Conductivity of type-II superconductors near the transition temperature

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The conductivity of type-II superconductors is investigated in the vicinity of the critical temperature. A broad region of magnetic field strengths near the critical field exists in which corrections in the following order in the conductivity should be taken into account, owing to the numerically small coefficient in the leading term.

1. INTRODUCTION

As has been shown in[1], there is a broad range of magnetic fields near the critical temperature, in the vicinity of the critical field H_{c2}, in which the order parameter Δ and all thermodynamical quantities can be found by simple expansion in the parameter $(1 - H/H_{c2})$ that measures the closeness of the field to critical, however, the conductivity turns out to be essentially a nonlinear functional of Δ. The numerical coefficient in the expression for the conductivity in a magnetic field near H_{c2} (in the region where there is no simple expansion in the order parameter Δ) turns out to be small. As a result, a numerically large region arises in which the corrections of next higher order in the conductivity turn out to be of the order of the contribution of the fundamental term, which increases rapidly with decrease in the field. We shall investigate below the behavior of all corrections to the conductivity.

2. BASIC EQUATIONS

We limit ourselves to the consideration of the most interesting case of superconductors with short free path length $l \ll \xi$ (ξ is the coherence length of the superconductor). For this case, general equations were obtained in[1] for finding the conductivity. In the vicinity of the critical temperature, the effective conductivity can be represented in the form[1]

$$\frac{\sigma_{\text{eff}}}{\sigma} - \frac{\langle H^2\rangle}{\langle H\rangle^2} = \frac{\pi}{8e^2DT}\frac{\langle|\mathbf{b}\partial_-\Delta|^2\rangle}{\langle \mathbf{A}_1\rangle^2} + \varphi_1 + \varphi_2,$$

$$\varphi_1 = -\frac{i}{8T\langle \mathbf{A}_1\rangle^2}\left\langle \mathbf{A}_1^2\int_{-i\infty}^{i\infty} d\omega(-1-\alpha\alpha_+ + \beta_+\beta^*)\right\rangle,$$

$$\varphi_2 = -\frac{1}{16e^2DT\langle \mathbf{A}_1\rangle^2}\left\langle \int_{-i\infty}^{i\infty} d\omega[f_1(\mathbf{b}\partial_+\Delta^*) + f_2(\mathbf{b}\partial_-\Delta)]\right\rangle$$

$$-\frac{i}{8eT\langle \mathbf{A}_1\rangle^2}\left\langle \mathbf{A}_1\int_{-i\infty}^{i\infty} d\omega\left[f_1\partial_+\beta^* + \beta_+\partial_+f_2 - ig_1\frac{\partial\alpha}{\partial\mathbf{r}} - i\alpha_+\frac{\partial g_1}{\partial\mathbf{r}}\right]\right\rangle \quad (1)$$

where $\partial_\pm = \partial/\partial\mathbf{r} \pm 2ie\mathbf{A}$, $\mathbf{A}_1$ is the amplitude of the alternating field. In zeroth approximation in frequency of the alternating field, the amplitude A_1 is expressed in terms of the local value of the magnetic field, according to the formula[1]

$$\mathbf{A}_1 = [\mathbf{H}(r)\times\mathbf{b}], \quad \mathbf{b} - \text{const.} \quad (2)$$

The functions

$$\alpha \equiv \alpha(\omega-\delta), \quad \alpha_+ \equiv \alpha(\omega+\delta), \quad \beta \equiv \beta(\omega-\delta), \quad \beta_+ \equiv \beta(\omega+\delta) \quad (3)$$

can be found at the Matsubara frequencies from the set of equations[1]

$$\frac{D}{2}\left[\alpha\partial_-^2\beta - \beta\frac{\partial^2\alpha}{\partial\mathbf{r}^2}\right] + \alpha\Delta - \beta\omega = \alpha\beta\tau_s^{-1},$$

