Polarization kinetics of particles in storage rings

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(Submitted September 22, 1972)

Zh. Eksp. Teor. Fiz. **64**, 1918-1929 (June 1973)

A closed description of the radiative kinetics of the polarization of charged particles with arbitrary spin and magnetic moment is presented, and takes spin-orbit coupling into account. The analysis is based on an investigation of the dynamics of spin motion in inhomogeneous fields [8,10]. For ultrarelativistic electrons (positrons) the paper combines the results of a number of investigations [1-8] and contains some new effects due to spin-orbit coupling in an inhomogeneous field. The time constant and degree of equilibrium polarization of a beam in storage rings with arbitrary fields are found for nonresonant conditions. The method developed in the paper may also be applied when the perturbing electromagnetic field is related to an "external" source.

1. INTRODUCTION

The polarizing action of the radiation of ultrarelativistic electrons (positrons) in a homogeneous magnetic field is well known [1-3]. To explain the real possibilities of obtaining polarized beams in accelerators and storage rings it is necessary to study the dependence of the degree of equilibrium polarization of the beam and the time necessary to establish this polarization on the inhomogeneities of the field that controls the motion of the particle, and on the particle spin. The change produced in the polarization state by radiation in an inhomogeneous field is a process that is in essence much more complicated than in a homogeneous field. The complications are due to the mutual dependence of the spin and orbital motions of the particle and in the absence of a constant direction of equilibrium polarization. In the existing papers on radiative polarization in storage rings, even when taken in their aggregate, there is still no exhaustive description of the polarization kinetics, this they have dealt only with individual aspects of the kinetics, examined separately and by different methods. Thus, in [4-6] they investigated the effects of depolarization in an inhomogeneous field due to the scattering of the particle trajectories by quantum fluctuations of synchrotron radiation. When spin resonances are approached, this effect becomes much stronger as a result of the increased spin-orbit coupling.

In [6-8] they obtained the average rate of change of the spin vector during radiation-formation times that are small in comparison with the characteristic periods of the motion in the leading field. An equation of this type gives an idea of the "instantaneous" character of the process, but to solve the problem of the behavior of the polarization over long periods of time in an inhomogeneous field it must also be supplemented by the equations of the orbital motion, with allowance for the spin dependence of the radiative deceleration force. Usually the relaxation times determined by the radiative processes are large in comparison with the characteristic periods of motion in the field of the storage ring. Under these conditions a consistent approach, which makes it possible to take into account all the essential effects, is to find the average rate of change and diffusion of the integrals of motion in the external field under the influence of the radiation. A nontrivial factor is in this case the determination of the quasiclassical stationary states of the particle with spin in an inhomogeneous field (of the alternating actions and phases in the operator method). A special role is played here by a concept, essentially introduced already in [5], of a moving quantization axis, the spin projection on which in the stationary state is a quantum number. This physical characteristic makes possible a closed and compact description of the kinetics of the polarization in inhomogeneous fields.

In an investigation of the interaction of a particle with an electromagnetic field we start, unlike many authors [1-3,6-8] who used the equations of quantum theory, with the classical equations of motion of a charged particle with arbitrary spin and magnetic moment and the equations of the radiation field. The quantum description is then derived by replacing the classical quantities (the spin vector and the field variables) with the corresponding operators in accordance with the usual rules. Such an approach, for all its lucidity and simplicity, is at the same time fully self-consistent and sufficient to consider radiation effects in classical external field, provided, as is usually the case, that the energy of the characteristic emitted quanta is low with the energy of the particle.

