

The relations (2.3) for $p\neq 0$ are satisfied by the function $l\propto\sqrt{\nu(t-t_0)}$ only when $b=A(t-t_0)^{-1}$, which again is at variance with the experimental dependence (2.1). For $p=0$, the third-order moment tensor vanishes identically—this approximation is not applicable at the intermediate stage of the decay. Our attempt at the direct processing of the initial data of the experiment [11] led to a considerable scatter in the values of the constant p obtained for the various experimental points.

Since Sedov's results are quite rigorous in the framework of the assumptions (1.4), we can conclude that the data of the experiments [11,12] do not corroborate Kármán's self-preserving decay law (1.4) at the intermediate stage of the decay of homogeneous isotropic turbulence.

3. If the self-preserving decay law for the second-order moments in the experimentally obtained form (2.1) is valid, then it is not necessary to postulate the self-preservation of the third-order moments—it is established by simply substituting (2.1) into the Kármán-Howarth equation (1.1) and then integrating it. The self-preserving decay law is then obtained in a form different from the one predicted by the Kármán hypothesis. It is clear, however, that Kármán's complete self-preservation hypothesis is not the most general self-preserving decay law. It is therefore natural to postulate the following more general self-preserving decay law for the second- and third-order moments:

$$B_{LL}=b(t)f(\chi),\quad B_{NN,L}=g(t)h(\chi),\quad \chi=r/l(t), \qquad (3.1)$$

where $b(t)$, $l(t)$, and $g(t)$ are certain functions of time, and, developing an approach similar to Sedov's approach, obtain all the solutions to the Kármán-Howarth equation (1.1) that are allowed by this self-preserving decay law. The formal difference between the hypothesis (3.1) and the Kármán hypothesis consists only in the fact that it is not assumed that $g(t)=[b(t)]^{3/2}$; in essence, however, the difference is significant—in contrast to (1.4), the validity of (3.1) implies the absence of an isogonal variation of the velocity field in the process of decay, and therefore we shall call the hypothesis (3.1) the hypothesis of incomplete self-preservation. Substituting (3.1) into Eq. (1.1), we obtain

$$\begin{aligned} &\left(\frac{b}{g}\frac{dl}{dt}\right)\chi f'(\chi)-\left(\frac{l}{g}\frac{db}{dt}\right)f(\chi)+\left(2\nu\frac{b}{gl}\right)\left[f''(\chi)+\frac{4}{\chi}f'(\chi)\right]\\ &-2\chi^{-4}\frac{d}{d\chi}\chi^4 h(\chi)=0. \end{aligned} \qquad (3.2)$$

Differentiating (3.2) with respect to t at constant χ, we have

$$\frac{d}{dt}\left(\frac{b}{g}\frac{dl}{dt}\right)\chi f'(\chi)-\frac{d}{dt}\left(\frac{l}{g}\frac{db}{dt}\right)f(\chi)+\frac{d}{dt}\left(2\nu\frac{b}{gl}\right)\left[f''(\chi)+\frac{4}{\chi}f'(\chi)\right]=0. \quad (3.3)$$

As in Sedov's treatment, two physically meaningful cases may be met here. The first case corresponds to the linear independence of the quantities $\chi f'(\chi)$, $f(\chi)$, and $f''(\chi)+4f'(\chi)/\chi$, so that all the coefficients attached to these quantities in (3.3) are equal to zero. Hence, we have

$$\frac{b}{g}\frac{dl}{dt}=\frac{c}{2n},\quad \frac{l}{g}\frac{db}{dt}=-c,\quad \frac{\nu b}{gl}=(mc)^{-1} \quad (3.4)$$