$$\alpha^2 + |\beta|^2 = 1, \quad (4)$$

$$\Delta = \frac{|\lambda|mp}{2\pi}T\sum_\omega\beta, \quad \mathbf{j} = \frac{iep^2l_{tr}}{6\pi}T\sum_\omega(\beta\partial_+\beta^* - \beta^*\partial_-\beta),$$

where $D = vl_{tr}/3$ is the diffusion coefficient, τ_s the time of flight of the electron with spin flip, $\mathbf{j}$ the current density. The functions $f_{1,2}$ and $g_{1,2}$ in Eq. (1) satisfy the set of equations[1]

$$D\frac{\partial}{\partial\mathbf{r}}\left[i(g_1-g_2)\frac{\partial\alpha}{\partial\mathbf{r}} + i\alpha_+\frac{\partial(g_1-g_2)}{\partial\mathbf{r}} - \beta_+\partial_+f_2 - f_1\partial_+\beta^* - f_2\partial_-\beta - \beta_+^*\partial_-f_1\right]$$

$$= -i(\mathbf{b}\partial_-\Delta)(\beta_+^* - \beta^*) - i(\mathbf{b}\partial_+\Delta^*)(\beta_+ - \beta) + e\mathbf{A}_1D[(\beta_+ - \beta)\partial_+(\beta_+^* + \beta^*) - (\beta_+^* - \beta^*)\partial_-(\beta_+ + \beta)],$$

$$2[-\Delta^*f_1 + \Delta f_2] + D\frac{\partial}{\partial\mathbf{r}}\left[i(g_1+g_2)\frac{\partial\alpha}{\partial\mathbf{r}} + i\alpha_+\frac{\partial(g_1+g_2)}{\partial\mathbf{r}} - \beta_+\partial_+f_2 - f_1\partial_+\beta^* + f_2\partial_-\beta + \beta_+^*\partial_-f_1\right] = i(\mathbf{b}\partial_-\Delta)(\beta_+^* + \beta^*)$$

$$- i(\mathbf{b}\partial_+\Delta^*)(\beta_+ + \beta) - e\mathbf{A}_1D\frac{\partial}{\partial\mathbf{r}}(2\alpha_+\alpha - \beta_+\beta^* - \beta_+^*\beta), \quad (5)$$

$$f_1 = \frac{i}{\alpha_+ - \alpha}(\beta g_1 + \beta_+ g_2), \quad f_2 = -\frac{i}{\alpha_+ - \alpha}(\beta_+^* g_1 + \beta^* g_2).$$

We use a gauge in which $\operatorname{div}\mathbf{A} = 0$.

In the derivation of Eq. (1), it was assumed that the quantity τ_s satisfies the condition $\pi T_c\tau_s \gg 1$. However, this condition is not essential and if it is not satisfied, then the change reduces to the following substitution for the coefficient of the first term in Eq. (1):

$$\frac{\pi}{8e^2DT} \to \frac{1}{4\pi e^2DT}\psi'\left(\frac{1}{2} + \frac{1}{2\pi T\tau_s}\right)$$

and the substitution $d\omega \to d\omega/\cos^2(\omega/2T)$, where ψ' is the derivative of the function ψ.

For the first term in Eq. (1), the important frequencies in the integration are $\omega \sim T$, and therefore, it depends weakly on the concentration of paramagnetic impurities; near H_{c2} it decomposes into a series in the parameter $1 - H/H_{c2}$. Near the critical field H_{c2}, this term gives the Maki correction[2] to the conductivity. The second and third terms in Eq. (1) are anomalous. They depend essentially on the concentration of paramagnetic impurities. Near the critical field H_{c2}, there is a broad region in which there is no simple expansion of these quantities in the parameter $1 - H/H_{c2}$. The second term in Eq. (1) is the Thompson correction to the conductivity.[3]

We now consider the behavior of the effective conductivity in the two regions with respect $T_c^{-1} \ll \tau_s\Delta^{-1}$ and $\tau_s \to \infty$ to the concentration of paramagnetic impurities.

