On the theory of two-dimensional stationary self-focusing of electromagnetic waves

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Effects due to the arbitrariness of wave polarization are taken into account exactly within the framework of the parabolic equation for media with cubic nonlinearity ($D_{NL} = \epsilon_{NL} |E|^2 E$). It is shown that as a result of intersection of homogeneous waveguide channels the radiation polarization in them may change. The thresholds for formation of waveguide channels with different structures following incidence of an intense wave on a screen with a gap are found.

INTRODUCTION

As is well-known,[1] the nonlinear change in the index of refraction of a medium, which is associated with the propagation of an intense electromagnetic wave in it, may cancel the diffraction divergence and lead to waveguide propagation of the radiation. The existence of wave packets that are stationary in time, and which propagate without any distortion of the shape of the wave envelope,[2] is possible for analogous reasons. The theory of such phenomena in two-dimensional geometry has been developed by Zakharov and Shabat[3] for waves having the same polarization everywhere. The basic physical results of their work are as follows. A definite fraction of the radiation entering the medium is captured in the self-focusing regime and is divided into a certain number of channels; the remaining radiation is scattered. Each channel has a definite direction and, generally speaking, a complicated structure. In the simplest cases the channels are uniform in the direction of its propagation (simple channels). The intensity of the radiation in a channel falls off exponentially with increasing distance from its axis. The channels may intersect; the picture of the intensity distribution of the radiation in the intersection region may be very complicated. However, all of the radiation subsequently collects in channels having the same directions and intensities as prior to the intersection.

The objective of the present work is to generalize the method used in [3] to the case of waves of arbitrary polarization (in the general case, the polarization will depend on the coordinates). It is found that the conclusions of article [3] expounded above carry over directly to the general case. However, the additional "degree of freedom"—namely, the polarization—leads to new effects. Let us enumerate the most important of these.

The waveguide channels which are formed have constant polarizations (the polarizations are generally different for different channels). Thus, a wave which had a varying polarization upon entry into the nonlinear medium is stratified in the medium into beams, where the polarization of the radiation in each beam is now constant. In this sense a nonlinear medium can play the role of a "polarization filter" (Sec. 1).

When two channels intersect, their polarizations change. This does not occur if the polarizations of the radiation in the channels are parallel or orthogonal. Intersection also leads to displacements of the channel axes. The magnitudes of these displacements intrinsically depend on the polarizations (Sec. 2). The directions and intensities of the channels remain the same as before (this result also pertains to the case of multi-channel intersections). Just as in [3], this result is related to the existence of an infinite set of conservation laws.

The problem of nonlinear diffraction by a slit (at zero angle) is treated as an illustration of the feasibility of the technique. The thresholds for the formation of waveguide channels of different structure are calculated (Sec. 3).

All of the above statements concerning two-dimensional stationary self-focusing directly pertain to the problem of one-dimensional self-modulation of electromagnetic waves of arbitrary polarization. In the latter problem stationary wave packets—solitons—play the role of simple waveguide channels. During soliton collisions their velocities and amplitudes do not change, but their polarizations do. The diffraction of a plane wave by a slit has its own analog in the problem of the decay of a rectangular wave train.

1. FORMULATION OF THE PROBLEM. SIMPLE CHANNELS

Let us consider a quasi-monochromatic electromagnetic wave. We represent its field in the form $\tilde{\mathbf{E}}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$, where $\mathbf{E}(\mathbf{r}, t)$ is a slowly varying function of $\mathbf{r}$ and t (the complex envelope) and $\omega = \omega(\mathbf{k})$ describes the dispersion law of the waves. In the case of two-dimensional stationary self-focusing, $\mathbf{E}$ doesn't depend on the time and satisfies the equation (see [4–6])

$$2ik\frac{\partial \mathbf{E}}{\partial z} + \frac{\partial^2 \mathbf{E}}{\partial x^2} = -k^2\frac{\delta n_{\text{nl}}}{n_0}|\mathbf{E}|^2\mathbf{E}. \qquad (1)$$

It is assumed that the index of refraction n behaves like $n = n_0 + \delta n_{nl}|\mathbf{E}|^2$ ("scalar nonlinearity"). The longitudinal self-modulation of an electromagnetic wave is described by a similar equation (see [6]):

$$i\left(\frac{\partial \mathbf{E}}{\partial t} + \omega_k'\frac{\partial \mathbf{E}}{\partial x}\right) + \frac{\omega_k''}{2}\frac{\partial^2 \mathbf{E}}{\partial x^2} = T|\mathbf{E}|^2\mathbf{E}. \qquad (2)$$

Here $T|\mathbf{E}|^2$ represents a nonlinear correction to the frequency of the wave.

Equations (1) and (2) lead to the standard dimensionless form:

$$i\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial^2 \mathbf{E}}{\partial x^2} + \varkappa|\mathbf{E}|^2\mathbf{E} = 0. \qquad (3)$$

We shall attach the meaning of time to the variable t. Self-focusing occurs when $\delta n_{nl} > 0$, that is $\kappa > 0$. This case is therefore treated below (for Eq. (2) this means that $\omega_k'' T < 0$; as is well known, a self-modulation instability of the monochromatic wave arises in this case).