(n, m, and c are constants). Integrating, we obtain

$$l^2=(mc^2/n)\nu(t-t_0),\quad b=A(t-t_0)^{-n}. \quad (3.5)$$

Comparing with (2.1), we see that this result agrees with experiment. Further, we obtain

$$g=\sqrt{mn\nu}\,A(t-t_0)^{-n-1/2}, \quad (3.6)$$

which does not agree with the Kármán hypothesis for $n\neq 1$, i.e., for all the values of n found. Substituting (3.4) into (3.2) and integrating, we find the expression for h in terms of f:

$$h=\left\{f'+\frac{mc^2\chi f}{4n}+\left[\frac{(2n-5)}{4\chi^4}\right]\int_0^\chi \chi^4 f\,d\chi\right\}\frac{1}{mc}. \quad (3.7)$$

It was shown in [11,12] that the experimental data for the correlation coefficient f can be represented to a very high degree of accuracy by the relation

$$f=[1+r_*^2/\alpha^2]^{-1},\quad r_*=r/\sqrt{\nu(t-t_0)} \quad (3.8)$$

in the investigated range of values of r_*: ($0\leq r_*<16$), the constant α being dependent upon n: $\alpha=3.16$ for $n=2$; $\alpha=3.40$ for $n=1.73$; $\alpha=3.85$ for $n=1.35$.

Let us compare these data with the consequences of the self-preserving decay law (3.1). Substituting the expansion of f in the vicinity of $\chi=0$ in the form $f=1-B\chi^2+\ldots$ into (3.7), we obtain

$$h=(-2B/mc+c/10)\chi+O(\chi^3). \quad (3.9)$$

As is well known [4], the expansion of $B_{NN,L}$ in the vicinity of $r=0$ should begin with the term $O(r^3)$, so that the coefficient in front of the first term in (3.9) vanishes, with the result that $B=mc^2/20$. Using (3.5), we obtain

$$f\approx 1-B\chi^2=1-nr^2/20\nu(t-t_0). \quad (3.10)$$

On the other hand, expanding the experimentally obtained expression (3.8) into a series for small r, we obtain the expression $f\approx 1-r^2/\alpha^2\nu(t-t_0)$, which coincides with (3.10) for $\alpha=\sqrt{20/n}$. This relation for α is in good agreement with the above-cited experimental data on the dependence $\alpha(n)$, which confirms the agreement with experiment of the incomplete self-preservation hypothesis (3.1). We note that the relation $\alpha=\sqrt{20/n}$ was obtained in [12] from other considerations. On account of the arbitrariness of the coefficient when choosing the scale, the constant c can be chosen arbitrarily; let us choose it so that $mc^2/n=1$. From (3.1), (3.5), and (3.7), we finally obtain the form of the self-preserving decay law which is allowed by (3.1) and agrees with experiment:

$$B_{LL}=A(t-t_0)^{-n}f(\chi),\quad B_{NN,L}=A\sqrt{\nu}(t-t_0)^{-n-1/2}h(\chi),$$
$$\chi=r/\sqrt{\nu(t-t_0)},\quad h=f'+(1/4)\chi f+[(2n-5)/4\chi^4]\int_0^\chi \chi^4 f(\chi)\,d\chi. \quad (3.11)$$

The second possible case is when the quantities $f(\chi)$, $\chi f'(\chi)$, and $f''(\chi)+4f'(\chi)/\chi$ are linearly dependent. This linear dependence can, without loss of generality, be represented in the form

$$f''(\chi)+\frac{4f'(\chi)}{\chi}+c\chi f'(\chi)+\frac{n}{2}f(\chi)=0 \quad (3.12)$$

