Electron-acoustic and drift instabilities in a finite-pressure plasma with a perpendicular current

J. P. Goedbloed[1)], A. I. Pyatak, and V. L. Sizonenko
Fysico-Technical Institute of the Ukrainian Academy of Sciences
(Submitted December 11, 1972)
Zh. Eksp. Teor. Fiz. 64, 2084-2096 (June, 1973)

Non-electrostatic instabilities are induced by a current flowing perpendicular to an external magnetic field in a finite-pressure plasma ($4\pi n_0 T \sim H_0^2$) with hot ions and cold electrons are investigated. Expressions are obtained for the frequencies and growth rates of the unstable oscillations. The level of turbulent electric field fluctuations in the saturation regime is estimated. Estimates are obtained for the turbulent friction force and the rate of increase of the temperature of the ions scattered by turbulent electric field noise.

1. INTRODUCTION

The interest in the investigation of high-frequency short-wavelength instabilities induced by a current perpendicular to an external magnetic field has recently grown considerably. This is caused by the understanding of the important role of analogous instabilities for the explanation of rapid turbulent heating, anomalous conductivity of plasma, anomalous diffusion, and plasma radiation in a large number of experiments, viz., theta-pinches, plasmoids entering an inhomogeneous magnetic field, excitation of large-amplitude ion-cyclotron and fast magnetosonic waves, rotating plasmas in a radial electric and axial magnetic field, etc. By now the theory of the excitation of longitudinal oscillations of a plasma with a perpendicular current is well developed (see, e.g.,[1-6]). However, it appears that in a plasma with sufficiently high kinetic pressure $[\beta \equiv 4\pi n_0(T_e + T_i)/H_0^2 \sim 1]$ and a perpendicular current there are excited in the main non-electrostatic oscillations which have received little study up to now.

In the present paper we investigate non-electrostatic instabilities excited by a perpendicular current in a high-pressure plasma ($\beta \sim 1$) with hot ions and cold electrons ($T_i \gg T_e$). It is shown that two branches of oscillations are excited at a current velocity u considerably larger than the drift velocity u_n but smaller than the thermal velocity v_{Ti} of the ions. To order of magnitude, the frequency ω and the wave vector **k** are determined by the relation

$$\omega \sim ku \sim (u/v_{Ti})\sqrt{\omega_{He}\omega_{Hi}}$$

whereas the relation for the growth rate is

$$\gamma \sim \frac{1}{16}\left(\frac{\pi}{2}\right)^{1/2}\frac{u^2}{v_{Ti}^2}(\omega_{He}\omega_{Hi})^{1/2}.$$

Here, one of the branches corresponds to waves that propagate nearly perpendicular to the external magnetic field with an angle θ, where $\cos\theta \sim (u/v_{Ti})(m_e/m_i)^{1/2}$. In a low-pressure plasma ($\beta \ll 1$) this branch coincides with the well-investigated electron-acoustic instability[2,7]. The second branch refers to waves which propagate strictly perpendicular to the magnetic field. In a low-pressure plasma it coincides with the usual drift oscillations and the growth rate is small compared to that of the electron-acoustic instability.

It should be noted that for the considered instabilities in a plasma with $\beta \sim 1$ an essential role is played by the magnetic drift of particles, which is caused by the inhomogeneity of the external magnetic field, whereas the Larmor drift associated with the inhomogeneity of the density turns out to be inessential.

An estimate was made of the noise level at saturation caused by the strong nonlinearity of the equations of motion for the electrons. In this case the amplitudes of the turbulent fluctuations of the electromagnetic fields **E** and **H** are of the form

$$|\mathbf{E}| \sim \frac{u}{c}H_0, \quad |\mathbf{H}| \sim \frac{u}{v_{Ti}}\left(\frac{m_e}{m_i}\right)^{1/2}H_0.$$

We find the turbulent friction force F acting on ions in the quasi-linear approximation to be of the order of

$$|\mathbf{F}| = m_i u \nu_{eff} \sim (m_e m_i)^{1/2}\omega_{Hi}\frac{u^2}{v_{Ti}^2}u,$$

and a determination of the heating rate of the ions scattered by the turbulent field fluctuations yields

$$\frac{dT_i}{dt} \sim \left(\frac{m_e}{m_i}\right)^{1/2}\omega_{Hi}\frac{u^2}{v_{Ti}^2}m_i u^2.$$

The possibility of stochastic heating of the plasma is also discussed.

2. LINEAR APPROXIMATION

We investigate first the excitation of oscillations in a plasma with cold electrons ($T_e = 0$) in the linear approximation. We assume that the external magnetic field H_0, which is directed along z, and the plasma density n_0 depend only on y and do not change appreciably over the length of a Larmor radius:

$$\frac{|\varkappa_{H,n}|v_{Ti}}{\omega_{Hi}} \ll 1, \quad \varkappa_H \equiv \frac{1}{H_0}\frac{\partial H_0}{\partial y}, \quad \varkappa_n \equiv \frac{1}{n_0}\frac{\partial n_0}{\partial y},$$

The ion temperature $T_i = m_i v_{Ti}^2$ is constant along y and the plasma current is directed along x. We choose the equilibrium distribution functions f_{0e} of the electrons and f_{0i} of the ions of the following form:

$$f_{0e} = n_0(y)\delta(\mathbf{v} - \mathbf{u}_e),$$
$$f_{0i} = \frac{n_0(y)}{(2\pi)^{3/2}v_{Ti}^3}\exp\left[-\frac{(\mathbf{v}-\mathbf{u}_i)^2}{2v_{Ti}^2}\right]\left(1 - \frac{v_x u_n}{v_{Ti}^2}\right),$$

where $u_n \equiv \kappa_n v_{Ti}^2/\omega_{Hi}$ and $u_{e,i}$ are the directed electron and ion velocities in the external electromagnetic fields. In the following we assume that $u_n < u \equiv |\mathbf{u}_i - \mathbf{u}_e| < v_{Ti}$ and for simplicity of the calculations we put $u_e = 0$.

