Waves and forces in a homogeneous medium with time-varying properties

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Exact expressions for the wave field in a homogeneous nondispersive medium with time-varying properties are presented; the dielectric constant or refractive index are assumed to vary linearly. Expressions are obtained for the transmitted and reflected waves. It is shown that variation in the properties of the medium may appreciably affect the averaged forces exerted by the wave field on charged particles or the medium. Some applications of the results to wave processes in nonstationary media are indicated. It is pointed out that the appearance of a reflected wave field and of averaged forces may be a factor that initiates various stimulated scattering processes. It is mentioned that similar effects may arise in the case of acoustic waves propagating in a medium with time-varying acoustic properties.

This paper considers an exact expression for electromagnetic waves in the simplest cases of practical interest where the properties of the medium are variable—linear increase of the dielectric constant and linear variation of the refractive index. We are thereby enabled to analyze a large class of observed effects associated with variations, from slow to very rapid, in the properties of a medium.[1)] We here consider averaged forces that arise under such conditions and act upon polarizable charged particles or upon the medium.

EXACT SOLUTIONS FOR AN ELECTROMAGNETIC WAVE PROPAGATING IN A MEDIUM WITH TIME-DEPENDENT PROPERTIES

Maxwell's equations

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c_0}\frac{\partial\mathbf{H}}{\partial t},\quad \operatorname{rot}\mathbf{H}=\frac{1}{c_0}\frac{\partial\mathbf{D}}{\partial t},\quad \operatorname{div}\mathbf{D}=0$$

for a nondispersive homogeneous medium with a time-dependent dielectric constant $\epsilon(t)$ yield for $\mathbf{D}=\epsilon(t)\mathbf{E}$ the equation

$$\Delta\mathbf{D}=\frac{\varepsilon(t)}{c_0^2}\frac{\partial^2\mathbf{D}}{\partial t^2}.$$

In the one-dimensional case, directing one of the coordinate axes along the vector $\mathbf{D}$, we obtain

$$\frac{\partial^2 D}{\partial z^2}=\frac{\varepsilon(t)}{c_0^2}\frac{\partial^2 D}{\partial t^2}.$$

Letting $D = Z(z)T(t)$, we have

$$c_0^2 Z''/Z=\varepsilon(t)\ddot{T}/T=-\varkappa^2,$$

that is,

$$Z=A_1e^{i\varkappa z/c_0}+A_2e^{-i\varkappa z/c_0}.$$

For an initial wave of the form $D = D_0e^{ikz}$ we have $\kappa = c_0k' = \omega_0 n_0$. Therefore, for T we obtain the equation[2)]

$$\ddot{T}+\frac{\omega_0^2\varepsilon_0}{\varepsilon(t)}T=0.$$

The greatest practical interest attaches to the two simplest cases:

$$1)\ \varepsilon(t)=\varepsilon_0+\dot{\varepsilon}t,\quad 2)\ n(t)=\varepsilon^{1/2}(t)=n_0+\dot{n}t.$$

For small variations of the parameters, $\dot{n}t \ll 1$, the two dependences coincide and $\dot{n} = \dot{\epsilon}/2n_0$. However, we shall be interested in all rates of variation of the properties of the medium.

1) Considering first $\epsilon = \epsilon_0 + \dot{\epsilon}t$, we introduce the new variable

$$x=\omega_0^2\varepsilon_0(\varepsilon_0+\dot{\varepsilon}t)/\dot{\varepsilon},$$

and obtain the equation

$$x\,d^2T/dx^2+T=0,$$

which reduces to a Bessel equation with the solution[5]

$$T=\sqrt{x}\{C_1J_1(2\sqrt{x})+C_2N_1(2\sqrt{x})\},$$

where J_1 and N_1 are first-order Bessel and Neumann functions. Hence

$$D={}^1\!/_2D_0e^{ikz}\xi\{C_1J_1(\xi)+C_2N_1(\xi)\},$$

where $\xi = 2\epsilon_0^{1/2}\omega_0(\epsilon_0+\dot{\epsilon}t)^{1/2}/\dot{\epsilon}$, and the constants C_1 and C_2 are determined by matching to D and $\dot{D}$ at t = 0:

$$C_1=\pi(N_0+iN_1),\qquad C_2=-\pi(J_0+iJ_1);$$

here J and N (omitting their argument ξ) are taken at t = 0, i.e., for $\xi_0 = 2\omega_0\epsilon_0/\dot{\epsilon}$. Using these constants, we obtain

$$D={}^1\!/_2D_0e^{ikz}\pi\xi\{[N_0J_1(\xi)-J_0N_1(\xi)]+i[N_1J_1(\xi)-J_1N_1(\xi)]\}.$$