Let us represent E as a sum of right- and left-hand polarized waves: $\mathbf{E} = E_1\mathbf{c}_R + E_2\mathbf{c}_L$, where $\mathbf{c}_R$ and $\mathbf{c}_L$ are the complex unit vectors corresponding to right-hand and left-hand polarizations. Using the orthogonality of $\mathbf{c}_R$ and $\mathbf{c}_L$, we obtain the following system of equations for E_1 and E_2:

$$\begin{gathered} i\frac{\partial E_1}{\partial t}+\frac{\partial^2 E_1}{\partial x^2}+\varkappa(|E_1|^2+|E_2|^2)E_1=0,\\ i\frac{\partial E_2}{\partial t}+\frac{\partial^2 E_2}{\partial x^2}+\varkappa(|E_1|^2+|E_2|^2)E_2=0. \end{gathered} \tag{4}$$

The possibility of a complete investigation of the system (4) is related to the fact that it can be represented in the form

$$\partial\hat{L}/\partial t = i[\hat{L}, \hat{A}], \tag{5}$$

where $\hat{L}$ and $\hat{A}$ are the following differential operators ($\kappa = 2/(1-p^2)$):

$$\begin{gathered} \hat{L} = i\begin{pmatrix} 1-p & 0 & 0 \\ 0 & 1+p & 0 \\ 0 & 0 & 1+p \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & E_1 & E_2 \\ E_1^* & 0 & 0 \\ E_2^* & 0 & 0 \end{pmatrix},\\ \hat{A} = -p\frac{\partial^2}{\partial x^2} + \begin{pmatrix} -(|E_1|^2+|E_2|^2)/(1-p) & -iE_{1x} & -iE_{2x} \\ iE_{1x}^* & |E_1|^2/(1+p) & E_2E_1^*/(1+p) \\ iE_{2x}^* & E_2^*E_1/(1+p) & |E_2|^2/(1+p) \end{pmatrix}. \end{gathered} \tag{6}$$

The relation (5) and the fact that the operator $\hat{A}$ is hermitian guarantee that the spectrum of the operator $\hat{L}$ will not change in time; in this connection the eigenfunctions of $\hat{L}$ satisfy the equation

$$i\partial\psi/\partial t = \hat{A}\psi + f(\hat{L})\psi, \tag{7}$$

where the function $f(\hat{L})$ of the operator $\hat{L}$ may be chosen from considerations of convenience.

The approach used here has acquired the name "the inverse scattering problem method". It was first applied by Gardner, Green, Kruskal, and Miura[7] to the well-known Korteweg-de Vries equation, describing nonlinear waves in weakly dispersive media. As Lax has shown,[8] their method is actually applicable to all equations which can be represented in the form (5) (in the case when the eigenfunctions of the operator $\hat{L}$ can be reconstructed from its scattering matrix). Zakharov and Shabat[3] subsequently applied the inverse scattering problem method to the equation $i\partial u/\partial t + u_{xx} + \kappa|u|^2 u + 0$, which is obtained from the system (4) by, for example, setting $E_2 = 0$.

Let us consider the problem about the eigenvalues of the operator $\hat{L}$: $\hat{L}\psi = \lambda\psi$ and we make the following substitution:

$$\psi_1 = (1+p)^{1/2}\exp\left(-i\frac{\lambda}{1-p^2}x\right)f_1, \quad \psi_{2,3} = (1-p)^{1/2}\exp\left(-i\frac{\lambda}{1-p^2}x\right)f_{2,3}.$$

The vector $\mathbf{f}(f_1, f_2, f_3)$ satisfies the equations

$$\begin{gathered} \partial f_1/\partial x + i\xi f_1 = q_1 f_2 + q_2 f_3, \quad \partial f_2/\partial x - i\xi f_2 = -q_1^* f_1,\\ \partial f_3/\partial x - i\xi f_3 = -q_2^* f_1, \end{gathered} \tag{8}$$

where

$$\xi = \frac{\lambda p}{1-p^2}, \qquad q_{1,2}(x) = \frac{i}{(1-p^2)^{1/2}}E_{1,2}.$$

If $q_{1,2}(x)$ fall off sufficiently rapidly as $x \to \pm\infty$, then each solution (8) is uniquely determined by one of its asymptotic forms as $x \to \pm\infty$. Let us consider two sets of solutions of (8), $\varphi_i(x, \xi)$ and $\psi_i(x, \xi)$ ($i = 1, 2, 3$), with the following asymptotic behavior

$$\begin{gathered} (\varphi_i)_k = \delta_{ik}\exp(-iI_k\xi x), \quad x \to -\infty,\\ (\psi_i)_k = \delta_{ik}\exp(-iI_k\xi x), \quad x \to +\infty. \end{gathered} \tag{9}$$

Here $I_1 = 1$, $I_2 = I_3 = -1$. We shall call the functions φ_i and ψ_i Jost functions. Since the vectors ψ_i form a complete set of solutions of the system (8), then

$$\varphi_i(x,\xi) = \sum_{k=1}^{3}\alpha_{ik}(\xi)\psi_k(x,\xi). \tag{10}$$

The relations (10) determine the scattering matrix $\hat{\alpha}$ of the system (8) in the basis of the Jost functions.