2. HAMILTONIAN AND EQUATIONS OF MOTION

We derive a Hamiltonian describing the quasiclassical motion of a particle with spin **s** and magnetic moment q**s** (q is the gyromagnetic ratio) in an electromagnetic field. There is a known classical Hamiltonian of a particle + radiation system neglecting spin interaction (we put throughout c = 1 for the speed of light):

$$\mathscr{H}_0 = \sqrt{(\mathbf{P} - e\mathbf{A})^2 + m^2} + eA^0 + \mathscr{H}_{rad} + e(\hat{A}^0 - \mathbf{v}\hat{\mathbf{A}}), \tag{2.1}$$

where $\mathscr{H}_{\text{rad}}$ is the Hamiltonian of the free wave field, A^i (i = 0, 1, 2, 3) is the potential of the external electromagnetic field **E** and **H**, and $\hat{A}^i$ is the potential of the radiation field $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$. The motion of the spin is described by the Bargmann-Michel-Telegdi equation for the spin vector in the rest system of the particle [8-11]:

$$\dot{\mathbf{s}} = [\mathbf{W}_\Sigma \mathbf{s}], \quad \mathbf{W}_\Sigma = \mathbf{W} + \mathbf{w},$$
$$\mathbf{W} = \left(\frac{q_0}{\gamma+1} + q'\right)[\mathbf{v}(\mathbf{E} + [\mathbf{vH}])] - \frac{q}{\gamma}\mathbf{H}_v - \frac{q}{\gamma^2}\mathbf{H}_{tr}, \qquad (2.2)^*$$

w pertains to the radiation field (it is expressed in terms of $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$ just as **W** is expressed in terms of **E**, **H**). Here $q_0 \equiv e/m$ and $q' = q - q_0$ is the anomalous part of q, $\gamma = (1 - \mathbf{v}^2)^{-1/2}$, $\mathbf{v} = \dot{\mathbf{r}}$ is the particle velocity; $\mathbf{H}_v = (\mathbf{H}\cdot\mathbf{v})\mathbf{v}/\mathbf{v}^2$, and $\mathbf{H}_{tr} = \mathbf{H} - \mathbf{H}_v$.

From (2.1) and (2.2) we obtain uniquely an expression for the Hamiltonian of a particle with spin in a classical field, with accuracy linear in the spin[1)]:

$$\mathscr{H} = \mathscr{H}_0 + \mathbf{W}_\Sigma \mathbf{s} \equiv \mathscr{H}_{ext} + \mathscr{H}_{rad} + V, \quad (2.3)$$

where

$$\mathscr{H}_{ext} = \sqrt{(\mathbf{P} - e\mathbf{A})^2 + m^2} + eA^0 + \mathbf{W}\mathbf{s} \quad (2.4)$$

is the Hamiltonian of the particle in an external field, and $V = e(\hat{A}^0 - \mathbf{v} \cdot \mathbf{A}) + \mathbf{w} \circ \mathbf{s}$ is the Hamiltonian of the interaction with the radiation field. Indeed, only this form of the spin dependence leads to Eq. (2.2).

The Hamiltonian (2.3) determines also the spin dependence of the acceleration of the particle in an electromagnetic field

$$\dot{\mathbf{p}} = m\frac{d}{dt}\gamma\mathbf{v} = e(\mathbf{E}_\Sigma + [\mathbf{v}\mathbf{H}_\Sigma]) + \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}}(\mathbf{W}_\Sigma \mathbf{s}) - \frac{\partial}{\partial \mathbf{r}}\mathbf{W}_\Sigma \mathbf{s}.$$

The equations of the radiation field (Maxwell's equations) can be written in canonical form

$$\dot{c}_\mathbf{k} = \frac{\partial c_\mathbf{k}}{\partial t} + \{\mathscr{H}, c_\mathbf{k}\} = \{V, c_\mathbf{k}\}, \quad (2.5)$$

where $c_\mathbf{k}$ and $c_\mathbf{k}^*$ are the canonical field variables—the amplitudes of the plane waves in the expansion of the radiation field potential $\hat{A}^i$, and $\{,\}$ are Poisson brackets. The quantum generalization of the Hamiltonian and of the equations of motion is by replacing the classical variables by operators with known commutation relations.