For example, with $\xi_0 \gg 1$ and with J and N expressed asymptotically in terms of trigonometric functions multiplied by coefficients which are power series in $1/\xi$ and $1/\xi_0$, by neglecting all except the first powers we obtain

$$D=D_0e^{ikz}\left(\frac{\xi}{\xi_0}\right)^{1/2}\left\{\left[1+\frac{i}{8}\left(\frac{1}{\xi_0}-\frac{3}{\xi}\right)\right]e^{-i(\xi-\xi_0)}+\frac{i}{4\xi_0}e^{i(\xi-\xi_0)}\right\},$$

whereby a reflected wave is seen to arise.[3)] For $\xi\to\infty$ ($\omega_0/\epsilon \gg 1$) we have

$$D=D_0\left[\frac{\varepsilon_0+\dot{\varepsilon}t}{\varepsilon_0}\right]^{1/4}\exp\left\{ik'z-i\frac{2\omega_0\sqrt{\varepsilon_0}}{\dot{\varepsilon}}[(\varepsilon_0+\dot{\varepsilon}t)^{1/2}-\sqrt{\varepsilon_0}]\right\},$$

or $\omega(t) = -\dot{\varphi}(t) = \omega_0\sqrt{\epsilon_0}/(\epsilon_0+\dot{\epsilon}t)^{1/2}$, which follows directly from conservation of the momentum of a quantum and conservation of the number of quanta.

A reflected wave also appears when the linear increase of ϵ terminates. [The appearance of reflected waves in association with rapid changes of the properties of a homogeneous medium was first pointed out in[1]. S. N. Stolyarov and A. Chigarev (private communication) found that the expression for fields when there are two "breaks" in ϵ is converted into the formulas of[1] for a jump as $\dot{\epsilon}\to\infty$.]

It also follows from the obtained solution that near $\xi\to 0$ ($\epsilon = 0$ for $t\to -\epsilon_0/\dot{\epsilon}$ with $\dot{\epsilon} < 0$) variations of the field are determined mainly by the Neumann functions because $N_1(\xi)|_{\xi\to 0}\approx -2/\pi\xi$, while $J_1(\xi)|_{\xi\to 0}\approx {}^1\!/_2\xi\to 0$. At this point D remains finite but E diverges: $E = D/\epsilon \to 0$.

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2) In the case of $n = n_0 + \dot{n}t = n_0(1 + \alpha t)$ where $\alpha = \dot{n}/n$, by introducing the variable $\tau = 1 + \alpha t$ and seeking a solution in the form of a power τ^k we obtain a characteristic equation for the exponent k:

$$k^2 - k + \Omega^2 = 0, \quad \Omega^2 = \omega_0^2/\alpha^2,$$

i.e.,

$$k_{1,2} = {}^1\!/_2[1 \pm \sqrt{1 - 4\Omega^2}];$$

or, for $k_1 \neq k_2$,

$$T = \sqrt{\tau}\{C_1\tau^{\sqrt{1-4\Omega^2}/2} + C_2\tau^{-\sqrt{1-4\Omega^2}/2}\},$$

and in the special case $k_1 = k_2 = {}^1\!/_2$:

$$T = \sqrt{\tau}\{C_1 \ln\tau + C_2\}.$$

For $\Omega = \omega_0/\alpha < {}^1\!/_2$, i.e., for sharp changes of the refractive index ($\dot{n}/\omega > 1$), we obtain aperiodic solutions.