3. SUPERCONDUCTORS WITHOUT PARAMAGNETIC IMPURITIES ($\tau_s \to \infty$)

The study of superconductors without paramagnetic impurities turns out to be very complicated, and simple answers have proved to be possible only in limiting cases. We shall consider below each term in Eq. (1) for the conductivity for various limiting cases.

In the neighborhood of the critical field $H_{c2} - H \ll H_{c2}$, we obtain

$$\alpha_+ = 1 - \frac{|\Delta|^2}{2(\omega+\lambda)^2}, \quad \beta_+ = \left(\omega - \frac{D}{2}\partial_-^2\right)^{-1}\left\{\Delta\left[1 - \frac{\omega|\Delta|^2}{2(\omega+\lambda)^2} + \frac{D}{4(\omega+\lambda)^3}\frac{\partial^2|\Delta|^2}{\partial \mathbf{r}^2}\right]\right\} \tag{6}$$

from the set of Eqs. (4), with accuracy to within terms of second order of smallness; here

$$\lambda = eH_{c2}D = 4\pi^{-1}(T_c - T).$$

Substituting these values for the functions α and β in Eq. (1), we obtain the following expression for φ_1:

$$\varphi_1 = \frac{\pi}{8T\lambda}\left\langle |\Delta|^2\left(1 + \frac{D}{16\lambda^3}\frac{\partial^2|\Delta|^2}{\partial \mathbf{r}^2}\right)\right\rangle = 2.5x\left[1 - 0.72x\frac{T_c}{T_c - T}\right], \quad x \ll (T_c - T)/T_c, \tag{7}$$

here

$$x = (1 - H/H_{c2})/(1 - 1/2\varkappa^2), \tag{8}$$

κ is the parameter of the Ginzburg-Landau theory.

It is easy to obtain an expression for φ_1 in the other limiting case

$$(T_c - T)/T_c \ll x \ll 1. \tag{9}$$

In the region of fields defined by the condition (9), we can neglect in the principal approximation the gradient terms in Eq. (4) for the functions α and β and get as a result

$$\alpha_+ = \frac{\omega}{((\omega+\delta)^2 + |\Delta|^2)^{1/2}}, \quad \beta_+ = \frac{\Delta}{((\omega+\delta)^2 + |\Delta|^2)^{1/2}}. \tag{10}$$

Substituting these values for the functions α and β in Eq. (1), we find, with logarithmic accuracy,

$$\varphi_1 = \frac{1}{5T}\left\langle |\Delta| \ln\left(\frac{\langle|\Delta|^2\rangle^{1/2}}{\lambda}\right)\right\rangle = \frac{\langle|\Delta|^2\rangle^{1/2}}{T}0.19\ln\left(\frac{\langle|\Delta|^2\rangle^{1/2}}{\lambda}\right) = 0.27\left(x\frac{T_c - T}{T_c}\right)^{1/2}\ln\left(\frac{5xT_c}{T_c - T}\right), \quad 1 \gg x \gg \frac{(T_c - T)}{T_c}. \tag{11}$$

The Thompson correction to the conductivity was studied in[4]. The numerical value of the coefficient in Eq. (11) differs from the corresponding limiting value of 4 by about 25%. This difference is connected with the fact that the interpolation formula of[4] $\{\beta = \Delta/[(\omega+\lambda)^2 + |\Delta|^2]^{1/2}\}$ gives a poor description of the Green's function in the threshold region. To find the Green's functions α and β in the region near the threshold $|\omega| = |\Delta|$, it is necessary to solve the set of equations (4). As was noted in[4], the Thompson correction to the conductivity is important only in the region $1 - H/H_{c2} \lesssim (T_c - T)T_c$ and in the region $1 - H/H_{c2} \gg (T_c - T)/T_c$ it becomes less than the Maki correction (the first term in Eq. (11)).

