Amplification of electromagnetic and gravitational waves scattered by a rotating "black hole"

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The amplification coefficient of electromagnetic and gravitational waves scattered by a rotating "black hole" whose gravitational field can be described by the Kerr metric is calculated. The condition of existence of the amplification effect is independent of the wave-field spin. The maximal magnitude of the effect rapidly increases with increase of the spin and is considerable for gravitational waves with a small multipolarity if the angular momentum of the "black hole" is close to the maximal value. The amplification coefficient is the same for all modes and this implies that the Kerr metric is stable with respect to generation of electromagnetic and gravitational waves.

In view of the fact that the recently discovered discrete x-ray sources (of the Cyg X-1 type) are, possibly, collapsars ("black holes"), the theoretical investigation of the physical processes that occur in the strong gravitational fields in the vicinity of such objects is quite essential. Of special interest are the processes leading to the extraction of energy from the rotating "black hole" owing to the decrease in its rotational energy and angular momentum.

The first such process, suggested by Penrose[1] and investigated by Christodoulou[2], is connected with the breakup into two parts of a particle that has flown into the exosphere of a rotating "black hole," one of the two parts being subsequently absorbed by the "black hole," while the other returns to infinity in space, carrying with it part of the rotational energy and angular momentum of the "black hole." As was recently shown by Bardeen, Press, and Teukolsky[3], this process is extremely unlikely under real astrophysical conditions: in order for it to occur, the two parts of the disintegrated particle should fly apart in definite directions and their relative velocity in their center-of-mass system should be $v \geq \frac{1}{2}c$. The Penrose process could occur only in the collision between two relativistic objects (a neutron star and a collapser) whose masses have the same order of magnitude.

Another process that also leads to the extraction of energy from a "black hole" was suggested by Zel'dovich[4] and, later, by Misner[5]. The essence of the process lies in the fact that a classical multipole wave with a definite angular-momentum value can, under certain conditions, be amplified upon being reflected from a rotating "black hole." The frequency of the wave does not change in the process, so that this effect has nothing in common with the Doppler effect. The process under consideration is astrophysically real, since the sources of such waves (electromagnetic and gravitational) can, for example, be bodies moving in stable orbits around the "black hole." Furthermore, it can be shown that if the test body revolves in a circular orbit in the equatorial plane of the "black hole" in the same sense as the latter and the radial coordinate of the orbit satisfies the condition[1)]

$$r > M^{1/3} r_1^{4/3} a^{-2/3}, \tag{1}$$

then the frequencies of all the harmonics emitted by the body satisfy the amplification condition (20). The condition (1) is satisfied by the coordinates of the majority of stable bound orbits, and, for a sufficiently fast rotation of the "black hole" ($a \geq 0.36$ M), by the coordinates of all such orbits. Waves with definite angular-momentum values also arise in the multipole expansion of a plane wave incident on the "black hole."

Zel'dovich's hypothesis is demonstrated for the model example of scalar waves by one of the present authors in[6]. The analytical formulas for the amplification factor obtained in[6], as well as the numerical computation carried out by Press and Teukolsky[7] for the particular case a = M, show that in the case of a scalar field the amplification effect is extremely small: the energy flux in the wave increases, upon reflection of the wave, by not more than 0.4%. In view of this, the question arises as to the magnitude of the effect for the really existing classical waves: electromagnetic and gravitational. This problem is solved in the present paper, which is therefore a continuation of the work[6].

In the calculations, the stationary gravitational field of the "black hole," a field which can be described by the Kerr metric (2), was considered as an external field, i.e., the inverse influence of the waves on the metric was not taken into account. The admissibility of such an approximation for real problems has been demonstrated in[6]. By gravitational waves we mean the weak perturbations which are applied to the Kerr metric and which have a wave character.