Inclusion of the particle interaction with the radiation in accordance with the scheme employed here is valid if the characteristic radiated quanta are soft: $\hbar\omega_{rad} \ll \gamma m$. The error in the obtained results will be of the order of $\hbar\omega_{rad}/\gamma m$ (for an ultrarelativistic particle we have $\omega_{rad} \sim \gamma^3|\dot{\mathbf{v}}|$). Under real conditions, this accuracy is perfectly satisfactory.

3. MOTION IN AN EXTERNAL FIELD

In this paper we confine ourselves to a situation in which the distortion of the particle trajectory and of the spin under the influence of radiation is small within the characteristic periods of motion in an external field. Under these conditions, the kinetic process will be determined by the average (over the phases of the dynamic motion) rate of change and diffusion of the action variables in the external field, and the distribution over the phases can be regarded as uniform[2)].

The action and phase variables of a particle with spin can be obtained if one knows the corresponding variables $I_0^\lambda(\mathbf{p}, \mathbf{r})$ and $\Phi_0^\lambda(\mathbf{p}, \mathbf{r})$ of the orbital motion, neglecting the spin degree of freedom and the motion of the spin on the given trajectory. Let $\boldsymbol{\xi}_\alpha(\mathbf{p}, \mathbf{r}, t)$ be some three orthonormal vectors satisfying Eq. (2.2) with w = 0 on a trajectory $\mathbf{p}(t)$, $\mathbf{r}(t)$ which is not perturbed by the spin-orbit coupling:

$$\dot{\boldsymbol{\xi}}_\alpha = [\mathbf{W}(\mathbf{p}(t), \mathbf{r}(t))\boldsymbol{\xi}_\alpha]. \quad (3.1)$$

The orthonormality on the trajectory is preserved by virtue of the obvious property of the solutions (3.1):

$$d(\boldsymbol{\xi}_\alpha \boldsymbol{\xi}_{\alpha'}) / dt = 0.$$

The Hamiltonian (2.4) takes, in terms of the variables I_0^λ and Φ_0^λ, the form

$$\mathscr{H}_{ext} = \mathscr{H}_{ext}^0(I_0^\lambda) + \mathbf{W}(I_0^\lambda, \Phi_0^\lambda)\mathbf{s}. \quad (3.2)$$

By solving the Hamiltonians of the equation for I_0^λ and Φ_0^λ by perturbation theory, we obtain the corrected canonical action and phase variables of the orbital motion:

$$I^\lambda = I_0^\lambda - {}^1/_2 \mathbf{s} \sum_{\alpha=1}^{3} [\boldsymbol{\xi}_\alpha \{\boldsymbol{\xi}_\alpha, I_0^\lambda\}], \quad (3.3)$$

$$\Phi^\lambda = \Phi_0^\lambda - {}^1/_2 \mathbf{s} \sum_{\alpha=1}^{3} [\boldsymbol{\xi}_\alpha \{\boldsymbol{\xi}_\alpha, \Phi_0^\lambda\}]. \quad (3.4)$$

Formulas (3.3) and (3.4) are easily generalized to an arbitrary function of I_0^λ and Φ_0^λ (of $\mathbf{p}$ and $\mathbf{r}$). In particular, for the coordinate of the particle as a function of the time we obtain the expression

$$r^\beta(t) = r_0^\beta(I^\lambda, \Phi^\lambda) + \frac{1}{2}\mathbf{s}\sum_{\alpha=1}^{3}\left[\boldsymbol{\xi}_\alpha, \frac{\partial}{\partial p_\beta}\boldsymbol{\xi}_\alpha\right],$$

where $\mathbf{r}_0(I^\lambda, \Phi^\lambda)$ corresponds to motion along the average trajectory, and the correction takes the spin modulation into account.