In fields that are close to critical, the expansion of the function φ_2 begins with a cubic term and its calculation turns out to be rather involved. We only give the result here:

$$\varphi_2 = 0.98\left(\frac{T_c}{T_c - T}\right)^2 x^3, \quad x \ll \frac{T_c - T}{T_c}, \tag{12}$$

where x is determined by Eq. (8).

The expression for φ_2 in the region $1 \gg 1 - H/H_{c2} \gg (T_c - T)/T_c$ in the principal approximation was obtained in[1]:

$$\varphi_2 = 0.18x^{3/2}(T_c/(T_c - T))^{1/2}, \quad (T_c - T)/T_c \ll x \ll 1. \tag{13}$$

However, because of the numerically small coefficient in Eq. (13), the next higher corrections to the function φ_2 are important. The region of frequencies $|\omega| > |\Delta|$ was important in obtaining Eq. (13)[1]. Account of the region $|\omega| < |\Delta|$ leads to the appearance of two more terms, one of which leads to cancellation of the logarithmic divergence which arises in the Maki correction in weak fields. The second of these divergences in square-root fashion near the "threshold" $|\omega| = |\Delta|$. We find it with accuracy to within a number of the order of unity. There also exists a contribution of a narrow region near the threshold, of width $\omega_c \sim |\Delta|(D/\xi^2|\Delta|)^{2/3}$. Unfortunately, we can make only a rough estimate of the contribution from this region. As a result, we obtain the following expression for the conductivity near the critical field:

$$\frac{\sigma_{\text{eff}}}{\sigma} - 1 = 5x - 1.8\frac{x^2T_c}{T_c - T}, \quad x \ll \frac{T_c - T}{T_c}; \tag{14}$$

$$\frac{\sigma_{\text{eff}}}{\sigma} - 1 = 0.18x^{3/2}\left(\frac{T_c}{T_c - T}\right)^{1/2} + 1.25x + 0.27\left(x\frac{T_c - T}{T_c}\right)^{1/2}\ln\left(\frac{5xT_c}{T_c - T}\right) + \alpha_1 x\left(\frac{T_c x}{T_c - T}\right)^{1/10} + 0.1\alpha_2 x\left(\frac{T_c x}{T_c - T}\right)^{1/5}, \quad \frac{T_c - T}{T_c} \ll x \ll 1. \tag{15}$$

In Eq. (15), the coefficients α_1 and α_2 are of the order of unity.

In weak fields $H \ll H_{c2}$, we find for the conductivity:

$$\frac{\langle H\rangle}{H_{c2}}\frac{\sigma_{\text{eff}}}{\sigma} = 1.1\left(1 - \frac{T}{T_c}\right)^{-1/2} + 0.81\left[1 + \alpha_3\left(\frac{T_c}{T_c - T}\right)^{1/10} + \alpha_4\left(\frac{T_c}{T_c - T}\right)^{1/5}\right]. \tag{16}$$

The coefficients α_3 and α_4 in Eq. (16) are of the order of unity. The numerical value of the coefficient for the first term in Eq. (16) was taken by us from the work of[5].

4. THE REGION $T_c^{-1} \ll \tau_s \ll \Delta^{-1}$

In the region $T_c^{-1} \ll \tau_s \ll \Delta^{-1}$, the set of equations (5) reduces to the rather simple form[1]

$$-D\frac{\partial^2}{\partial \mathbf{r}^2}(g_1 - g_2) = \frac{2\omega}{\omega^2 - \Gamma^2}\left(\mathbf{b}\frac{\partial|\Delta|^2}{\partial \mathbf{r}}\right),$$