We look for unstable oscillations in the frequency and wave-vector regions

$$\omega_{Hi} \ll \omega \ll \omega_{He}, \quad \frac{\omega_{Hi}}{v_{Ti}} \ll k, \quad \cos^2\theta \equiv \frac{k_z^2}{k^2} \ll 1. \tag{2.1}$$

For the description of the motion of the electrons in the electric and magnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ of the unstable oscillations we employ the equations of magnetohydrodynamics

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{e}{m_e}\left(\mathbf{E}+\frac{1}{c}[\mathbf{v},\mathbf{H}_0+\mathbf{H}]\right), \quad (2.2)^*$$

$$\frac{\partial n}{\partial t}+\nabla(n\mathbf{v})=0. \quad (2.3)$$

Linearizing these equations with respect to small perturbations of the velocity $\mathbf{v}$ and the density $n' = n - n_0$, and employing the usual Fourier transformation, we obtain

$$-i\omega\mathbf{v}=-\frac{e}{m_e}\left\{\mathbf{E}+\frac{1}{c}[\mathbf{v},\mathbf{H}_0]\right\}, \quad (2.4)$$

$$-i\omega n'+i(\mathbf{kv})n_0+n_0\frac{\partial v_y}{\partial y}+v_y\frac{\partial n_0}{\partial y}=0. \quad (2.5)$$

In (2.4) and (2.5) we take into account a slow dependence of the Fourier-component of the density and velocity on the coordinate y. Using, according to Eq. (2.1), the smallness of the parameter ω/ω_{He}, we can then easily find the expressions for $\mathbf{v}$ and n':

$$\mathbf{v}_\perp=\frac{c}{H_0^2}[\mathbf{EH}_0]+i\frac{c}{H_0}\frac{\omega}{\omega_{He}}\mathbf{E},$$
$$v_z=-i\frac{c}{H_0}\frac{\omega_{He}}{\omega}E_z, \quad (2.6)$$
$$n'=n_0\frac{c}{H_0\omega}\left\{[\mathbf{kE}]_z+i\frac{\omega}{\omega_{He}}(\mathbf{k}_\perp\mathbf{E}_\perp)-i\frac{\omega_{He}}{\omega}k_zE_z-i(\varkappa_H-\varkappa_n)E_x\right\}$$

and for the electron current density $\mathbf{j}_e = -en_0\mathbf{v}$.

The motion of the ions in the electromagnetic field of the oscillations is found by linearizing the Vlasov equation with respect to the small field amplitude. The conditions γ, $\omega \gg \omega_{Hi}$, and $kv_{Ti} \gg \omega_{Hi}$ allow us here to regard the motion of the ions as unmagnetized. The Fourier component of the oscillating ion distribution function is then

$$f_i=-\frac{i}{\omega-\mathbf{kv}}\frac{e}{m_i}\left\{\mathbf{E}+\left[\frac{\mathbf{v}}{c}\mathbf{H}\right]\right\}\frac{\partial f_{0i}}{\partial \mathbf{v}}. \quad (2.7)$$

From this expression one then easily calculates the ion contribution to the charge density $\rho_i = e\int f_i\,d\mathbf{v}$ and the current density $\mathbf{j}_i = e\int \mathbf{v}f_i\,d\mathbf{v}$.

Substituting the thus found expressions in the Maxwell equations, one obtains the dispersion equations for the unstable oscillations (see Appendix I):

$$(1+q)\left\{1+q+\frac{q}{\beta}\left[1+i\sqrt{\frac{\pi}{2}}\frac{\omega-\mathbf{ku}+k_xu_n}{kv_{Ti}}\right]-\frac{k_xu}{\omega}q\left(1-\frac{u_n}{u\beta}\right)\right\}-\frac{\omega_{He}^2}{\omega^2}\cos^2\theta=0, \quad (2.8)$$

where $q \equiv \omega_{pe}^2/k^2c^2$ and $\omega_{pe}^2 \gg \omega_{He}^2$.

In the derivation of (2.8) we have used the condition

$$z_i\equiv\frac{\omega-ku}{\sqrt{2}\,kv_{Ti}}\sim\frac{u}{v_{Ti}}<1$$

and the equality $\kappa_H = 4\pi e n_0 u/cH_0$, which follows from the Maxwell equation $\mathrm{curl}_x H_0 = 4\pi c^{-1}en_0u$. We notice that at $q \ll 1$ and $u \gg u_n$ Eq. (2.8) transforms into the well-known[2] dispersion equation for electrostatic electron-acoustic oscillations, and at $q \gg 1$ it coincides with the dispersion equation for helicon waves[8]. If $\cos\theta = 0$, $q \ll 1$, $u \sim u_n$ and $\beta \ll 1$, Eq. (2.8) coincides with the dispersion equation for drift waves excited by a perpendicular current[6].

From (2.8) one easily obtains the expressions for the frequency and the growth rate:

$$\omega=\frac{1}{2}\left(1+q+\frac{q}{\beta}\right)^{-1}\left\{qk_xu\left(1-\frac{u_n}{u\beta}\right)\pm\left[q^2k_x^2u^2\left(1-\frac{u_n}{u\beta}\right)^2+4\left(1+q+\frac{q}{\beta}\right)\frac{\omega_{He}^2\cos^2\theta}{1+q}\right]^{1/2}\right\}, \quad (2.9)$$

$$\gamma=\mp\left(\frac{\pi}{2}\right)^{1/2}\frac{q}{\beta}\frac{\omega-\mathbf{ku}+k_xu_n}{kv_{Ti}}\omega^2\left[q^2k_x^2u^2\left(1-\frac{u_n}{u\beta}\right)^2+4\left(1+q+\frac{q}{\beta}\right)\frac{\omega_{He}^2\cos^2\theta}{1+q}\right]^{-1/2}. \quad (2.10)$$

We now find the maximum growth rate. To do so we first have to put $k_y = 0$. Noticing that usually $\kappa_n/\kappa_H < 0$ and, therefore, $1 - u_n/u\beta > 0$, it is easy to verify that the branch with the upper sign in (2.9) is unstable at $u > 0$ and the branch with the lower sign at $u < 0$, so that these branches transform into one another when u is replaced by $-u$ and vice versa.