The case of not very rapid changes in n ($\Omega = \omega_0/\alpha > {}^1\!/_2$) corresponds to complex roots of n and therefore to oscillating solutions. In this case it is more convenient to write

$$D = D_0 e^{ik'z}\tau^{1/2}\{C_1 e^{i\gamma\ln\tau} + C_2 e^{-i\gamma\ln\tau}\},$$

where

$$\gamma = {}^1\!/_2(4n_0^2\omega_0^2/\dot{n}^2 - 1)^{1/2}, \quad \tau = 1 + \dot{n}t/n_0,$$

and, for $t > 0$,

$$\dot{D} = D_0 e^{ik'z}\frac{\dot{n}}{n_0}\tau^{-1/2}\left\{\left(\frac{1}{2} + i\gamma\right)C_1 e^{i\gamma\ln\tau} + \left(\frac{1}{2} - i\gamma\right)C_2 e^{-i\gamma\ln\tau}\right\}.$$

Matching D and $\dot{D}$ at t = 0 to the initially given simplest unperturbed functions

$$\tilde{D} = D_0 e^{i(kz-\omega_0 t)}, \quad \dot{\tilde{D}} = -i\omega_0\tilde{D},$$

we obtain equations for C_1 and C_2 that yield

$$C_1 = \frac{1}{2} + \frac{i}{4\gamma} - \frac{\omega_0 n_0}{2\dot{n}\gamma}, \quad C_2 = \frac{1}{2} - \frac{i}{4\gamma} + \frac{\omega_0 n_0}{2\dot{n}\gamma},$$

i.e., in the general case we have a direct and a reflected wave.

In the case of a non-sharply varying refractive index, $\dot{n}/\omega \to 0$ ($\gamma \to n_0\omega_0/\dot{n} \to \infty$), we obtain

$$C_1 \approx i\dot{n}/4n_0\omega_0 \to 0, \quad C_2 \approx 1 - i\dot{n}/4n_0\omega_0,$$

i.e., the amplitude of the reflected wave increases with the rate of change of n(t). The "instantaneous" frequency is then determined from the condition

$$\frac{1}{2}\left(\frac{4n_0^2\omega_0^2}{\dot{n}^2} - 1\right)^{1/2}\ln\left(1 + \frac{\dot{n}}{n_0}t\right) \approx \int_0 \omega(t)\,dt,$$

i.e.,

$$\omega(t) = \frac{1}{2}\left(\frac{4n_0^2\omega_0^2}{\dot{n}^2} - 1\right)^{1/2}\left(1 + \frac{\dot{n}}{n_0}t\right)^{-1}\frac{\dot{n}}{n_0}.$$

AVERAGED FORCES INDUCED BY THE WAVE FIELD DURING OR AFTER CHANGES IN THE PROPERTIES OF THE MEDIUM

Time dependence of the properties of a medium can strongly alter the forces acting in a wave field upon particles in the medium or upon volume elements of the medium itself. We shall be especially interested in a force averaged over many oscillation periods of a wave. This force has the form

$$F = \alpha\nabla\overline{(\mathrm{Re}\,E)^2},$$

where α is the polarizability of the object; for a particle of charge e and mass m we have $\alpha \approx -e^2/m\omega^2$, for a polarizable particle of radius a we have $\alpha \approx a^3(\epsilon - \epsilon_2)/ \times (\epsilon + 2\epsilon_2)$, or for the force per unit volume of the medium we have $\alpha \approx \rho\partial\epsilon/\partial\rho$.

For given amplitudes A_p and A_r of the progressive and reflected waves the force is

$$F = \alpha\nabla\overline{(\mathrm{Re}\,E)^2} = 2ik\{(A_1A_2 - A_1^*A_2^*)\cos 2kz + i(A_1A_2 + A_1^*A_2^*)\sin 2kz\}.$$

For example, in the case of a non-sharp change of $n(\dot{n}/\omega \ll 1)$,

$$\frac{A_p}{E_0} \approx C_2 = 1 - i\frac{\dot{n}}{4n_0\omega_0}, \quad \frac{A_r}{E_0} \approx C_1 = i\frac{\dot{n}}{4n_0\omega_0}$$

and we have

$$F \approx -\alpha k\frac{\dot{n}E_0^2}{n_0\omega_0}\cos 2kz \approx -\alpha\frac{\dot{n}}{c}E_0^2\cos 2kz,$$

i.e., the dimension of the gradient region is represented by $l = c/\dot{n} \sim cT$, where T is the characteristic time required for an appreciable change in the parameters of the medium.