$$\left[|\Delta|^2 - \frac{\Gamma^2 - \omega^2}{2\Gamma}D\frac{\partial^2}{\partial \mathbf{r}^2}\right](g_1 + g_2) = \Delta(\mathbf{b}\partial_+\Delta^*) - \Delta^*(\mathbf{b}\partial_-\Delta), \tag{17}$$

$$f_1 = \frac{i\Delta}{2}\left(-\frac{g_1}{\omega - \Gamma} + \frac{g_2}{\omega + \Gamma}\right), \quad f_2 = -\frac{i\Delta^*}{2}\left(\frac{g_1}{\omega + \Gamma} - \frac{g_2}{\omega - \Gamma}\right),$$

where $\Gamma = \tau_s^{-1}$. In the principal approximation, we have for the functions α and β

$$\alpha_+ = -\alpha = 1, \quad \beta_+ = \Delta/(\omega + \Gamma), \quad \beta = -\Delta/(\omega - \Gamma). \tag{18}$$

Substituting the expressions (17) and (18) in Eq. (1), we easily obtain

$$\varphi_1 = \frac{\pi\tau_s}{8T}\frac{\langle \mathbf{A}_1^2|\Delta|^2\rangle}{\langle \mathbf{A}_1\rangle^2},$$

$$\varphi_2 = \frac{\pi\tau_s(\langle|\Delta|^4\rangle - \langle|\Delta|^2\rangle^2)}{64e^2D^2T\langle H\rangle^2} - \frac{i}{64e^2DT\langle \mathbf{A}_1\rangle^2}\int_{-i\infty}^{i\infty} d\omega\left(\frac{1}{\omega+\Gamma} - \frac{1}{\omega-\Gamma}\right)^2 \times\left\langle[\Delta(\mathbf{b}\partial_+\Delta^*) - \Delta^*(\mathbf{b}\partial_-\Delta)]\left(\frac{2\Gamma|\Delta|^2}{\Gamma^2 - \omega^2} - D\frac{\partial^2}{\partial \mathbf{r}^2}\right)^{-1}[\Delta(\mathbf{b}\partial_+\Delta^*) - \Delta^*(\mathbf{b}\partial_-\Delta)]\right\rangle. \tag{19}$$

We now find the expression for the conductivity near the critical field in the region $1 - H/H_{c2} \ll 1$. In this region,

$$\Delta(\mathbf{b}\partial_+\Delta^*) - \Delta^*(\mathbf{b}\partial_-\Delta) = \frac{i}{H_{c2}}\left(\mathbf{A}_1 \frac{\partial|\Delta|^2}{\partial\mathbf{r}}\right). \quad (20)$$

We represent the operator which enters into the expression for φ_2 in the form of a series

$$\left(\frac{2\Gamma|\Delta|^2}{\Gamma^2-\omega^2} - D\frac{\partial^2}{\partial\mathbf{r}^2}\right)^{-1} = \left[1 - \left(\frac{2\Gamma\langle|\Delta|^2\rangle}{\Gamma^2-\omega^2} - D\frac{\partial^2}{\partial\mathbf{r}^2}\right)^{-1} \times \frac{2\Gamma(|\Delta|^2 - \langle|\Delta|^2\rangle)}{\Gamma^2-\omega^2} + \ldots\right]\left(\frac{2\Gamma\langle|\Delta|^2\rangle}{\Gamma^2-\omega^2} - D\frac{\partial^2}{\partial\mathbf{r}^2}\right)^{-1}. \quad (21)$$