(n and c are constants). Comparing (3.12) and (3.3), we obtain

$$\frac{d}{dt}\left(\frac{b}{g}\frac{dl}{dt}-2c\nu\frac{b}{gl}\right)=0,\quad \frac{d}{dt}\left(\frac{l}{g}\frac{db}{dt}+n\frac{\nu b}{gl}\right)=0 \quad (3.13)$$

whence

$$\frac{b}{g}\frac{dl}{dt}=2c\nu\frac{b}{gl}+p,\quad \frac{l}{g}\frac{db}{dt}=-\frac{n\nu b}{gl}+q, \quad (3.14)$$

where p and q are constants. Substituting (3.14) into (3.2) and integrating, we find

$$h=\frac{p}{2\chi^4}\int_0^\chi \chi^5 f'(\chi)\,d\chi-\frac{q}{2\chi^4}\int_0^\chi \chi^4 f(\chi)\,d\chi. \quad (3.15)$$

The condition that $h=O(\chi^3)$ for $\chi\to 0$ yields $q=0$. In this case, in contrast to the previous one, quite a definite differential equation is obtained for the function $f(\chi)$, but then the three functions l, b, and g turn out to be connected only by two relations. If we set, in accord with experiment, $l=\sqrt{\nu(t-t_0)}$, then the relations (3.14) (for $q=0$) yield a system of equations from which the functions b and g can be completely determined:

$$b=A(t-t_0)^{-n},\quad g=A\sqrt{\nu}(t-t_0)^{-n-1/2}[(1-4c)/p], \quad (3.16)$$

so that in this case, as in the preceding one, the decay of the correlation moments in time is found to be in agreement with experiment and at variance with Kármán's similarity law. By means of a simple transformation of the independent variable Eq. (3.12) can be reduced to the equation for the confluent hypergeometric function. Assuming natural boundary conditions, we obtain

$$f(\chi)=M(n/4c,\ 5/2,\ -n\chi^2/8), \quad (3.17)$$

where M(a, b, z) stands for the confluent hypergeometric function. Thus, a simple comparison with experiment should determine whether or not the constant c can be chosen such that the whole curve for the correlation coefficient can be described in a unified manner. Experimental data on the correlation coefficient have been obtained up to considerable values of χ (up to $\chi\sim 15$), when the function (3.17) should already be representable to a fairly high degree of accuracy by the first term of the asymptotic form for large values of the argument. Using the asymptotic representation of the confluent hypergeometric function for large values of the argument [13], we find $f\sim\chi^{-n/2c}$. But according to (3.8) $f\sim\chi^{-2}$ for $\chi\to\infty$, so that $n=4c$. For the case of the fixed grid, $n=2$, whence $c=1/2$ and we obtain $f(\chi)=M(1,\ 5/2,\ -\chi^2/4)$, which, as inspection shows, is at variance with experiment. Thus, there is in fact a natural first case when a definite equation for the function f cannot be obtained, and only a relation between the functions f and h can be found from the equation for the propagation of the correlations.

4. Ling and Wan[12] noted that the exponent n in the decay law for the second-order moments tends to unity as the grid Reynolds number is increased. If this is so, then one can expect Kármán's complete self-preservation hypothesis to be applicable at large grid Reynolds numbers. The Kistler-Vrebalovich experiments [14],

which were performed at very large Reynolds numbers, can serve as known confirmation of this. Apparently, the viscous terms in the equation for the propagation of the correlations become unimportant at such large Reynolds numbers: the applicability of the complete self-preservation hypothesis at the initial stage of the decay of turbulence is in complete agreement with Kármán's original idea [5].

The appearance of self-preserving asymptotic forms for correlation moments of different orders at the intermediate stage of the decay of homogeneous isotropic turbulence is a typical "self-preservation-of-the-second-kind" effect [15]. Indeed, the decisive dimensional parameters of the decay of turbulence behind a grid are: the mean velocity U, the dimension M of the grid mesh, the kinematic viscosity ν, the distance r of correlatable points, the time $t-t_0$, the amplitude V_p and frequency ω of the velocity of the tips of the agitated bars of the grid (in the case of the fixed grid the last two parameters drop out). We can, on the basis of dimensional considerations, write the moments (correlation functions) in the form