One can show that the functions $\varphi_1(x, \xi)$, $\psi_2(x, \xi)$, and $\psi_3(x, \xi)$ can be analytically continued into the upper half-plane of the complex variable ξ for every x; the functions $\varphi_2(x, \xi)$, $\varphi_3(x, \xi)$, and $\psi_1(x, \xi)$ are analytic in the lower half-plane of ξ. As is shown in the Appendix, analyticity of the Jost functions implies analyticity of $\alpha_{11}(\xi)$ in the region $\mathrm{Im}\,\xi \geq 0$ and analyticity of α_{22}, α_{23}, α_{32}, and α_{33} in the lower half-plane of ξ. The zeros of the function $\alpha_{11}(\xi)$ in the upper half-plane correspond to the eigenvalues of the problem (8) with zero conditions as $x \to \infty$. In this connection

$$\varphi_1(x,\xi) = c_{12}\psi_2(x,\xi) + c_{13}\psi_3(x,\xi) \quad (\alpha_{11}(\xi) = 0). \tag{11}$$

All of the Jost functions, and together with them the "potential" $q_{1,2}(x)$ as well, are reconstructed from the scattering matrix $\alpha_{ik}(\xi)$ and the values of c_{12}, c_{13} at each zero of $\alpha_{11}(\xi)$. Let us determine the time dependence of these quantities. In order to do this we choose the $f(\hat{L})$ in Eq. (7) in such a way that the definition (9) of the Jost functions is conserved in time. Having done this we obtain

$$\begin{gathered} \alpha_{ik}(\xi, t) = \alpha_{ik}(\xi, 0)\exp[-2i\xi^2(I_k - I_i)t],\\ c_{1k}(t) = c_{1k}(0)\exp(4i\zeta^2 t) \quad (\alpha_{11}(\zeta) = 0). \end{gathered} \tag{12}$$

Thus, the Cauchy problem for the system of nonlinear equations (4) reduces to the systematic investigation of linear systems. In the first place it is necessary to find the scattering matrix for the system (8), having substituted the initial condition $q(x, 0)$ in place of $q(x)$; the values of $\alpha_{ik}(\xi)$ at subsequent moments of time are obtained from Eqs. (12). The second nontrivial step consists of the solution of the inverse scattering problem (Eqs. (A.5)–(A.9)), which enables us to determine the "potential" $q_{1,2}(x, t)$ (given by Eq. (A.10)).

Particular solutions of the system (4), for which $\alpha_{11}(\xi)$ has only a single zero ζ and the remaining elements in the first row of the scattering matrix are equal to zero, play a fundamental role in the theory. It is precisely these solutions which describe the simple waveguide channels in the theory of self-focusing and the solitary envelope waves in the problem of self-modulation; in accordance with what was said above, their form must be completely determined by three complex constants, the position ζ of the zero and the values of c_{12} and c_{13} (see Eq. (11)). One can easily find the explicit expression for these solutions since the system of equations (occurring in the inverse scattering problem) for the matrix of the indicated form reduces to a system of linear algebraic equations. Simple calculations give

$$\mathbf{q}^*(x,t) = -2\eta\frac{\mathbf{c}\exp[2i\xi x + 4i(\xi^2-\eta^2)t]}{\mathrm{ch}[2\eta(x-x_0)+8\xi\eta t]}. \tag{13}$$

Here

$$\begin{gathered} \xi + i\eta = \zeta, \quad x_0 = \frac{1}{4\eta}\ln|\mathbf{S}|^2,\\ \mathbf{c} = \mathbf{S}/|\mathbf{S}|, \quad S_1 = c_{12}(0), \quad S_2 = c_{13}(0). \end{gathered} \tag{14}$$

We shall call the solutions of Eq. (3) having this form "solitons." A soliton is characterized by two arbitrary constants ξ and η, which determine its velocity $v = -4\xi$ and amplitude, respectively; x_0 denotes the coordinate of the soliton at $t=0$, and the unit vector $\mathbf{c}$ determines its polarization (and phase). In the theory of self-focusing, the soliton has the meaning of a waveguide channel that makes an angle $\theta=-\tan^{-1}(4\xi)$ to the direction of the wavevector $\mathbf{k}$ of the initial wave; the integrated intensity of the radiation in the channel is given by $I=\int|q|^2dx=4\eta$.

The special role of the solitons lies in the fact that they determine, in a certain sense, the asymptotic state ($t\to\pm\infty$) of an arbitrary solution of Eq. (3). Namely, one can show that for any initial condition the solution of Eq. (3) tends to zero on the straight lines $x-vt=\text{const}$ as $|t|\to\infty$ if the quantity $-v/4$ does not coincide with the real part of some zero of $\alpha_{11}(\xi)$ in the upper half-plane of ξ.[9] In particular this means that a definite part of the radiation entering the medium is "laminated" in it into a certain number of beams. One can show, in the same way as in [3] that the solitons are stable in the sense that a small change of the initial conditions causes a small change of the soliton parameters.

In the next section we shall need the explicit form of the scattering matrix $\hat{\alpha}$ for the potential given by Eq. (13) (the single-soliton scattering matrix). One can easily find the expression for α_{ik} from Eqs. (A.5)–(A.9) and from the analytic properties of $\hat{\alpha}$:

$$\alpha_{11}(\xi,\zeta)=\frac{\xi-\zeta}{\xi-\zeta^*},$$
$$\alpha_{ik}(\xi,\zeta,S)=\delta_{ik}+\frac{\zeta-\zeta^*}{\xi-\zeta}c^*_{i-1}c_{k-1}. \tag{15}$$

Here c_i denote the components of the unit polarization vector (see (14)). The remaining elements of the scattering matrix vanish.