The calculations show that: a) the condition (20) for the existence of the amplification effect does not depend on the spin of the wave field (i.e., it is the same for scalar, electromagnetic, and gravitational waves); b) the maximum magnitude of the effect rapidly increases with increasing spin. The amplification factor for electromagnetic waves[2)] does not exceed a few percent, although it is nevertheless roughly an order of magnitude larger than for the scalar field. Gravitational waves, on the other hand, can be amplified on reflection under optimum conditions by more than a factor of two (the reflection coefficient $R \approx 2.4$ for $l = n = 2$; $a \to M$; $\omega \to n\Omega$). Thus, the effect under consideration turns out to be quite considerable for gravitational waves and, in particular, it significantly influences the dynamics and magnitude of the gravitational radiation of particles moving in the vicinity of a "black hole." For example, it may turn out that there exist in the exosphere of a "black hole" the so-called "floating orbits" (for their definition, see[7]), which are absent for a point particle in the case of scalar waves. Notice that the process under consideration is irreversible: if the incident wave is amplified on reflection ($R > 1$), then, as shown below, the surface area of the event horizon of the "black hole" always increases, which agrees with Hawking's theorem [8].

The obtained formulas are also applicable to the cases when ω and n do not satisfy the condition (20) and they allow us to compute the partial cross sections for the capture by a "black hole" of waves with such parameters. Besides being of interest in their own right, the obtained reflection coefficients are essential for the computation of the amount of electromagnetic and gravitational radiation emitted by a particle revolving around a "black hole," the probability for the creation of a photon or graviton pair in the Kerr metric, etc. Finally, the coefficient of amplification of the waves upon reflection from the "black hole" is finite for all modes, from which follows the important conclusion that the Kerr metric is stable against spontaneous generation of classical electromagnetic and gravitational waves.

2. The gravitational field of a rotating "black hole" can be described by the Kerr metric (it is assumed in the paper that $G = \hbar = c = 1$):

$$ds^2 = \frac{1}{\rho^2}(\Delta - a^2\sin^2\theta)\,dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2\,d\theta^2 - \frac{1}{\rho^2}\sin^2\theta\,[\,(r^2+a^2)^2 - \Delta a^2\sin^2\theta\,]\,d\varphi^2 + \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\varphi, \quad (2)$$

where $\rho^2 = r^2 + a^2\cos^2\theta$; $\Delta = r^2 - 2Mr + a^2$; M is the mass of the "black hole," $L = Ma$ ($0 \le a \le M$) is its angular momentum oriented in the direction $\theta = 0$; $r_{1,2} = M \pm \sqrt{M^2 - a^2}$; the electric charge of the "black hole" $e = 0$. The amplification condition (20) preserves its form in the case of $e \neq 0$, but then in the formula (19) $r_1 = M + \sqrt{M^2 - a^2 - e^2}$.

The equation of the surface S_{hor} of the event horizon is given by $r = r_1$, or $\Delta = 0$; the exosphere is located between the surfaces $\Delta = 0$ and $\Delta = a^2\sin^2\theta$.

To derive the equations describing the electromagnetic and gravitational perturbations in the Kerr metric, it is convenient to use the Newman-Penrose formalism [9]. It consists in the following:

a) An isotropic tetrad, consisting of two real vectors, l^μ and n^μ, and one complex vector m^μ, is introduced. Only two of all the possible scalar products of these vectors are different from zero: $l^\mu n_\mu = -m^\mu m^*_\mu = 1$. We can recover the initial metric from the given tetrad, using the formula:

$$g_{\mu\nu} = 2[\,l_{(\mu}n_{\nu)} - m_{(\mu}m_{\nu)}^*\,], \quad (3)$$

where the round brackets denote symmetrization in the pair of indices. The specific form of these tetrads in the Kerr metric was found by Kinnersley [10]:

$$l^\mu = \left[\frac{r^2+a^2}{\Delta},\ 1, 0, \frac{a}{\Delta}\right],\quad n^\mu = [r^2+a^2, -\Delta, 0, a]\frac{1}{2\rho^2},$$
$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right], \quad (4)$$