$$B_{LL}=MU(t-t_0)^{-1}f_1(\xi,\eta,\mathrm{Re},\lambda,\mu),\quad B_{NN,L}=[MU(t-t_0)^{-1}]^{3/2}f_2(\xi,\eta,\mathrm{Re},\lambda,\mu), \tag{4.1}$$

where the decisive dimensionless parameters can be represented in the following manner:

$$\xi=\frac{r}{\sqrt{\nu(t-t_0)}},\quad \eta=\frac{U(t-t_0)}{M},\quad \mathrm{Re}=\frac{UM}{\nu},\quad \lambda=\frac{V_p}{U},\quad \mu=\frac{M\omega}{U}. \tag{4.2}$$

The details of the initial conditions should cease to be felt at large distances from the grid, i.e., as $\eta\to\infty$. If, as $\eta\to\infty$, the functions f_1 and f_2 tend to finite limits, then we have the Kármán case of complete self-preservation. Comparison with the results of the experiments [11,12] indicated to us that this is, generally speaking, not the case, so that as $\eta\to\infty$, the functions f_1 and f_2 do not tend to finite limits, but have power asymptotic forms in η:

$$f_1\approx\eta^{\alpha}F_1(\xi,\mathrm{Re},\lambda,\mu),\quad f_2\approx\eta^{\beta}F_2(\xi,\mathrm{Re},\lambda,\mu), \tag{4.3}$$

where the exponents α and β depend, generally speaking, on the parameters λ, μ, and Re. Therefore, as $\eta\to\infty$, the parameter η remains important (including the case when the grid is fixed) no matter how large its magnitude is. However, owing to the power character of the asymptotic forms (4.3), as $\eta\to\infty$, the decay of the correlation functions becomes self-similar. In spite of the fact that the variation in time of the velocity field does not lead to the variation of the velocity and length scales, and, in this sense, is not self-similar, the correlation coefficients vary in self-similar fashion. In this sense, we speak of incomplete self-similarity of the problem as a whole.

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[1]G. I. Taylor, Proc. Roy. Soc. **A151**, 421, 444 (1935).
[2]A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).
[3]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics), Gostekhizdat, 1953 (Eng. Transl., Addison-Wesley Publ. Co., Reading, Mass., 1959), Ch. 3.
[4]A. S. Monin and A. M. Yaglom, Statisticheskaya gidromekhanika (Statistical Fluid Mechanics), Vol. 2, Nauka, 1967.
[5]Th. de Kármán and L. Howarth, Proc. Roy. Soc. **A164**, 192 (1938).
[6]M. D. Millionshchikov, Dokl. Akad. Nauk SSSR **22**, 236 (1939).
[7]L. G. Loĭtsyanskiĭ, Tr. TsAGI (Transactions of the N. E. Zhukovskiĭ Central Aero-Hydrodynamics Institute), No. 440, 1939.
[8]A. N. Kolmogorov, Dokl. Akad. Nauk. SSSR **31**, 538 (1941).
[9]L. I. Sedov, Metody podobiya i razmernosti v mekhanike (Similarity and Dimensional Methods in Mechanics), 1st ed., Gostekhizdat, 1944; 7th ed., Nauka, 1972 (Eng. Translation from the 4th Russ. ed., Academic Press, New York, 1959).
[10]H. L. Dryden, Quart. Appl. Math. **1**, 7 (1943).
[11]S. C. Ling and T. T. Huang, Phys. Fluids **13**, 2912 (1970).
[12]S. C. Ling and C. A. Wan, Phys. Fluids **15**, 1363 (1972).
[13]M. Abramowitz and I. Stegun (ed.), Handbook of Mathematical Functions, Nat. Bureau of Standards, 1965.
[14]A. L. Kistler and T. Vrebalovich, J. Fluid Mech. **26**, 37 (1966).
[15]G. I. Barenblatt and Ya. B. Zel'dovich, Ann. Rev. Fluid Mech. **4**, 285 (1972).

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