2. INTERSECTION OF CHANNELS (INTERACTION OF SOLITONS

If the directions of two waveguide channels do not coincide, then sooner or later they will approach each other enough so that their interaction becomes important. The problem arises of determining the result of such a "collision" of the channels. In this connection it is obvious beforehand that after the intersection channels appear with the same directions and intensities as before the intersection—since these characteristics of the channels are determined by the eigenvalues of the operator $\hat{L}$ which do not change in time. Since the integrated intensities of radiation in the channels are conserved, one can confirm that after an intersection the radiation again collects in channels. Thus, only the polarizations of the channels and the positions of their axes can vary as a result of the intersection. Everything said above obviously also pertains to the intersection of an arbitrary number of channels.

Leter it will be convenient for us to use the "soliton language". The problem of an N-soliton collision can be investigated with the aid of a scattering matrix of special form. Namely, let $\alpha_{12}(\xi)=\alpha_{13}(\xi)=0$ for all real ξ, and let $\alpha_{11}(\xi)$ have N simple zeroes $\zeta_1, \zeta_2, \ldots, \zeta_N$ in the upper half of the complex ξ-plane. In this connection the quantity $\alpha_{11}(\xi)$ can be represented in the form

$$\alpha_{11}(\xi)=\prod_{i=1}\frac{\xi-\zeta_i}{\xi-\zeta_i^*}. \tag{16}$$

For a matrix $\hat{\alpha}$ of the indicated form, the system of equations for the inverse scattering problem (Eqs. (A.5)–(A.9)) reduces to a system of linear algebraic equations. By solving this system one can, in principle, find an explicit expression for the N-soliton solution. However, the expressions which arise are complicated and difficult to analyze; for this reason they are not useful in practice. Only the asymptotic states, which appear as $t\to\pm\infty$, can be investigated relatively simply. However, this is quite adequate for solving the problem of the scattering of solitons by one another. If there are no equal values among the $\operatorname{Re}\zeta_i$,[1)] then (as one can easily ascertain from the system of Eqs. (A.5)–(A.9)) for $t\to\pm\infty$ the solution of Eq. (3) can be represented in the form $\mathbf{E}=\Sigma\mathbf{E}_i$, where $\mathbf{E}_i$ has the form (13), i.e., the field breaks up into individual solitons.

Let us fix the values of the soliton parameters for $t\to-\infty$, i.e., for each soliton ζ_i we assign a vector $\mathbf{S}_i^-$ (see (14)) which completely determines $\mathbf{E}_i$. For $t\to+\infty$ we denote the corresponding vectors by $\mathbf{S}_i^+$. In order to be definite, let us assume $\operatorname{Re}\zeta_1>\operatorname{Re}\zeta_2>\ldots>\operatorname{Re}\zeta_N$. Then as $t\to-\infty$ the solitons are distributed along the x axis in the order of decreasing values of their number; the order of the soliton sequence is reversed as $t\to+\infty$.

In order to determine the result of a collision between solitons, i.e., to calculate $\mathbf{S}_i^+$ from given $\mathbf{S}_i^-$, we proceed in the same way as in [10], that is, we trace the passage of the Jost functions through the asymptotic states $\mathbf{E}$. We denote the soliton coordinates at the instant of time t by $x_i(t)$ ($|t|$ is assumed to be large enough so that one can talk about individual solitons). If $t\to-\infty$, then $x_N\ll x_{N-1}\ll\ldots\ll x_1$. The function $\boldsymbol{\varphi}_1(x,\zeta_i)$ has the form $\boldsymbol{\varphi}_1(x,\zeta_i)=\mathbf{e}_1\exp(-i\zeta_i x)$ in the region $x\ll x_N$ (here $(\mathbf{e}_i)_k=\delta_{ik}$). After passing through the soliton with number N we obtain the following result for $\boldsymbol{\varphi}_1$:

$$\boldsymbol{\varphi}_1(x,\zeta_i)=\alpha_{11}(\zeta_i,\zeta_N)\exp(-i\zeta_i x)\mathbf{e}_1.$$

Repeating the argument, we find that in the region $x_{i+1}\ll x\ll x_i$

$$\boldsymbol{\varphi}_1(x,\zeta_i)=\prod_{n>i}\alpha_{11}(\zeta_i,\zeta_n)\exp(-i\zeta_i x)\mathbf{e}_1.$$

$\mathbf{e}_1\exp(-i\zeta_i x)$ is the asymptotic form of the eigenfunction for the i-th soliton; therefore for $x_i\ll x\ll{}_{i-1}$ we have

$$\boldsymbol{\varphi}_1(x,\zeta_i)=\prod_{n>i}\alpha_{11}(\zeta_i,\zeta_n)[(\mathbf{S}_i^-)_1\mathbf{e}_2+(\mathbf{S}_i^-)_2\mathbf{e}_3]\exp(i\zeta_i x).$$

Upon passing through the $(i-1)$-st soliton, $\mathbf{e}_2\exp(i\zeta_i x)$ is changed into

$$[\alpha_{22}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)\mathbf{e}_2+\alpha_{23}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)\mathbf{e}_3]\exp(i\zeta_i x);$$

furthermore $\mathbf{e}_3\exp(i\zeta_i x)$ changes into

$$[\alpha_{32}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i1}^-)\mathbf{e}_2+\alpha_{33}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)\mathbf{e}_3]\exp(i\zeta_i x).$$