The series on the right side of Eq. (21) converges rapidly in the region

$$T_c\tau_s(1 - H/H_{c2}) \leqslant 1. \quad (22)$$

In the region defined by the condition (22), we keep only two terms in the expansion (21). After simple calculations, we get the following expression for the conductivity:

$$\frac{\sigma_{\text{eff}}}{\sigma} - 1 = 0.32T_c\tau_s x^2 + 2.5x + 3.18(T_c - T)\tau_s x - 0.73x\left[\left(1 - \frac{1}{\sqrt{1+z}}\right) + 0.08\frac{z}{\sqrt{1+z}}\left(\frac{2}{1+\sqrt{1+z}} - \frac{1}{1+z}\right)\right], \quad (23)$$

$$T_c\tau_s(1 - H/H_{c2}) \leqslant 1,$$

where

$$x = \frac{1 - H/H_{c2}}{1 - 1/2\varkappa^2}, \quad z = \frac{\sqrt{3}\pi^2}{14\zeta(3)\beta_A}T_c\tau_s x = 0.88T_c\tau_s x,$$

$$\beta_A = \langle|\Delta|^4\rangle / \langle|\Delta|^2\rangle^2 \approx 1.16.$$

The first term in Eq. (23) arises from the first component in ρ_2[1] (Eq. (19)), the second is the Maki correction, the third the Thompson correction. The Thompson correction is always small. The relative contribution of the last component in (23) increases with decrease in the field.

In the region of fields

$$T_c\tau_s(1 - H/H_{c2}) \gg 1$$

we find for the conductivity (using Eqs. (1) and (19)),

$$\frac{\sigma_{\text{eff}}}{\sigma} - \frac{\langle H^2\rangle}{\langle H\rangle^2} = \frac{\pi\tau_s(\langle|\Delta|^4\rangle - \langle|\Delta|^2\rangle^2)}{64e^2D^2T\langle H\rangle^2} + \frac{\pi}{32e^2DT\langle\mathbf{A}_1\rangle^2}\left\langle\frac{1}{|\Delta|^2}\left(\mathbf{b}\frac{\partial|\Delta|^2}{\partial\mathbf{r}}\right)^2\right\rangle. \quad (24)$$

The last term in Eq. (24) is the sum of the Maki correction and the last component in Eq. (19) for φ_2. From Eq. (24), we easily obtain

$$\sigma_{\text{eff}}/\sigma - 1 = 0.32(T_c\tau_s)x^2 + 1.25x, \quad (T_c\tau_s)^{-1} \ll x \ll 1. \quad (25)$$

The first term in Eq. (25) was obtained in[1]. However, the condition of smallness of the second term in comparison with the first, because of the numerical smallness of the coefficient of the first term, imposes a very rigid condition on the closeness of the temperature to critical. Replacing τ_S by its limiting value Δ^{-1}, we transform the condition of smallness of the second term in Eq. (25) in comparison with the first to the form

$$T_c x / (T_c - T) > 150.$$

In weak fields $H \ll H_{c2}$ we get from (24)

$$\frac{\sigma_{\text{eff}}}{\sigma} = 0.85\frac{H_{c2}}{\langle H\rangle}\left[\pi T\tau_s\left(1 + \frac{4.31}{\pi T_c\tau_s}\right) + 0.95\right]. \quad (26)$$

The results of[6] were used in finding the numerical coefficients in Eq. (26). The first term in Eq. (24) in the region $H \ll H_{c2}$ is identical with the result of[7]. We note that the logarithmically diverging contribution at large distances in the Maki correction (the first term in Eq. (11) is cancelled in weak fields by the second component in the expression (19) for φ_2.

5. CONCLUSION

The conductivity of type-II superconductors in a magnetic field can be represented in the form of the sum of three terms. One of them—the Maki correction—leads to a rather simple form and can be studied easily for any field. The numerical coefficient in the Maki correction is anomalously large and turns out to be important over a wide range in the magnetic field. The second term—the Thompson correction—depends strongly on the concentration of paramagnetic impurities as well as on the value of the magnetic field. It is not small only in superconductors without paramagnetic impurities in a narrow region close to the critical point. The third, a significantly anomalous term, increases rapidly with decrease in the field. Unfortunately, it is impossible to limit its calculation to the principal approximation, for numerical reasons, and the corrections of the next higher order turn out to be significant for practically all regions of the magnetic field.

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