Since the system is conservative[3)], it is easy to find also the spin variables of the action and the phases. With the aid of (3.3) we can rewrite the Hamiltonian (3.2) in the form

$$\mathscr{H}_{ext} = \mathscr{H}_{ext}^0(I^\lambda) + \boldsymbol{\Omega}\mathbf{s}, \quad \boldsymbol{\Omega} = -\frac{1}{2}\sum_{\alpha=1}^{3}\left[\boldsymbol{\xi}_\alpha, \frac{\partial}{\partial t}\boldsymbol{\xi}_\alpha\right]. \quad (3.5)$$

The vector $\boldsymbol{\Omega}$ and its direction $\mathbf{n} = \boldsymbol{\Omega}/\Omega$ satisfy obviously Eq. (3.1). Since $\partial\mathscr{H}_{ext}/\partial t = 0$, $\partial I^\lambda/\partial t = 0$, it follows that $\mathbf{n}$ is a function of only $\mathbf{p}$ and $\mathbf{r}$ and does not depend explicitly on the time, while Ω can depend only on I^λ and has the meaning of the frequency of the spin precession about $\mathbf{n}$.

It is easy to find two other solutions of (3.1), $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ ($\boldsymbol{\eta}\mathbf{n} = 0$, $\mathbf{n}\partial\boldsymbol{\eta}/\partial t = 0$), that are orthogonal to $\mathbf{n}$ (and to each other):

$$\boldsymbol{\eta}(\mathbf{p}, \mathbf{r}, t) \equiv \boldsymbol{\eta}_1 + i\boldsymbol{\eta}_2 = [\mathbf{l}_1(\mathbf{p}, \mathbf{r}) + i\mathbf{l}_2(\mathbf{p}, \mathbf{r})]e^{-i\Omega t} \equiv \mathbf{l}e^{-i\Omega t}.$$

In the basis $(\mathbf{l}_1, \mathbf{l}_2, \mathbf{n})$ the general solution for the spin $\mathbf{s}(t)$ is

$$\mathbf{s}(t) = s_n\mathbf{n} + \mathrm{Re}\, s_\eta \mathbf{l} e^{-i\Omega t} \equiv s_n\mathbf{n} + \sqrt{\mathbf{s}^2 - s_n^2}\,\mathrm{Re}\,\mathbf{l}e^{-i\psi},$$

$$s_n \equiv \mathbf{s}\mathbf{n} = \mathrm{const}, \quad s_\eta \equiv \mathbf{s}\boldsymbol{\eta}^* = \mathrm{const}, \quad \dot{\psi} = \Omega.$$

Thus, the projection $\mathbf{s} \cdot \mathbf{n}(\mathbf{p}, \mathbf{r})$ and the phase ψ of the spin precession about $\mathbf{n}$ (reckoned from $\mathbf{l}_1$) are the spin variables of the action and of the phase. The direction $\mathbf{n}$ plays the role of the quantization axis. In stationary states, I^λ and $\mathbf{s}_n$ are quantum numbers (in this case formula (3.5) defines the energy).

In the simplest case of motion in a homogeneous magnetic field, as can be easily verified, we have

$$\mathbf{n} \to \mathbf{n}(\mathbf{p}) = \mathbf{H}_c / H_c, \quad \mathbf{H}_c = \gamma\mathbf{H}_{tr} + \mathbf{H}_v. \quad (3.6)$$

The vector $\mathbf{n}$ becomes indeterminate only at the resonance points $\Omega = m_\lambda\Omega^\lambda$, when Ω coincides with a whole-number combination of frequencies of the orbital motion $\Omega^\lambda = \partial\mathscr{H}_{ext}/\partial I^\lambda$. We shall show that if $\Omega \neq m_\lambda\Omega^\lambda$ the solution $\mathbf{n}$ is unique, i.e., it does not depend on the choice of $\boldsymbol{\xi}_\alpha$. Let $\mathbf{n}'$ be a vector obtained in formula (3.5) in a basis $\boldsymbol{\xi}'_\alpha \neq \boldsymbol{\xi}_\alpha$. Then for $C = \boldsymbol{\eta}' \cdot \mathbf{n}$ we obtain the equation

$$\frac{\partial C}{\partial t} = -i\Omega C, \quad \frac{\partial C}{\partial t} \equiv \dot{C} - \Omega^\lambda\frac{\partial C}{\partial\Phi_0^\lambda} = -\Omega^\lambda\frac{\partial C}{\partial\Phi_0^\lambda}.$$