Thus, for $x_{i-1}\ll x\ll x_{i-2}$ we obtain

$$\boldsymbol{\varphi}_1(x,\zeta_i)=\prod_{n>i}\alpha_{11}(\zeta_i,\zeta_n)\exp(i\zeta_i x)\{[\alpha_{22}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)(\mathbf{S}_i^-)_1$$
$$+\alpha_{32}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)(\mathbf{S}_i^-)_2]\mathbf{e}_2+[\alpha_{23}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)(\mathbf{S}_i^-)_1$$
$$+\alpha_{33}(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)(\mathbf{S}_i^-)_2]\mathbf{e}_3\}.$$

Continuing this process as long as we do not go out into the region $x\gg x_1$, and comparing the obtained result with the relation

$$\boldsymbol{\varphi}_1=\sum_{\gamma=2,3}c_{1\gamma}^{(i)}\mathbf{e}_\gamma\exp(i\zeta_i x),$$

we find that

$$\mathbf{c}^{(i)}=\prod_{n>i}\alpha_{11}(\zeta_i,\zeta_n)\hat{\alpha}^T(\zeta_i,\zeta_1,\mathbf{S}_1^-)\hat{\alpha}^T(\zeta_i,\zeta_2,\mathbf{S}_2^-)\ldots\hat{\alpha}^T(\zeta_i,\zeta_{i-1},\mathbf{S}_{i-1}^-)\mathbf{S}_i^-,$$

where the vector with components $c_{12}^{(i)}(0)$, $c_{13}^{(i)}(0)$ is denoted by $c^{(i)}$. The superscript T indicates the operation of transposition.

Proceeding in a similar fashion for $t \to +\infty$, we obtain

$$c^{(i)} = \prod_{n<i} \alpha_{11}(\zeta_i, \zeta_n) \hat{\alpha}^T(\zeta_i, \zeta_N, S_N^+) \ldots \hat{\alpha}^T(\zeta_i, \zeta_{i+1}, S_{i+1}^+) S_i^+,$$

whence it follows that

$$S_N^+ = \prod_{n<N} \alpha_{11}^{-1}(\zeta_N, \zeta_n) \hat{\alpha}^T(\zeta_N, \zeta_1, S_1^-) \ldots \hat{\alpha}^T(\zeta_N, \zeta_{N-1}, S_{N-1}^-) S_N^-,$$

$$S_i^+ = \prod_{k>i} \alpha_{11}(\zeta_i, \zeta_k) \prod_{n<i} \alpha_{11}^{-1}(\zeta_i, \zeta_n) \hat{\alpha}^*(\zeta_i^*, \zeta_{i+1}, S_{i+1}^+) \ldots \tag{17}$$

$$\ldots \hat{\alpha}^*(\zeta_i^*, \zeta_N, S_N^+) \hat{\alpha}^T(\zeta_i, \zeta_1, S_1^-) \ldots \hat{\alpha}^T(\zeta_i, \zeta_{i-1}, S_{i-1}^-) S_i^-.$$

Equations (17) give the solution of the soliton collision problem.

Let us consider the collision of two solitons ζ_1, S_1^- and ζ_2, S_2^- ($\mathrm{Re}\,\zeta_1 > \mathrm{Re}\,\zeta_2$) in more detail. Solitons with the vectors S_1^+ and S_2^+ appear as a result of the collision:

$$S_2^+ = \alpha_{11}^{-1}(\zeta_2, \zeta_1) \hat{\alpha}^T(\zeta_2, \zeta_1, S_1^-) S_2^-,$$
$$S_1^+ = \alpha_{11}(\zeta_1, \zeta_2) \hat{\alpha}^*(\zeta_1^*, \zeta_2, S_2^+) S_1^-. \tag{18}$$

These formulas are clearly not symmetric with respect to interchange of the subscripts 1 and 2. However, such a notation expresses the invariance of the system (4) under the substitutions $t \to -t$, $E \to E^*$ (here a "fast" soliton becomes a "slow" soliton and vice versa).

As follows from Eqs. (18), during the scattering of the solitons there is a change of their polarizations and a displacement of the "coordinates of the solitons for $t = 0$," x_{0i}. According to (14) the magnitudes of the latter displacements are given by

$$\Delta x_{0i} = x_{0i}^+ - x_{0i}^- = (1/4\eta_i) \ln (|S_i^+|^2 / |S_i^-|^2).$$

Let us denote the ratio $|S_2^+|/|S_2^-|$ by χ. One can easily determine the quantity χ from (18):

$$\chi = \left| \frac{\zeta_1 - \zeta_2^*}{\zeta_1 - \zeta_2} \right| \left\{ 1 + \left| \frac{(\zeta_1 - \zeta_1^*)(\zeta_2^* - \zeta_2)}{|\zeta_1 - \zeta_2|^2} \right|^2 |(c_1^* c_2)|^2 \right\}^{1/2}. \tag{19}$$

Formula (19) determines the displacement Δx_0 for the fast soliton: $\Delta x_{02} = (4\eta_2)^{-1} \ln \chi$. Its value is always positive, $\chi > 1$. By direct calculation we find $\Delta x_{01} = -(4\eta_1)^{-1} \ln \chi$. Thus, in the collision the fast soliton receives an additional forward displacement by the amount Δx_{02}, while the slow soliton is displaced backwards. In this connection $\eta_1 \Delta x_{01} + \eta_2 \Delta x_{02} = 0$, which is a consequence of the fact that $\int |E|^2 dx$ is independent of the time.