In the following we shall consider the oscillation branch that is unstable at $u > 0$. It is easily seen that when $\cos^2\theta$ varies in the interval from zero to unity the growth rate (2.10) decreases monotonically if the condition

$$q\left[7\beta-2-\frac{u_n}{u}(7-2\beta)\right]>2\beta\left(1-\frac{u_n}{u}\right). \quad (2.11)$$

is satisfied. In this case the growth rate is the largest (but not extremal) for $\cos^2\theta = 0$ and equals

$$\gamma=\gamma_0=(2\pi)^{1/2}\beta\frac{\omega_{He}\omega_{Hi}}{kv_{Ti}^3}\frac{u_\varkappa(u-u_n-2u_\varkappa)}{[\beta+q(\beta+1)]}, \quad (2.12)$$

where

$$u_\varkappa=\frac{u}{2}\frac{q(\beta-u_n/u)}{\beta+q(\beta+1)}.$$

If, on the other hand, the inequality

$$q\left[7\beta-2-\frac{u_n}{u}(7-2\beta)\right]<2\beta\left(1-\frac{u_n}{u}\right), \quad (2.13)$$

is satisfied, then the growth rate has one maximum in the interval $0 \le \cos^2\theta \le 1$ at

$$\cos^2\theta=\frac{m_e}{m_i}\frac{u^2}{v_{Ti}^2}\frac{(1+q)[\beta+q(\beta+1)]}{q}\left(\alpha^2-\frac{u_\varkappa^2}{u^2}\right) \quad (2.14)$$

and equals

$$\gamma=\gamma_1=\sqrt{\frac{\pi}{8}}\frac{u}{v_{Ti}}\frac{qku}{\alpha}\left(\frac{u_\varkappa}{u}+\alpha\right)^2\left(1-\frac{u_n}{u}-\frac{u_\varkappa}{u}-\alpha\right)[\beta+q(\beta+1)]^{-1},$$
$$\alpha=\frac{1}{4}\left\{1-\frac{u_n}{u}-\frac{u_\varkappa}{u}+\left[\left(1-\frac{u_n}{u}-\frac{u_\varkappa}{u}\right)\left(1-\frac{u_n}{u}-9\frac{u_\varkappa}{u}\right)\right]^{1/2}\right\}. \quad (2.15)$$

We notice that the growth rate (2.12) also exists if the inequality (2.13) is satisfied, so that it is useful to compare it with the growth rate (2.15). By varying k we therefore find the maximum value of γ_0.

$$\max\gamma_0=\left(\frac{2\pi}{\beta+1}\right)^{1/2}\left(\frac{u-u_n}{v_{Ti}}\right)^2\frac{\mu x_0(x_0^2+1-2\mu)}{(x_0^2+1)^3}(\omega_{He}\omega_{Hi})^{1/2};$$
$$x_0^2={}^1/_3[-1+5\mu+(4-16\mu+25\mu^2)^{1/2}],\quad \mu=\frac{1}{2}\frac{u\beta-u_n}{(1+\beta)(u-u_n)} \quad (2.16)$$

We next find the maximum value of γ_1. After a straightforward but tedious calculation we obtain

$$\max\gamma_1=\left(\frac{\pi}{8(\beta+1)}\right)^{1/2}\left(\frac{u-u_n}{v_{Ti}}\right)^2\frac{x_1(3x_1^2-1)}{(5x_1^2-1)^2}(\omega_{He}\omega_{Hi})^{1/2},$$
$$x_1^2={}^1/_6[4+5\mu+(4+28\mu+25\mu^2)^{1/2}]. \quad (2.17)$$

For $u \gg |u_n|$ and $\beta \lesssim |u_n|/u$ it turns out that $u \sim u_n/u \ll 1$, so that the growth rate (2.16) is smaller by a factor

$1/\mu$ than the growth rate (2.17). However, if $\beta \gg |u_n|/u$ then $\mu = \beta/2(1+\beta)$ and the growth rates γ_0 and γ_1 transform to $\gamma_0 = \gamma_p f_0(\beta)$ and $\gamma_1 = \gamma_p f_1(\beta)$, where γ_p is the maximum growth rate of electrostatic oscillations ($\beta \ll 1$) and is determined by the expression[2]

$$\gamma_p = \frac{1}{16}\left(\frac{\pi}{2}\right)^{1/2}\frac{u^2}{v_{Ti}^2}(\omega_{He}\omega_{Hi})^{1/2}. \qquad (2.18)$$

The functions $f_{0,1}(\beta)$ are shown in the figure. It follows that at $\beta < 0.1$ there are excited primarily electrostatic waves with $\cos^2\theta \sim m_e/m_i$, for which the inhomogeneity of the plasma and the magnetic field are inessential. However, if $\beta > 0.25$ the waves become essentially non-electrostatic ($q \sim 1$) and the waves that propagate strictly perpendicular to the magnetic field now grow faster. It should be stressed that for these oscillations the inhomogeneity of the magnetic field plays an important role, whereas at the same time the inhomogeneity of the plasma density can be neglected at $u \gg u_n$. As $\beta \to \infty$, the growth rates of both branches decrease monotonically.

We notice that in the region of small β ($\beta \ll |u_n|/u$) the growth rate (2.16) coincides with the maximum growth growth rate of the drift waves investigated in[6]. Under these circumstances, however, drift waves apparently do not play an essential role because the growth rate (2.17) is u/u_n times larger than the growth rate (2.16).[2)]

3. ESTIMATE OF THE LEVEL OF TURBULENCE

The exponentially growing amplitude of the oscillations of the electromagnetic field becomes so large that it becomes necessary to take into account the nonlinear terms in the dispersion equation for unstable waves. Because the unstable oscillations investigated in the present work are similar to the unstable electron-acoustic oscillations, one can expect the nonlinearity in the present problem to have the same character as the nonlinearity investigated in[3-5].