b) The three complex functions

$$\varphi_0 = F_{\mu\nu}l^\mu m^\nu,\quad \varphi_1 = {}^1/_2 F_{\mu\nu}(l^\mu n^\nu - m^\mu m^{*\nu}),\quad \varphi_2 = F_{\mu\nu}m^{*\mu}n^\nu \quad (5)$$

are introduced in place of the six different components of the electromagnetic-field tensor $F_{\mu\nu}$. Similarly, instead of the ten different components of the Weyl conformal tensor $C_{\alpha\beta\gamma\delta}$ (which coincides with the Riemann tensor in empty space), one introduces five complex functions, of which we write out two:

$$\psi_0 = -C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta,\quad \psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha m^{*\beta}n^\gamma m^{*\delta}. \quad (6)$$

As was recently shown by Teukolsky [11], in the Kerr metric, in contrast to the spherically symmetric Schwarzschild metric, it is possible to obtain wave equations with separable variables only for certain components φ_α and ψ_α, to wit, for the so-called radiation parts, φ_0, φ_2, ψ_0, and ψ_4, of the wave field. This is however sufficient for the solution of the set problem, since it is precisely in terms of these field components that the energy flux at infinity (in space) is expressed. For the electromagnetic field, the flux in a unit solid angle $d\Omega$

$$\left(\frac{d^2E}{dt\,d\Omega}\right)_{in} = \lim_{r\to\infty}\frac{r^2}{8\pi}|\varphi_0|^2,\quad \left(\frac{d^2E}{dt\,d\Omega}\right)_{out} = \lim_{r\to\infty}\frac{r^2}{2\pi}|\varphi_2|^2. \quad (7)$$

To compute the energy flux of the gravitational field, we can use the Landau-Lifshitz pseudotensor, which yields ($\psi_0, \psi_4 \sim e^{-i\omega t}$):

$$\left(\frac{d^2E}{dt\,d\Omega}\right)_{in} = \lim_{r\to\infty}\frac{r^2}{64\pi\omega^2}|\psi_0|^2,\quad \left(\frac{d^2E}{dt\,d\Omega}\right)_{out} = \lim_{r\to\infty}\frac{r^2}{4\pi\omega^2}|\psi_4|^2. \quad (8)$$

We shall henceforth be interested in only the components φ_0 and ψ_0. The solution corresponding to the wave with the frequency ω ($\omega > 0$) a Z component n of angular momentum has the form

$$\varphi_0,\ \psi_0 = F(r)P(\theta)e^{in\varphi - i\omega t}, \quad (9)$$

where $F(r)$ and $P(\theta)$ satisfy the homogeneous equations [11]:

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dF}{dr}\right) + \{[\,(r^2+a^2)^2\omega^2 - 4aMn\omega r + a^2n^2 + 2ia(r-M)ns - 2iM(r^2-a^2)\omega s\,]\Delta^{-1} + 2ir\omega s - \lambda\}F = 0, \quad (10)$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \left(a^2\omega^2\sin^2\theta + \frac{n^2}{\sin^2\theta} + 2a\omega s\cos\theta + \frac{2ns\cos\theta}{\sin^2\theta} + s^2\operatorname{ctg}^2\theta - s\right)P + \lambda P = 0, \quad (11)$$

where $s = 1$ for electromagnetic and $s = 2$ for gravitational waves. For $s = 0$, these equations go over into the well-known equations for scalar waves (see, for example, [6]). Formally, s can assume any integral value.

The equation (11), together with the boundary conditions $|P(0)| < \infty$, $|P(\pi)| < \infty$, constitutes an eigenvalue problem for the eigenvalues

$$\lambda = {}_s\lambda_l^n(a\omega) = {}_s\lambda_l^{-n},$$

where l is a whole number: $l \ge \max(|n|, s)$. For $a\omega = 0$

$${}_s\lambda_l^n(0) = (l-s)(l+s+1),$$

and the eigenfunctions are spinor spherical harmonics (see [12]). For $a\omega \neq 0$, the eigenvalues λ are not analytically expressible in terms of l, n, and $a\omega$. The calculation shows that for $a\omega \ll 1$ and for any s:

$${}_s\lambda_l^n = (l-s)(l+s+1) - a\omega\frac{2ns^2}{l(l+1)} + a^2\omega^2\left\{\frac{2}{3}\left[1 + \frac{3n^2 - l(l+1)}{(2l-1)(2l+3)}\right] - \frac{2s^2}{l(l+1)}\frac{3n^2 - l(l+1)}{(2l-1)(2l+3)} + 2s^2\left[\frac{(l^2-s^2)(l^2-n^2)}{l^3(2l-1)(2l+1)} - \frac{[(l+1)^2-n^2][(l+1)^2-s^2]}{(l+1)^3(2l+1)(2l+3)}\right]\right\}. \quad (12)$$

It remains to establish the boundary conditions for Eq. (10). The boundary condition for $r \to +\infty$ has the form

$$F(r) = \frac{A}{r}e^{-i\omega r} + \frac{B}{r}\frac{e^{i\omega r}}{(\omega r)^{2s}}. \quad (13)$$

The first and second terms describe the incident and reflected waves respectively. To find the reflection coefficient R for the wave, it is sufficient to find $|B/A|^2$. In fact, we can show with the aid of the Maxwell equations that if for $r \to \infty$

$$\varphi_0 = \left[\frac{A}{r}e^{-i\omega r} + \frac{B}{r}\frac{e^{i\omega r}}{(\omega r)^2}\right] P_{ln}(\theta)e^{in\varphi - i\omega t}, \quad (14)$$

then

$$\varphi_2 = \left[-2\frac{B}{r}e^{i\omega r} + O\left(\frac{A}{r^3}\right)\right] P_{ln}(\pi-\theta)e^{in\varphi - i\omega t}(-1)^{l-n}[(\lambda - 2an\omega + 2)^2 + 4a\omega(n - a\omega)]^{-1/2}, \quad (15)$$

so that we obtain for electromagnetic waves (s = 1)

$$R = \frac{(dE/dt)_{out}}{(dE/dt)_{in}} = 4\frac{|\varphi_2|^2}{|\varphi_0|^2} = \frac{16}{(\lambda - 2an\omega + 2)^2 + 4a\omega(n - a\omega)}\left|\frac{B}{A}\right|^2. \quad (16)$$

Using the Bianchi identities, we can derive a similar expression for R in the case of gravitational waves (s = 2).

The physical boundary condition for Eq. (10) for $r \to r_1$ follows from the fact that the surface $r = r_1$ is a surface-trap which can only capture physical objects. In[6] this condition was chosen in such a way that the group velocity of the wave for $r \to r_1$[3)] was directed towards the surface-trap, i.e.,

$$F(r) \sim e^{-i(\omega - n\Omega)y}\frac{1}{\Delta^s} \quad (17)$$

for $r \to r_1$ ($y \to -\infty$), where y is determined by the condition

$$\frac{dy}{dr} = \frac{r^2 + a^2}{\Delta}, \quad -\infty < y < \infty, \quad (18)$$

and Ω is the angular velocity of the rotating "black hole":

$$\Omega = a/(r_1^2 + a^2). \quad (19)$$

For a discussion of this boundary condition from a different point of view, see[3]. The divergence of F(r) as $r \to r_1$ ($\Delta \to 0$) is not physical; it arises as a result of the singularity in the tetrad (4) at $\Delta = 0$. All physical quantities measured in a freely falling reference system remain finite as $r \to r_1$; this was shown by Price[13] in the Schwarzschild metric (a = 0) for the case of waves with arbitrary integral spin.

3. Investigation of the radial equation (10) with the boundary conditions (13) and (17) shows that for s = 1, 2, as well as for s = 0, the reflection coefficient R > 1 (i.e., the wave is amplified) if

$$\omega < n\Omega \quad (20)$$

(we recall that the phase of the wave has been chosen so that $\omega > 0$). Thus, the condition for existence of the amplification effect is the same for scalar, electromagnetic, and gravitational waves. It is evident that the effect does not exist in the Schwarzschild metric ($a = \Omega = 0$).