The values of the displacements significantly depend on the polarizations of the solitons. Solitons with identical polarizations interact most strongly. For identical polarizations Δx_{02} is maximal and is given by

$$(\Delta x_{02})_{max} = \frac{1}{2\eta_2} \ln \left| \frac{\zeta_1 - \zeta_2^*}{\zeta_1 - \zeta_2} \right|.$$

For fixed amplitudes and velocities of the solitons, the minimal value of Δx_{02} is achieved in the case of orthogonal polarizations: $(\Delta x_{02})_{min} = (1/2)(\Delta x_{02})_{max}$.

The unit polarization vectors after the collision can be expressed in terms of the parameters characterizing the colliding solitons in the following way:

$$c_2' = \frac{1}{\chi} \frac{\zeta_1^* - \zeta_2}{\zeta_1 - \zeta_2} \left(c_2 + \frac{\zeta_1 - \zeta_1^*}{\zeta_2 - \zeta_1} (c_1^* c_2) c_1 \right),$$
$$c_1' = \frac{1}{\chi} \frac{\zeta_1^* - \zeta_2}{\zeta_1^* - \zeta_2^*} \left(c_1 + \frac{\zeta_2 - \zeta_2^*}{\zeta_2^* - \zeta_1^*} (c_2^* c_1) c_2 \right). \tag{20}$$

It is clear that the soliton polarizations do not change only in the case when their initial polarizations are parallel or orthogonal. In particular, the polarizations are preserved if the radiation in the solitons was circularly polarized.

Comparison of relations (17) and (18) indicates that an N-soliton collision does not, in general, reduce to a pair collision. This is clear, for example, from the fact that the expression for S_k^+ contains S_j^+ with $j > k$, which depend on the initial parameters of all the remaining solitons. In the case when all of the solitons have identical polarizations, that is, all the S_i^- are parallel, the effect of the matrix α on S_i reduces to multiplication of the latter by a number, and we have the picture of soliton scattering described in [3].

3. DIFFRACTION BY A SLIT

In this section we shall find the thresholds for the formation of waveguide channels of various structure and the intensities of the radiation captured in these channels associated with the diffraction of the wave by a slit. The appearance of self-channelized propagation of the radiation corresponds to the appearance of a zero for the scattering matrix element $\alpha_{11}(\xi)$ in the upper half-plane (see Sec. 1). In the problem under consideration, $\alpha_{11}(\xi)$ can be found directly from the solution of the system of equations (8).

Let the radiation be incident from the left on the screen, which is located in the plane $z = 0$, and let the slit be described by the band $0 < x < l$. We shall regard the field E in the plane of the screen as given:

$$E(x) = \begin{cases} 0, & x < 0, \quad x > l, \\ E_0, & 0 < x < l. \end{cases} \tag{21}$$

If $\partial E_0/\partial x = 0$, one can assume $E_2 = 0$ without any loss of generality. In this connection the system (8) is simplified:

$$\frac{\partial f_1}{\partial x} + i\xi f_1 = q(x) f_2, \quad \frac{\partial f_2}{\partial x} - i\xi f_2 = -q^*(x) f_1. \tag{22}$$

The problem of scattering by the "rectangular well" (21) for the system (22) is easily solved. We obtain the following result for $\alpha_{11}(\xi)$:

$$\alpha_{11}(\xi) = \frac{1}{2k} [(k+\xi) e^{-ikl} + (k-\xi) e^{ikl}] e^{i\xi l},$$
$$k = k(\xi) = \mathrm{sign}\,\xi (\xi^2 + |q|^2)^{1/2}. \tag{23}$$

In spite of the fact that $k(\xi)$ has a cut in the complex plane, expression (23) obviously possesses the necessary analytic properties.

It is easy to verify that all of the zeroes of $\alpha_{11}(\xi)$ lie on the imaginary axis (this means that the waveguide channel which arises has the direction of the incident wave). Let us set $\xi = i\eta|q|$. Then the equation $\alpha_{11}(\xi) = 0$ can be rewritten in the form

$$\frac{(1-\eta^2)^{1/2} + i\eta}{(1-\eta^2)^{1/2} - i\eta} = -\exp\{-2il|q|(1-\eta^2)^{1/2}\}$$

or

$$\arcsin \eta = l|q|(1-\eta^2)^{1/2} - \pi/2. \tag{24}$$

Here we must consider the values of the $\arcsin \eta$ only in the intervals $[2\pi n, 2\pi n + \pi/2]$, $n = 0, 1, \ldots$ The first positive root of Eq. (24) appears when $l|q| > \pi/2$. The value $|q| = \pi/2l$ corresponds to the threshold for the creation of a simple channel. In dimensional variables the corresponding threshold value of the integrated intensity $I = |E|^2 l$ is given by

$$I_{cr}=\frac{\pi}{2l}\frac{n_0}{k^2\delta n_{nl}}. \tag{25}$$

For small excesses above threshold, the integrated intensity of the radiation captured in the self-channelization regime is $2(I-I_{cr})$. Here the width of the channel behaves like $1/(I-I_{cr})$. With a further increase in the power of the incident wave, the radiation intensity in the channel increases and the channel contracts. When the integrated intensity reaches the value $25 I_{cr}$, the second root of Eq. (24) appears, and the channel ceases to be homogeneous (this corresponds to the appearance of a bound soliton state; a complete description of this bound state is given in [3]). The structure of the channel begins to periodically vary along its axis. A further increase in the power of the incident wave leads to an increase in the number of "harmonics" in the channel. If $(4n+1)^2 I_{cr} < I < (4(n+1)+1)^2$, then all of the channel's parameters are described by certain conditionally-periodic functions,[3] which are characterized by n incommensurable frequencies. Thus, a "turbulent channel" appears for very large excesses above threshold; the characteristics of the radiation then vary in a rather random fashion along its axis.