After a sharp jump of n the appearance of a reflected wave also leads to the formation of a standing wave and to averaged forces

$$F \approx \alpha k A_r A_p \sin(2kz + \varphi).$$

The averaged forces can accelerate particles and can also deform the medium. Specifically, a reflected wave field and averaged forces arising due to changes of the medium's properties with time can serve as the initiating factors of various induced scattering processes, such as Mandelstam-Brillouin scattering etc., associated with changes in the density and temperature distributions of the medium.

CONCLUSION AND COMMENTS

The results obtained from the investigation of changes in a wave that is propagating in a medium with time-dependent properties are of interest for many physical problems. This is the case, for example, when variations of the wave are used to determine changes occurring in a medium (when Doppler, interferometric, or holographic investigating techniques are used), or when speically induced changes of the medium's properties are utilized to change the wave amplitude and frequency, or when spontaneous changes of the medium's properties result from the radiation itself (these last processes are characteristic of the high-power emissions that have become very common in recent years; examples are spectral changes and reflection accompanying self-focusing), and in many other cases of dynamic and nonstationary processes occurring within media in the presence of a wave field.

A dielectric constant can be changed rapidly under an external influence. For example, an electric field E_0 applied to a medium with anisotropic molecules induces a change of ϵ because the polarizability of the medium is changed. In the case of polar molecules with an intrinsic dipole moment p_0 the change of the refractive index is $\Delta n \sim KE_0^2 \approx 10^{-4}$ for $E_0 \sim 30$ kV/cm, where $K \sim (p_0/kT)^2$.

The change of ϵ in a rapidly alternating field can be associated with the so-called high-frequency Kerr effect. The orientation time of molecules is sufficiently short for inclusion in the range of rapid changes of the medium. It is also possible to utilize the Pockels effect, for which the change is $\Delta n(E_0) \sim E_0$.

Various nonstationary processes in a medium (variations of density, temperature, degree of excitation, closeness to resonance, etc.) can induce rapid changes of the dielectric constant. For example, a simultaneous change of ϵ can be induced by powerful radiation transverse to a propagating wave (thus a powerful light pulse can alter the transmission of radio waves). For a plasma and uhf waves, variations of the dielectric constant that are associated with compression, expansion, or formation of the plasma can lead to strong changes in the wave, especially close to plasma resonances.

We note that changes of a wave can be enhanced manyfold by passing it many times through a changing medium; for example, a resonator filled with a varying medium can be used.

On a cosmic scale the considered effects can account for some characteristics of variations in light, radiowave, and particle spectra.

Nonstationary effects can occur in the case of acoustic waves propagating in a medium with time-dependent acoustic properties. Specifically, a) variations of sound velocity due to variations of temperature or of carrier concentration, or to the appearance of phase-transformation nuclei, etc., or b) variations in the density of the medium, can lead to changes of wave amplitude and frequency and to reflection. When the density change is small and the considered effects are associated mainly with a variation of sound velocity, the equations for acoustic waves are of the same form as in the electromagnetic case with $c_0/\sqrt{\epsilon(t)}$ replaced by the velocity of sound, $c_s(t)$.

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1)Several papers [1-3] have been published dealing with the problem of wave propagation in a time-dependent homogeneous medium. However, the investigated changes in the parameters of the medium were slow, abrupt, or periodic; linear variations were not considered.

2)These equations are analogous to equations that describe wave reflection from an inhomogeneous medium [with $\epsilon(x)$] [4] when x is replaced by t and $\epsilon(x) = \varphi(x)$ is replaced by $\epsilon(t) = 1/\varphi(t)$.

3)High reflective efficiency is associated with a short period of time (compared with the wave period) during which changes occur in the prpperties of the medium or their derivatives. Thus the derived expressions correspond to a jump of the derivative $\dot{\epsilon}$ (or n). Reflection is reduced sharply if $\dot{\epsilon}$ changes smoothly. This is easily seen when variation in the medium is represented by the expansion $\Delta\epsilon = f't + \frac{1}{2}f''t^2 + \ldots$ etc.; for the reflected wave we then have $A_r \propto f'(0)/\omega + f''(0)/\omega^2$. If there is no jump in f' at $t = 0$ (i.e., $f'(0) = 0$), we have

$$A_r \propto f''(0)/\omega^2 \sim f''(0)t/(t\omega)\omega \sim f'(t)/(t\omega)\omega \sim A_r f'(t)/t\omega,$$

i.e., a non-steep form for $f'(t)$ leads to greatly diminished reflection if $t\omega \gg 1$.

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