We shall show that in the process under consideration M and a decrease in such a way that the surface area S_{hor} of the event horizon of the "black hole" increases[4)]. Let the energy flux in an incident wave of frequency ω and multipole order n be equal to W. Then the energy flux in the reflected wave is equal to RW and

$$\frac{dM}{dt} = -(R-1)W, \quad \frac{dL}{dt} = -\frac{n}{\omega}(R-1)W. \quad (21)$$

Let us use the Christodoulou formula[2], which expresses the mass M of the "black hole" in terms of its angular momentum L and "undiminishable" mass M_0 connected with the surface area of the event horizon by the formula

$$S_{hor} = 4\pi(r_1^2 + a^2) = 16\pi M_0^2, \quad M^2 = M_0^2 + L^2/4M_0^2. \quad (22)$$

Then

$$dM = \left(\frac{\partial M}{\partial M_0^2}\right)_L d(M_0^2) + \left(\frac{\partial M}{\partial L}\right)_{M_0} dL, \quad (23)$$

and $(\partial M/\partial L)_{M_0} \equiv \Omega$. Consequently,

$$\frac{dS_{hor}}{dt} = 16\pi\frac{d(M_0^2)}{dt} = 32\pi\frac{M}{1 - a^2/r_1^2}\left(\frac{dM}{dt} - \Omega\frac{dL}{dt}\right) = 32\pi\frac{MW}{1 - a^2/r_1^2}\left(\frac{n\Omega}{\omega} - 1\right) \quad (R-1) \geq 0. \quad (24)$$

The process is reversible ($dS_{hor}/dt = 0$) only when $a < M$ and $\omega = n\Omega$, but in that case, as will be shown below, R = 1 and dM/dt = 0. However, by choosing the frequency ω close to $n\Omega$, we can make the wave-amplification process as close to being reversible as we wish.

Below we investigate the behavior of R in the vicinity of the points $\omega = 0$ and $\omega = n\Omega$. In the remaining region the reflection coefficient R is clearly analytic with respect to ω and is consequently finite. The quantity R does not depend on the polarization of the incident wave. The method used to obtain the values of R is similar to the method employed in[6] by one of the present authors for the case s = 0. All the formulas cited below are formally valid for any integral s, although only the values s = 0, 1, and 2 have a direct physical meaning. These formulas are also valid for ω and n that do not satisfy the condition (20); in this case R < 1, i.e., the incident wave is partly reflected and partly absorbed by the "black hole."

For $\omega \to 0$, we can assume in the first approximation that ${}_s\lambda_l^n = (l - s)(l + s + 1)$. R then has the form (for some details of the derivation, see the Appendix):

$${}_sR_{ln} - 1 = ({}_0R_{ln} - 1)\left[\frac{(l-s)!(l+s)!}{(l!)^2}\right]^2, \quad (25)$$

where ${}_0R_{ln}$ is the reflection coefficient for scalar waves that was found in[6]:

$${}_0R_{ln} - 1 = 4Q[\omega(r_1 - r_2)]^{2l+1}\frac{(l!)^4}{[(2l)!]^2[(2l+1)!!]^2}\prod_{k=1}^{l}\left(1 + \frac{4Q^2}{k^2}\right); \quad Q = \frac{r_1^2 + a^2}{r_1 - r_2}(n\Omega - \omega). \quad (26)$$

The formulas (25) and (26) include all the particular cases, including a = 0 and a = M, the region of applicability of these formulas being defined by the condition: $\omega M \ll 1$. For fixed l, the quantity $|{}_sR_{ln} - 1|$ increases with increasing s; for $l \gg s^2$, ${}_sR_{ln} \approx {}_0R_{ln}$. For $n \neq 0$ and $\omega \to 0$, the quantity $|{}_sR_{ln} - 1| \sim \omega^{2l+1}$. The various particular cases of the formula (26) are thoroughly investigated in[6].