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APPENDIX

THE INVERSE SCATTERING PROBLEM FOR THE OPERATOR $\hat{L}$

It is convenient to represent the system of equations (9) in the form

$$\partial \mathbf{f}/\partial x + i\xi \hat{I}\mathbf{f} = \hat{Q}\mathbf{f}, \tag{A.1}$$

where one can easily determine the matrices $\hat{I}$ and $\hat{Q}$ by comparing Eq. (A.1) with Eq. (9). In this connection it is obvious that $I^+=I$, $Q^+=-Q$. For any two solutions $\mathbf{f}_1$ $\mathbf{f}_2$ of Eq. (A.1) with $\xi=\xi_1, \xi_2$, respectively, we have

$$\frac{\partial}{\partial x}(\mathbf{f}_2^+\mathbf{f}_1) + i(\xi_2^* - \xi_1)\mathbf{f}_2^+ I \mathbf{f}_1 = 0.$$

Thus, for real values of ξ we have

$$\frac{\partial}{\partial x}(\mathbf{f}_2^+(x,\xi),\ \mathbf{f}_1(x,\xi)) = 0.$$

By utilizing expressions (10) and (11) we find that

$$\alpha_{ik}(\xi) = \psi_k^+(x,\xi)\varphi_i(x,\xi), \tag{A.2}$$

$$\sum_{k=L}^{3} \alpha_{ik}^*(\xi)\alpha_{lk}(\xi) = \delta_{il}. \tag{A.3}$$

By considering the derivatives of the Wronskians of the sets of solutions φ_i and ψ_i, one can verify that $\det\hat{\alpha}=1$. Thus, the scattering matrix $\hat{\alpha}$ is unitary and unimodular. Using the analytic properties of the Jost functions enumerated in section 1, relationship (A.2) permits us to analytically continue $\alpha_{11}(\xi)$ into the upper half of the complex ξ-plane, and to continue α_{22}, α_{23}, α_{32}, and α_{33} into the lower half. We further note that if the "potential" $\hat{Q}$ is finite, i.e., if $q_{1,2}(x)$ vanishes outside a certain finite interval, then all of the Jost functions and, together with them the scattering matrix as well, are analytic in the whole complex ξ-plane.

Now let us consider the problem of reconstructing the potential $\hat{Q}$ from the set of quantities $\alpha_{ik}(\xi)$ and c_{1k}, assuming for simplicity that $\alpha_{11}(\xi)$ has only simple zeroes at certain points $\zeta_1, \zeta_2, \ldots, \zeta_N$ in the upper half-plane.

We denote the elements of the matrix which is the inverse of $\begin{pmatrix}\alpha_{22} & \alpha_{23}\\ \alpha_{32} & \alpha_{33}\end{pmatrix}$ by $\Omega_{\gamma\delta}$ (below Greek indices take the values 2 and 3; summation is to be carried out over repeated indices). The relations of unitarity and analyticity for the matrix $\hat{\alpha}$ give $\det\{\alpha_{\beta\gamma}(\xi)\} = \alpha_{11}^*(\xi^*)$; thus, the matrix Ω is analytic in the lower half of the complex ξ-plane with the exception of the points $\zeta_1^*, \zeta_2^*, \ldots, \zeta_N^*$, where it has simple poles.

Now let us consider the following piecewise-analytic functions:

$$\Phi_1(x,\xi) = \begin{cases} \dfrac{\varphi_1(x,\xi)e^{i\xi x}}{\alpha_{11}(\xi)}, & \operatorname{Im}\xi > 0 \\ \psi_1(x,\xi)e^{i\xi x}, & \operatorname{Im}\xi < 0 \end{cases}$$

$$\Phi_\gamma(x,\xi) = \begin{cases} \psi_\gamma(x,\xi)e^{-i\xi x}, & \operatorname{Im}\xi > 0 \\ \Omega_{\gamma\delta}\varphi_\delta(x,\xi)e^{-i\xi x}, & \operatorname{Im}\xi < 0 \end{cases}. \tag{A.4}$$

The functions Φ_1 and Φ_γ have discontinuities on the real axis. We denote these discontinuities by $\tilde{\Phi}_1(x, \xi)$ and $\tilde{\Phi}_\gamma(x, \xi)$ respectively:

$$\tilde{\Phi}_1(x,\xi) = \frac{\alpha_{1\gamma}(\xi)}{\alpha_{11}(\zeta)}\psi_\gamma(x,\xi)e^{i\xi x},$$

$$\tilde{\Phi}_\gamma(x,\xi) = \frac{\alpha_{1\gamma}^*(\xi)}{\alpha_{11}^*(\xi)}\psi_1(x,\xi)e^{-i\xi x}. \tag{A.5}$$