We neglect the contribution of the ions to the nonlinear part of the dispersion equation. Retaining the nonlinear terms in Eq. (2.2) and recognizing that $\omega \ll \omega_{He}$ and $H_0 \gg |\mathbf{H}|$, we obtain

$$\mathbf{v} \approx \mathbf{v}_0 + \mathbf{v}_1, \quad |\mathbf{v}_1| \ll |\mathbf{v}_0|,$$

where the perpendicular component of $\mathbf{v}$ equals

$$\mathbf{v}_{\perp 0} = \frac{c}{H_0^2}[\mathbf{E}\mathbf{H}_0], \qquad (3.1)$$

$$\mathbf{v}_{\perp 1} = \frac{1}{H_0\omega_{He}}\left\{\left[\mathbf{H}_0\frac{\partial \mathbf{v}_0}{\partial t}\right] + [\mathbf{H}_0, (\mathbf{v}_0\nabla)\mathbf{v}_0] + \frac{e}{m_e c}[\mathbf{H}_0[\mathbf{v}_0\mathbf{H})]\right\}, \qquad (3.2)$$

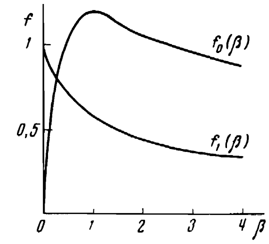
and v_{z0} is determined from the equation

$$\frac{\partial v_{z0}}{\partial t} + (\mathbf{v}_0\nabla)v_{z0} = -\frac{e}{m_e}E_z - \frac{e}{m_e c}[\mathbf{v}_{\perp 0}\mathbf{H}]_z. \qquad (3.3)$$

Substituting now the obtained values of $\mathbf{v}$ in the continuity equation (2.3) we find that the oscillatory part of the electron density n' ($|n'| \ll n_0$) is determined by the equation

$$\frac{\partial n'}{\partial t} + (\mathbf{v}_{\perp 0}\nabla)n' + \frac{\varkappa_H c n_0}{H_0}E_x - \frac{n_0}{H_0}\frac{\partial H_z}{\partial t} + n_0 \operatorname{div}\mathbf{v}_{\perp 1} + n_0\frac{\partial v_{z0}}{\partial z} = 0, \qquad (3.4)$$

In the derivation of this equation we have neglected terms that are small compared to $\partial n'/\partial t$. It should be stressed that the quantities $\mathbf{v}_{\perp 0}$ and $\mathbf{v}_{\perp 1}$ give comparable contributions to Eq. (3.4) even though the condition $|\mathbf{v}_{\perp 0}| \gg |\mathbf{v}_{\perp 1}|$ is satisfied. At the same time the perpendicular part of the electron current density is practically deter-



mined only by the quantity $\mathbf{v}_{\perp 0}$. It equals

$$\mathbf{j}_\perp = -en_0\mathbf{v}_{\perp 0} = -\frac{en_0 c}{H_0^2}[\mathbf{E}\mathbf{H}_0]. \qquad (3.5)$$

For the estimate of the contribution of the nonlinear terms to the dispersion equation it would be necessary to expand the quantities n', v_{z0}, and $\mathbf{v}_{\perp 1}$ in the small field amplitude and to assume the nonlinearity of Eqs. (3.2)–(3.4) to be weak. To the thus-obtained nonlinear dispersion equation one should apply the method of averaging over an ensemble of random phases, employed in the theory of weak turbulence, to find next the nonlinear part of the growth rate. Comparing now the nonlinear part of the growth rate with the linear part, one can estimate the level of noise at which saturation occurs. However, the calculation carried out in[4] shows that for the given type of instabilities the nonlinearity in the growth rate becomes essential when the nonlinear terms in the equations of motion of the electrons become comparable with the linear ones, so that no weak nonlinearity can be assumed in this case. As shown in[4], this fact is the result of the absence of resonant electrons and of corresponding terms in the linear dispersion equation. From this one can expect that the investigated instability may be stabilized at a noise level such that the nonlinear terms in the (3.2)–(3.4) are comparable in order of magnitude with the linear terms. In that case the quantity u', e.g., becomes a strongly nonlinear function of the oscillation amplitude, although for its estimate one may still use the formulas of the linear theory. The components ϵ_{00}, ϵ_{02}, ϵ_{03} of the dielectric tensor also become strongly nonlinear functions of the oscillation amplitude, but they retain their previous order of magnitude. Formally analogous changes occur in all the components ϵ_{ij} except ϵ_{20}, which does not change because of the relation (3.5).

Thus, the given instability can saturate only in the strongly turbulent regime, when the charge and current densities of the electrons become strongly nonlinear functions of the oscillation amplitude. In that case one easily obtains the following estimates for the turbulence level:

$$|\mathbf{v}_0| \sim u, \qquad |\psi| \sim \frac{u v_{Ti}(m_e m_i)^{1/2}}{e}, \qquad |\mathbf{E}_\perp| \sim \frac{u}{c}H_0,$$
$$|\mathbf{H}| \sim \left(\frac{m_e}{m_i}\right)^{1/2}\frac{u}{v_{Ti}}H_0, \qquad |n'| \sim n_0\frac{\omega}{\omega_{He}} \sim n_0\left(\frac{m_e}{m_i}\right)^{1/2}\frac{u}{v_{Ti}}. \qquad (3.6)$$

4. HEATING OF THE IONS

The strong-turbulence regime of the electrons and electromagnetic waves remains a weakly nonlinear process in the equations of motion of the ions. Since the contribution of the ions to the nonlinear interaction of the waves can be neglected, only the slow scattering of ions by the turbulent fluctuations (3.6) remains to be taken into account. To this end we employ the well-known approximation of the quasilinear theory. Averaging the Vlasov equation over the rapidly oscillating variations, we then obtain for the background ion distribution function f_{0i} the equation

$$\frac{\partial f_{0i}}{\partial t}+\mathbf{v}\frac{\partial f_{0i}}{\partial \mathbf{r}}-\omega_{Hi}\frac{\partial f_{0i}}{\partial \Phi}+\frac{e}{m_i}\mathbf{E}_0\frac{\partial f_{0i}}{\partial \mathbf{v}}=\frac{\partial}{\partial v_i}D_{ij}(\mathbf{v})\frac{\partial f_{0i}}{\partial v_j}, \tag{4.1}$$

where

$$D_{ij}(\mathbf{v})=\pi\frac{e^2}{m_i^2}\sum_{\mathbf{k}}k_ik_j|\psi_{\mathbf{k}}|^2\delta(\omega-\mathbf{kv}),$$