In the particular case when a = 0, i.e., for the Schwarzschild metric, the formula (25) for s = 1 coincides with Khar'kov's result[14], while for s = 2 it coincides with Fackerell's result[15].

Let us now consider the case $\omega \to n\Omega$. Let $\alpha = 1 - \omega/n\Omega$. It can be shown that if $a < M$, then ${}_sR_{ln} - 1 \sim \alpha$ in the region $|\alpha|Q_1 \ll 1$, where $Q_1 = an/(r_1 - r_2) > 0$. Under the supplementary conditions $a \ll M$ and $n \ll M/a$ (in which case $n\Omega M \ll 1$), the formulas (25) and (26) become applicable in the region under consideration and enable us to determine the correct coefficient attached to α.

Two substantially different cases can occur when a = M and $\omega \to n\Omega$. Let

$$\delta^2 = 2n^2 - \lambda - (s + {}^1\!/_2)^2, \tag{27}$$

then for $\delta^2 < 0$ the reflection coefficient R is continuous and varies monotonically in the vicinity of the point $\alpha = 0$:

$$ {}_sR_{ln} - 1 = \operatorname{sign}\alpha\cdot|\alpha|^{2|\delta|}4|\delta|^2(2n^2)^{2|\delta|} \times \frac{|\Gamma({}^1\!/_2+s+|\delta|+in)|^2|\Gamma({}^1\!/_2-s+|\delta|+in)|^2}{\Gamma^4(1+2|\delta|)} e^{\pi n(1-\operatorname{sign}\alpha)}. \tag{28}$$

If, on the other hand, $\delta^2 > 0$ (and this condition is satisfied, in particular, by all ${}_1\lambda_n^n$ with $n \geq 1$ and all ${}_2\lambda_n^n$ with $n \geq 2$), then in the region $|\alpha| \ll n^{-4}\max(|\alpha|^2; 1)$ we shall have

$$({}_sR_{ln} - 1)^{-1} = \operatorname{sign}\alpha\left\{\frac{\operatorname{ch}^2\pi(n-\delta)}{\operatorname{sh}^2 2\pi\delta} e^{\pi(n+\delta)(\operatorname{sign}\alpha-1)} + \frac{\operatorname{ch}^2\pi(n+\delta)}{\operatorname{sh}^2 2\pi\delta} e^{\pi(n-\delta)(\operatorname{sign}\alpha-1)} - \frac{2\operatorname{ch}\pi(n-\delta)\operatorname{ch}\pi(n+\delta)}{\operatorname{sh}^2 2\pi\delta} e^{\pi n(\operatorname{sign}\alpha-1)}\cos[\gamma_0 - 2\delta\ln(2n^2|\alpha|)]\right\}, \tag{29}$$

where

$$\gamma_0(\delta) = 4\arg\Gamma(1+2i\delta) + 2\arg\Gamma({}^1\!/_2+s+in-i\delta) + 2\arg\Gamma({}^1\!/_2+s-in-i\delta).$$

In the vicinity of the point $\alpha = 0$ the reflection coefficient R has an infinite number of oscillations in the region $|\alpha| n^2 \ll 1$ (provided δ is not small in comparison with unity, which is an exceptional case). These oscillations are, in terms of amplitude, important only if n = 1 and $\pi\delta \lesssim 1$; in the opposite case when $\alpha > 0$, we have

$${}_sR_{ln} - 1 \approx e^{2\pi(\delta-n)}. \tag{30}$$

For $\alpha < 0$, we have $\min {}_sR_{ln} = 0$, i.e., the barrier can be totally transparent. The reflection coefficient is discontinuous at the point $\alpha = 0$.

If $a \neq M$, but $M - a \ll M$ and $n \ll \sqrt{M/(M-a)}$, then the reflection coefficient is described by the formulas (28) and (29) in the region $Q_1^{-1} \ll |\alpha| \ll n^{-2}$, while, as was indicated above, for $|\alpha| \ll Q_1^{-1}$, $R - 1 \sim \alpha$. Thus, the reflection coefficient is continuous at the point $\alpha = 0$ when $a \neq M$.