In addition, $\Phi_1(x, \xi)$ has simple poles at the points $\zeta_1, \ldots, \zeta_N$; the residues of Φ_1 at this points are given by

$$\Gamma_1^{(n)} = \frac{c_{1\gamma}^{(n)}\psi_\gamma(x,\zeta_n)\exp(i\zeta_n x)}{\alpha_{11}'(\zeta_n)}. \tag{A.6}$$

The functions $\Phi_\gamma(x, \xi)$ also have simple poles at the points $\zeta_1^*, \zeta_2^*, \ldots, \zeta_N^*$. We denote the residues of these functions at the point ζ_n^* by $\Gamma_\gamma^{(n)}$. In order to determine the $\Gamma_\gamma^{(n)}$ we cut off the potential at a certain finite radius. For a finite potential the whole scattering matrix is analytic in the complex ξ-plane. Therefore, one can rewrite $\Omega_{\gamma\delta}\varphi_\delta$ in the form

$$\Omega_{\gamma\delta}\varphi_\delta = \sum_{i=1}^{3}\Omega_{\gamma\delta}\alpha_{\delta i}\psi_i = \psi_\gamma + \Omega_{\gamma\delta}\alpha_{\delta 1}\psi_1.$$

The unitarity and unimodularity relations enable us to determine $\Omega_{\gamma\delta}\alpha_{\delta 1}$:

$$\Omega_{\gamma\delta}\alpha_{\delta 1}(\xi) = -\alpha_{1\gamma}^*(\xi^*)/\alpha_{11}^*(\xi^*).$$

Not letting the cut-off radius of the potential tend to infinity, we obtain

$$\Gamma_\gamma^{(n)} = -\frac{c_{1\gamma}^{(n)*}\psi_1(x,\zeta_n^*)\exp(-i\zeta_n^* x)}{(\alpha_{11}'(\zeta_n))^*}. \tag{A.7}$$

We introduce unit vectors $\mathbf{e}_i$ with components $(\mathbf{e}_i)_k = \delta_{ik}$. It is obvious that $\lim_{\xi\to\infty}\Phi_i(x,\xi) = \mathbf{e}_i$. The piece-wise analytic functions Φ_i are reconstructed from their boundary values, residues, and discontinuities on the real axis:

$$\Phi_i(x,\xi) = \mathbf{e}_i + \sum_{n=1}^{N}\frac{\Gamma_i^{(n)}}{\xi-\zeta_n^{(i)}} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\Phi}_i(x,\xi')d\xi'}{\xi'-\xi}.$$

Here $\zeta_n^{(i)}$ is equal to ζ_n if $i=1$, and is equal to ζ_n^* if $i=2$ or 3. Thus, we obtain

$$\psi_\gamma(x,\zeta_n)\exp(-i\zeta_n x) = \mathbf{e}_\gamma + \sum_{m=1}^{N}\frac{\Gamma_\gamma^{(m)}}{\zeta_n-\zeta_m^*} + \frac{1}{2\pi i}\int\frac{\tilde{\Phi}_\gamma(x,\xi')d\xi'}{\xi'-\zeta_n}, \tag{A.8a}$$

$$\psi_1(x,\zeta_n^*)\exp(i\zeta_n^* x) = \mathbf{e}_1 + \sum_{m=1}^{N}\frac{\Gamma_1^{(m)}}{\zeta_n^*-\zeta_m} + \frac{1}{2\pi i}\int\frac{\tilde{\Phi}_1(x,\xi')d\xi'}{\xi'-\zeta_n^*}. \tag{A.8b}$$

Examining the limiting values of $\Phi_i(x, \xi)$ on the real axis, we obtain

$$\psi_1(x,\xi)e^{i\xi x}=\mathbf{e}_1+\sum_{n=1}^{N}\frac{\Gamma_1^{(n)}}{\xi-\zeta_n}+\frac{1}{2\pi i}\int\frac{\tilde{\Phi}_1(x,\xi')d\xi'}{\xi'-\xi+i0},$$

$$\psi_\gamma(x,\xi)e^{-i\xi x}=\mathbf{e}_\gamma+\sum_{n=1}^{N}\frac{\Gamma_\gamma^{(n)}}{\xi-\zeta_n^*}+\frac{1}{2\pi i}\int\frac{\tilde{\Phi}_\gamma(x,\xi')d\xi'}{\xi'-\xi-i0}. \quad \text{(A.9)}$$

Relations (A.5)–(A.9) constitute the complete system of equations for the inverse scattering problem. All of the Jost functions are reconstructed from the first row of the scattering matrix ($\alpha_{11}(\xi)$, $\alpha_{12}(\xi)$, and $\alpha_{13}(\xi)$) and from the set of quantities $c_{12}^{(n)}$, $c_{13}^{(n)}$, n = 1, 2, ..., N. We still have to determine the potential $q_1(x)$, $q_2(x)$, which is easily done by comparing the asymptotic expansion of $\Phi_i(x, \xi)$ in powers of $1/\xi$, determined from Eq. (A.9), with the same quantities found directly from the system (9). This gives the following expressions for $q_1(x)$ and $q_2(x)$:

$$q_{1,2}^*(x)=2i\left(\sum_{n=1}^{N}\Gamma_1^{(n)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\tilde{\Phi}_1(x,\xi)d\xi\right). \quad \text{(A.10)}$$

[1)]Bound states of solitons appear in the opposite case (in terms of the self-focusing theory, these correspond to "composite channels"). Their structure is not considered in the present work.

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