$\mathbf{E}_0$ is the external electric field, $v_x = v_\perp \cos\Phi$, and $v_y = v_\perp \sin\Phi$. From (4.1) one can obtain the continuity equation and the equations of motion for the ions:

$$\frac{\partial n_0}{\partial t}+\operatorname{div} n_0\mathbf{u}=0, \tag{4.2}$$

$$\frac{\partial u_j}{\partial t}+(\mathbf{u}\nabla)u_j-\frac{e}{m_i}[\mathbf{u}\mathbf{H}_0]_j-\frac{e}{m_i}E_{0j}+\frac{1}{m_in_0}\frac{\partial p_{lj}}{\partial x_l}=\frac{1}{m_i}F_j, \tag{4.3}$$

where

$$\mathbf{F}\equiv-\pi\frac{e^2}{m_in_0}\sum_{\mathbf{k}}\mathbf{k}|\psi_{\mathbf{k}}|^2\int d\mathbf{v}\left(\mathbf{k}\frac{\partial f_{0i}}{\partial \mathbf{v}}\right)\delta(\omega-\mathbf{kv}),$$

$$p_{lj}\equiv m_i\int d\mathbf{v}(v_l-u_l)(v_j-u_j)f_{0i},\qquad n_0=\int f_{0i}\,d\mathbf{v},\qquad n_0\mathbf{u}=\int \mathbf{v}f_{0i}\,d\mathbf{v}.$$

Thus, Eq. (4.3) differs from the equation of motion of the ions in the absence of instability by the presence of the turbulent friction force $\mathbf{F}$ in the right-hand side. The presence of this force should naturally influence the damping of the current in the plasma. However, this problem is not investigated here because it calls for knowledge of the external fields exciting the current in the plasma and the geometry of the system. We therefore give only an estimate of the effective collision frequency ν_{eff} of the ions with the turbulent field fluctuations (3.6)

$$\nu_{eff}\equiv\frac{|\mathbf{F}|}{m_iu}\sim\omega_{Hi}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{u}{v_{Ti}}. \tag{4.4}$$

To estimate the heating of the ions, Eq. (4.1) should be written in the reference system where the average velocity of the ions vanishes. Writing $\mathbf{v} = \mathbf{u} + \mathbf{v}'$, we obtain the following equation for f_{0i} in terms of the variables $\mathbf{v}'$, $\mathbf{r}$ and t:

$$\frac{\partial f_{0i}}{\partial t}+(\mathbf{v}'+\mathbf{u})\frac{\partial f_{0i}}{\partial \mathbf{r}}+\frac{\partial f_{0i}}{\partial v_j'}\left\{-(\mathbf{v}'\nabla)u_j+\frac{e}{m_ic}[\mathbf{v}'\mathbf{H}_0]\right.$$
$$\left.+\frac{1}{m_in_0}\frac{\partial p_{lj}}{\partial x_l}-\frac{1}{m_i}F_j\right\}=\frac{\partial}{\partial v_i'}D_{ij}(\mathbf{v}'+u)\frac{\partial f_{0i}}{\partial v_j'}. \tag{4.5}$$

Recognizing that

$$|(\mathbf{v}'\nabla)\mathbf{u}|\sim\varkappa_nv_{Ti}u\ll v_{Ti}\omega_{Hi},\quad (m_in_0)^{-1}\cdot|\partial p_{lj}/\partial x_l|\sim\varkappa_nv_{Ti}^2\ll v_{Ti}\omega_{Hi},$$

we observe that the term with the magnetic field in the curly brackets in Eq. (4.5) is decisive. Neglecting then the small terms in Eq. (4.5) (i.e., neglecting the diffusion of ions in ordinary space), we obtain

$$\frac{\partial f_{0i}}{\partial t}-\omega_{Hi}\frac{\partial f_{0i}}{\partial \Phi}=\frac{\partial}{\partial v_i'}D_{ij}(\mathbf{v}'+\mathbf{u})\frac{\partial f_{0i}}{\partial v_j'}+\frac{1}{m_i}\mathbf{F}\frac{\partial f_{0i}}{\partial \mathbf{v}'}. \tag{4.6}$$

It is easy to verify that at the turbulence level (3.6) the term $\omega_{Hi}\partial f_{0i}/\partial\Phi$ in Eq. (4.6) is the main one. Therefore the quantity f_{0i} can be represented in the form $f_{0i} = \bar{f}_{0i} + \tilde{f}_{0i}$, where $\bar{f}_{0i}$ does not depend on the azimuthal angle Φ in velocity space. Averaging then with respect to the angle Φ and neglecting the contribution of $\tilde{f}_{0i}$ in the right-hand side, we obtain the following equation for the distribution function $\bar{f}_{0i}$:

$$\frac{\partial \bar{f}_{0i}}{\partial t}=\frac{v_{Ti}^2}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp\nu_i\frac{\partial \bar{f}_{0i}}{\partial v_\perp}\right), \tag{4.7}$$

where

$$\nu_i=\frac{e}{m_i}\frac{1}{v_\perp{}^3v_{Ti}^2}\sum_{\mathbf{k}}\frac{(\omega-\mathbf{ku})^2}{k_\perp}|\psi_{\mathbf{k}}|^2.$$

Consequently, for the noise level given by (3.6) the order of magnitude of ν_i is

$$\nu_i\sim\omega_{Hi}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{u^4}{v_{Ti}^4}, \tag{4.8}$$

and for the heating rate of the ions we obtain the estimate

$$\frac{dT_i}{dt}\sim\omega_{Hi}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{u}{v_{Ti}}\right)^2m_iu^2 \tag{4.9}$$

In concluding this section, we stop to discuss the diffusion of particles in ordinary space and the associated cooling of the plasma. For a plasma cylinder placed in an external magnetic field one can employ the model equation (see, e.g., [5])

$$\frac{\partial \bar{f}_{0i}}{\partial t}=\frac{v_{Ti}}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp\nu_i\frac{\partial \bar{f}_{0i}}{\partial v_\perp}+\frac{1}{r}\frac{\partial}{\partial r}r\nu_i\frac{v_{Ti}^4}{u^2\omega_{Hi}^2}\frac{\partial \bar{f}_{0i}}{\partial r}. \tag{4.10}$$

Recognizing now that $\partial/\partial r \sim 1/r \sim \kappa_n$, we easily verify that the rate of decrease of the ion temperature due to spatial diffusion of the particles is smaller by a factor $(u/u_n)^2$ than the rate of ion heating due to particle diffusion in velocity space. In conclusion, we notice that the heating of ions obviously continues until the drift velocity $u_n \sim v_{Ti}$ becomes comparable to the current velocity u. However, we indicated already that the given theory becomes inapplicable in that case.