Calculation of the magnitude of the coefficient R shows that for electromagnetic waves, $R - 1 < 10\%$ and, in particular, for $l = n = s = 1$, a = M, and $\omega = \Omega - 0$, the quantity ${}_1R_{11} - 1 \approx 2\%$. For gravitational waves, when $l = n = s = 2$, a = M, and $\omega = 2\Omega - 0$, we shall have ${}_2R_{22} - 1 \approx 1.37$, i.e., a gravitational wave can be amplified on reflection by a factor of more than two. For fixed s and for $n \to \infty$, the effect decreases according to an n-power exponential law; for example, ${}_sR_{nn} - 1 \approx e^{-\pi n(2-\sqrt{3})}$ when a = M, $\omega = n\Omega - 0$, and $n \gg s^2$.

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APPENDIX

The derivation of the formulas (25), (28), and (29) is similar to the derivation given in [6]. In particular, to obtain the formula (25), we must find the solution to the equation

$$[x(x+1)]^{1-s}\frac{d}{dx}\left\{[x(x+1)]^{s+1}\frac{dF}{dx}\right\} + [Q^2 + iQs(1+2x) - (l-s)(l+s+1)x(x+1)]F = 0, \tag{A.1}$$

where $x = (r - r_1)/(r_1 - r_2)$ ($a \neq M$), which satisfies the boundary condition (17) for $r \to r_1$ ($x \to 0$). The Eq. (A.1) is obtained by neglecting in Eq. (10) all the terms containing ω except the one which enters into Q (this can be done when $\omega M \ll 1$ and $x \ll l/\omega(r_1 - r_2)$).

The required solution has the form

$$F = \left(\frac{x}{x+1}\right)^{iQ}\frac{1}{[x(x+1)]^s}G(-l-s, l-s+1, 1-s-2iQ;\ x+1), \tag{A.2}$$

where $G(\alpha, \beta, \gamma; z)$ is the hypergeometric function.

For $x \gg \max(l, Q)$, we obtain[5)]

$$F = C_1x^{l-s} + C_2x^{-l-s-1}, \tag{A.3}$$

where

$$C_1 = (-1)^{l+s}\frac{(2l)!}{(l-s)!}\frac{\Gamma(1-s-2iQ)}{\Gamma(l+1-2iQ)},$$

$$C_2 = \frac{1}{2}\frac{(l+s)!}{(2l+1)!}\frac{\Gamma(1-s-2iQ)}{\Gamma(-2iQ-l)}.$$

The solution (A.2) should be matched in the region $\max(Q, l) \ll x \ll l/\omega(r_1 - r_2)$ with the solution to the equation

$$\frac{d^2F}{dx^2} + \frac{2s+2}{x}\frac{dF}{dx} + \left[\omega^2(r_1-r_2)^2 + \frac{2is\omega(r_1-r_2)}{x} - \frac{(l-s)(l+s+1)}{x^2}\right]F = 0, \tag{A.4}$$

that is expressible in terms of the confluent hypergeometric functions, or in terms of indefinite integrals of Bessel functions.

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1) For the explanation of the notation, see below after the formula (2).

2) By definition, the amplification factor is equal to the difference between the reflection coefficient R and unity if R > 1.

3) For $\omega \neq n\Omega$, the quasi-classical approximation is applicable near the point $r = r_1$ in Eq. (10) and, therefore, a group velocity can properly be introduced for a wave of any frequency $\omega \neq n\Omega$.

4) A similar method was used by Zel'dovich in his paper [4] to prove the existence of the effect of amplification of waves on reflection from a rotating conducting cylinder; see also Bekenstein's paper [16].

5) In [6] a somewhat more complicated expression for C_2 in the form $\tilde{C}_2 = C_2 + \beta C_1$, $\operatorname{Im}\beta = 0$, was used which also led to the correct answer, since only $\operatorname{Im}(C_2/C_1)$ enters into the formula for R.

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