5. STOCHASTIC HEATING

Let us now consider the possibility of stochastic heating in the strong turbulence regime examined above. The ideas underlying stochastic heating[10] also apply in the case of instabilities of a plasma with a perpendicular current[11]. Therefore, it makes sense to estimate the rate of stochastic heating of electrons and ions, although this problem seems problematical to us.

Since the correlation time τ of the Fourier components of the potential $\psi_{\mathbf{k}}$ for ions is finite, one may use the modified equations (4.1), (4.3), (4.7) of the quasilinear theory, in which the quantities $\mathbf{F}$ and ν_i are of the following form:[3)]

$$\mathbf{F}=-\frac{e^2}{\sqrt{\pi}\,m_in_0}\sum_{\mathbf{k}}\mathbf{k}|\psi_{\mathbf{k}}|^2\int d\mathbf{v}\left(\mathbf{k}\frac{\partial f_{0i}}{\partial \mathbf{v}'}\right)\tau\mathscr{E}(\tau),$$

$$\nu_i=\left(\frac{e}{m_i}\right)^2\frac{1}{\sqrt{\pi}\,v_{Ti}^2v_\perp^2}\sum_{\mathbf{k}}|\psi_{\mathbf{k}}|^2\langle(\mathbf{kv}')^2\tau\mathscr{E}(\tau)\rangle_{\text{av}}, \tag{5.1}$$

where $\mathscr{E}(\tau) = \exp[-(\omega - \mathbf{k}\cdot\mathbf{u} - \mathbf{k}\cdot\mathbf{v}')^2\tau^2]$, and the brackets $\langle\rangle_{\text{av}}$ mean averaging over the angle Φ. It is reasonable to determine the quantity τ from the following considerations. At the noise level (3.6), the oscillations are in the strongly nonlinear regime in which the frequency and the growth rate depend essentially on the amplitude of the waves, but their order of magnitude remains determined by the equations of the linear theory. In that case the time dependence of the potential ψ becomes obviously so complicated that the oscillations "forget" their own history within a time of the order of the period of the oscillations, i.e., $\tau \sim 1/\omega$. But then the quantities $\mathbf{F}$ and ν_i are of the same order as the corresponding estimates obtained from the quasilinear theory. Hence, the change of the ion temperature is estimated by (4.9) also in the case of stochastic heating.

For the description of stochastic heating of the electrons we use the method developed in[11]. For the background electron distribution functions f_{0e} we then obtain the equation (see Appendix II)

$$\frac{\partial f_{0e}}{\partial t}=\frac{\partial}{\partial v_z}D\frac{\partial f_{0e}}{\partial v_z}, \quad (5.2)$$

where

$$D=\frac{e}{m_e}\left\langle\left(E_z+\frac{E_\perp H_\perp}{H_0}\right)u^\sim\right\rangle,$$

$$\frac{\partial u^\sim}{\partial t}+\frac{e}{m_e\omega_{He}}\left(E_y\frac{\partial u^\sim}{\partial x}-E_x\frac{\partial u^\sim}{\partial y}\right)=\frac{e}{m_e}\left(E_z+\frac{E_\perp H_\perp}{H_0}\right). \quad (5.3)$$

At the turbulence level (3.6) the nonlinear terms of (5.3) are of the same order as the linear ones. Therefore the velocity $u^\sim$ is connected with the intensity of the electromagnetic fields **E** and **H** in a complicated manner. For that reason it seems impossible to calculate the diffusion coefficient D, and moreover one cannot even determine its sign. Under these conditions it still makes sense to estimate the order of magnitude in the following manner:

$$D\sim\frac{e}{m_e}\left|E_z+\frac{E_\perp H_\perp}{H_0}\right||u^\sim|\sim\frac{u^3}{v_{Ti}}(\omega_{He}\omega_{Hi})^{1/2},$$

so that we obtain for the increase of the longitudinal electron temperature $T_\parallel$ the estimate

$$\frac{dT_\parallel}{dt}\sim\frac{u}{v_{Ti}}\left(\frac{m_e}{m_i}\right)^{1/2}\omega_{Hi}m_iu^2. \quad (5.4)$$

This relation allows us to estimate the maximum possible rate of electron heating, if that heating takes place at all. From the comparison of Eqs. (4.9) and (5.4) it follows that the longitudinal electron temperature will not increase faster than the perpendicular ion temperature.

Finally, the authors express their gratitude to K. N. Stepanov for valuable advice.

APPENDIX I

For the derivation of the dispersion relation it is convenient to employ the method proposed by Mikhaĭlovskiĭ[12]. For this we express the fields **E** and **H** in terms of scalar and vector potentials according to the relations

$$\mathbf{E}=-\nabla\psi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\to-i\mathbf{k}\psi+\frac{i\omega}{c}\mathbf{A},$$

$$\mathbf{H}=\operatorname{rot}\mathbf{A}\to i[\mathbf{kA}],$$

where the potential **A** is subjected to the Coulomb gauge condition (div $\mathbf{A}\to i\mathbf{k}\cdot\mathbf{A}\approx i\mathbf{k}_\perp\cdot\mathbf{A}_\perp=0$). The perpendicular components $\mathbf{A}_\perp$ can be characterized by a single quantity $A_\perp$ defined by the relation

$$A_\perp=\frac{1}{k_\perp}[\mathbf{k}_\perp\mathbf{A}]_z,\qquad A_x=-\frac{k_y}{k_\perp}A_\perp,\qquad A_y=\frac{k_x}{k_\perp}A_\perp.$$

In that case the charge density depends on ψ, $A_\perp$, and A_z, i.e., $\rho=\rho(\psi)+\rho(A_\perp)+\rho(A_z)$, and Poisson's equation takes the form

$$\varepsilon_{00}\psi=\frac{4\pi}{k^2}[\rho(A_\perp)+\rho(A_z)]. \quad (I.1)$$

From the Maxwell equations we obtain the two missing equations:

$$k^2A_\perp=\frac{4\pi}{c}j_\perp+\frac{\omega^2}{c^2}A_\perp, \quad (I.2)$$

$$k^2A_z=\frac{4\pi}{c}j_z+\frac{\omega^2}{c^2}A_z, \quad (I.3)$$

where $j_\perp=\mathbf{k}_\perp\times\mathbf{j}_z/k_\perp$. Further, using the relations

$$\{\rho(A_\perp),\rho(A_z)\}=\frac{k_\perp\omega}{4\pi c}\{\varepsilon_{02}A_\perp,\varepsilon_{03}A_z\},$$

$$j_\perp=\frac{\omega}{4\pi}\left\{-k\varepsilon_{20}\psi+\frac{\omega}{c}(\varepsilon_{22}-1)A_\perp+\frac{\omega}{c}\varepsilon_{23}A_z\right\},$$

$$j_z=\frac{\omega}{4\pi}\left\{-k\varepsilon_{30}\psi+\frac{\omega}{c}\varepsilon_{32}A_\perp+\frac{\omega}{c}(\varepsilon_{33}-1)A_z\right\}, \quad (I.4)$$

where $\epsilon_{ij}=\delta_{ij}+\epsilon_{ij}^e+\epsilon_{ij}^i$, we obtain from the (I.1)–(I.3) the dispersion equation in the local approximation (see, e.g.,[12]):

$$\begin{vmatrix}\varepsilon_{00} & \varepsilon_{02} & \varepsilon_{03}\\ \varepsilon_{20} & \varepsilon_{22}-N^2 & \varepsilon_{23}\\ \varepsilon_{30} & \varepsilon_{32} & \varepsilon_{33}-N^2\end{vmatrix}=0. \quad (I.5)$$

where $N=kc/\omega$. Using the relations (2.6), we find the electron contribution to the expression for the current $\mathbf{j}_e=-en_0\mathbf{v}$ and the charge density $\rho_e=-en'$. Next, substituting these expressions in (I.1)–(I.4), we easily find the components ϵ_{ij}^e:

$$\varepsilon_{00}^e=\frac{\omega_{pe}^2}{\omega_{He}^2}-\frac{k_x(\varkappa_H-\varkappa_n)\omega_{pe}^2}{k^2\omega\omega_{He}}-\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta,$$

$$\varepsilon_{02}^e=-\delta\varepsilon_{20}^e=-i\frac{\omega_{pe}^2}{\omega\omega_{He}},\qquad \varepsilon_{03}^e=\varepsilon_{30}^e=-\frac{\omega_{pe}^2}{\omega^2}\cos\theta,$$

$$\varepsilon_{22}^e=\omega_{pe}^2/\omega_{He}^2,\quad \varepsilon_{23}^e=\varepsilon_{32}^e=0,\quad \varepsilon_{33}=-\omega_{pe}^2/\omega^2,$$

$$\varepsilon_{00}^i=\frac{\omega_{pi}^2}{k^2v_{Ti}^2}\left[1+i\sqrt{\pi}z_i\left(1+\frac{k_xu_n}{\omega-\mathbf{ku}}\right)\right],$$

$$z_i=\frac{\omega-\mathbf{ku}}{\sqrt{2}kv_{Ti}}.$$

APPENDIX II

For simplicity, we discard in the Vlasov equation for the electron distribution function f_e the term with the external electric field, which is responsible for the presence of an average electron velocity. We then obtain

$$\frac{\partial f_e}{\partial t}+\mathbf{v}\frac{\partial f_e}{\partial\mathbf{r}}+\omega_{He}\frac{\partial f_e}{\partial\Phi}-\frac{e}{m_e}\left(\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f_e}{\partial\mathbf{v}}=0. \quad (II.1)$$

We represent the function f_e in the form

$$f_e=F+\tilde{F},\quad |\tilde{F}|\ll F,$$

where F does not depend on the angle Φ and $\tilde{F}$ vanishes after averaging over Φ. Recognizing now that ω, $kv_{Te}\ll\omega_{He}$, we expand the function $\tilde{F}$ in powers of ω_{He}^{-1}:

$$\tilde{F}=F_0+F_1+\ldots,$$

$$F_0=\hat{l}_1F\sin\Phi+\hat{l}_2F\cos\Phi,$$

$$F_1=\frac{1}{\omega_{He}}\frac{\partial}{\partial t}(\hat{l}_1\cos\Phi-\hat{l}_2\sin\Phi)F+\ldots,$$

where

$$\hat{l}_1=-\frac{1}{\omega_{He}}\left[v_\perp\frac{\partial}{\partial x}-\frac{e}{m_e}E_x\frac{\partial}{\partial v_\perp}+\omega_{He}\frac{H_y}{H_0}\left(v_z\frac{\partial}{\partial v_\perp}-v_\perp\frac{\partial}{\partial v_z}\right)\right],$$

$$\hat{l}_2=\frac{1}{\omega_{He}}\left[v_\perp\frac{\partial}{\partial y}-\frac{e}{m_e}E_y\frac{\partial}{\partial v_\perp}-\omega_{He}\frac{H_x}{H_0}\left(v_z\frac{\partial}{\partial v_\perp}-v\frac{\partial}{\partial v_z}\right)\right].$$

We have omitted here from the expression for F_1 the terms proportional to cos 2Φ and sin 2Φ, which give no contribution to the equations for F and the current density. Averaging Eq. (II-1) over Φ and using the found expressions for F_0 and F_1, we obtain the following equation for F:

$$\frac{\partial F}{\partial t}+v_z\frac{\partial F}{\partial z}+\frac{e}{2m_e}\frac{v_\perp}{\omega_{He}^2}\frac{\partial}{\partial t}\operatorname{div}\mathbf{E}\frac{\partial F}{\partial v_\perp}-\frac{e}{m_e}\frac{\partial F}{\partial v_z}\left[E_z+\frac{E_\perp H_\perp}{H_0}+\frac{v_\perp^2}{2c\omega_{He}}\frac{\partial H_z}{\partial t}\right]+\frac{e}{m_e\omega_{He}}\left[\nabla F,\left(\mathbf{E}+\frac{1}{c}[\mathbf{vH}]\right)\right]_z+\frac{H_z}{H_0}(\nabla_\perp F\mathbf{v}_\perp)$$
$$+\frac{v_\perp}{2H_0}\frac{\partial F}{\partial v_\perp}\left(\frac{\partial H_z}{\partial t}+v_z\frac{\partial H_z}{\partial z}\right)-\frac{\varkappa_Hv_\perp}{2}\hat{l}_1F=0. \quad (II.2)$$

In the derivation of this equation we have discarded from the expressions proportional to ω_{He}^{-2} the terms proportional to cos θ and sin θ which make smaller contributions than the retained ones.

Resolving now the function F into a background part

f_{0e} and an oscillatory part $\tilde{f}_e$ ($|\tilde{f}_e| \ll f_{0e}$), we obtain from Eq. (II.2) by the method employed in the quasi-linear theory

$$\frac{\partial \tilde{f}_e}{\partial t} - \frac{e}{m_e}\left(E_z + \frac{\mathbf{E}_\perp \mathbf{H}_\perp}{H_0}\right)\frac{\partial f_{0e}}{\partial v_z} - \frac{e}{m_e}\frac{1}{\omega_{He}}\left(\frac{\partial \tilde{f}_e}{\partial y}E_x - \frac{\partial \tilde{f}_e}{\partial x}E_y\right) = 0, \quad \text{(II.3)}$$

$$\frac{\partial f_{0e}}{\partial t} - \frac{e}{m_e}\frac{\partial}{\partial v_z}\left\langle\left(E_z + \frac{\mathbf{E}_\perp \mathbf{H}_\perp}{H_0}\right)\tilde{f}_e\right\rangle = 0 \quad \text{(II.4)}$$

where $\langle\ \rangle$ indicates averaging over the rapid oscillations. It should be stressed that in Eq. (II.3) we have discarded also small terms $\sim \kappa_H$, κ_n, $\cos\theta$, which make no contributions after substitution of $\tilde{f}_e$ in Eq. (II.4) (however, they are essential for the derivation of the dispersion equation!).

One easily observes now that Eq. (II.3) admits of the solution

$$\tilde{f}_e = \frac{\partial f_{0e}}{\partial v_z} u^{\sim}(\mathbf{r}, t),$$

where $u^{\sim}(\mathbf{r}, t)$ satisfies the equation

$$\frac{\partial u^{\sim}}{\partial t} + \frac{e}{m_e \omega_{He}}\left(\frac{\partial u^{\sim}}{\partial x}E_y - \frac{\partial u^{\sim}}{\partial y}E_x\right) = \frac{e}{m_e}\left(E_z + \frac{\mathbf{E}_\perp \mathbf{H}_\perp}{H_0}\right).$$

Eq. (II.4) then takes the form (5.2).

*$[\mathbf{v}, \mathbf{H}_0 + \mathbf{H}] \equiv \mathbf{v} \times (\mathbf{H}_0 + \mathbf{H})$.

1) On leave of absence from FOM-Instituut voor Plasma-Fysica, Jutphaas, The Netherlands.

2) The results of the present work and of [6] can be used only if $u \gg u_n$ because at $u \sim u_n$ the inequality $\gamma \gg \omega_{Hi}$ is not satisfied in practice. However, in the case $\omega \sim k v_{Ti} \gg \omega_{Hi}$ and $\gamma \leqslant \omega_{Hi}$ the influence of the magnetic field on the ions can not be neglected (see, e.g., [9]).

3) We use here an auto-correlation function R(t) of the form $R(t) = \exp(-t^2/\tau^2)$.

[1] A. B. Mikhaĭlovskiĭ and V. S. Tsypin, ZhETF Pis. Red. **3**, 247 (1966) [JETP Lett. **3**, 158 (1966)].

[2] V. L. Sizonenko and K. N. Stepanov, Nucl. Fusion **7**, 131 (1967).

[3] V. L. Sizonenko, K. N. Stepanov, and I. Teichmann, 2-eme Colloque Intern. sur les interact. entre les champs oscillants et les plasmas, Saclay, 15–19 January, **5**, 77 (1968).

[4] V. L. Sizonenko and K. N. Stepanov, Preprint FTI UKr. Ac. Sci. No. 218, Kharkov (1968); Zh. Eksp. Teor. Fiz. **56**, 316 (1969) [Sov. Phys.-JETP **29**, 174 (1969)].

[5] V. I. Aref'ev, I. A. Kovan, and L. I. Rudakov, ZhETF Pis. Red. **7**, 286 (1968) [JETP Lett. **7**, 223 (1968)].

[6] A. Hirose and I. Alexeff, Phys. Rev. Lett. **28**, 1176 (1972).

[7] A. B. Mikhaĭlovskiĭ, Zh. Tekh. Fiz. **37**, 1365 (1967) [Sov. Phys.-Tech. Phys. **12**, 993 (1968)].

[8] V. V. Parail, Zh. Tekh. Fiz. **41**, 485 (1971) [Sov. Phys.-Tech. Phys. **16**, 376 (1971)].

[9] V. L. Sizonenko and K. N. Stepanov, UKr. Fiz. Zh. **16**, 936 (1971).

[10] F. G. Bass, Ya. B. Faĭnberg, and V. D. Shapiro, Zh. Eksp. Teor. Fiz. **49**, 329 (1965) [Sov. Phys.-JETP **22**, 230 (1960)].

[11] V. V. Demchenko, V. L. Sizonenko, and K. N. Stepanov, Preprint FTI UKr. Ac. Sci., No. 70-62 (1970) T 16602; Nucl. Fusion **11**, 245 (1971).

[12] A. B. Mikhaĭlovskiĭ, in: Voprosy Teorii plasmy **3**, Atomizdat (1963) p. 141 [Rev. of Plasma Physics **3**, Consultants Bureau, New York, 1967, p. 159].

Translated by J. P. Goedbloed
223