Since C should be a periodic function of the constant phases $\Phi_0^\lambda - \Omega^\lambda t$, it follows that if $\Omega \neq m_\lambda\Omega^\lambda$ the last equation has the unique solution C = 0, i.e., $\mathbf{n} = \mathbf{n}'$. At $\Omega = m_\lambda\Omega^\lambda$ any solution of (3.1) does not depend explicitly on the time.

This representation of the spin motion was used earlier in[5] (without a rigorous justification), where explicit expressions for $\mathbf{n}(\mathbf{p}, \mathbf{r})$ were constructed for various specific situations.

4. POLARIZATION KINETICS

In a storage ring the particles move with small deviations along an equilibrium closed orbit on which n is periodic in the generalized particle azimuth θ [10]: $\mathbf{n} \to \mathbf{n}(\mathbf{p}_S, \mathbf{r}_S) = \mathbf{n}_S(\theta) = \mathbf{n}_S(\theta + 2\pi)$. In the nonresonant situation the scatter of $\mathbf{n}$ is also small: $|\mathbf{n} - \mathbf{n}_S| \ll 1$. The average polarization of the particles at a given azimuth θ, owing to phase mixing, is directed along $\mathbf{n}_S(\theta)$: $\langle \mathbf{s} \rangle_\theta = \langle s_n \rangle \mathbf{n}_S(\theta)$. The radiation causes $\langle s_n \rangle$ to vary slowly and approach a certain equilibrium value. Let us obtain the average rate of this change. From the definition of s_n we obtain an expression for the instantaneous velocity $\dot{s}_n$:

$$\dot{s}_n = [\mathbf{ws}]\mathbf{n} + \mathbf{s}(\mathbf{f}\partial / \partial \mathbf{p})\mathbf{n} \equiv [\boldsymbol{\omega}\mathbf{s}]\mathbf{n} \equiv -\operatorname{Im} \omega\eta s_\eta, \tag{4.1}$$

$$\boldsymbol{\omega} = \mathbf{w} - \left[\mathbf{n}, \left(\mathbf{f}\frac{\partial}{\partial \mathbf{p}}\right)\mathbf{n}\right]$$

$$= \frac{[\mathbf{vf}]}{m}\left(\frac{q'}{q_0} + \frac{1}{\gamma+1}\right) - \frac{q}{\gamma}\hat{\mathbf{H}}_v - \frac{q}{\gamma^2}\hat{\mathbf{H}}_{tr} - \left[\mathbf{n}, \left(\mathbf{f}\frac{\partial}{\partial \mathbf{p}}\right)\mathbf{n}\right], \tag{4.2}$$

where $\mathbf{f} = e(\hat{\mathbf{E}} + \mathbf{v} \times \mathbf{H})$. The term $\mathbf{w}$ in (4.2) corresponds to the direct action of the radiation field on the spin, and the second, due to the spin-orbit coupling in the leading field, decreases the perturbation of the particle trajectory.

We begin with a classical treatment of the effect of the radiation. To obtain $\dot{s}_n$ it is then necessary to substitute in $\boldsymbol{\omega}$ the radiation field induced by the particle motion and then average $\boldsymbol{\omega} \cdot \boldsymbol{\eta}$ over the phases of the unperturbed motion. It is then obviously necessary to take into account in $\boldsymbol{\omega}$ only the part of the radiation field which is modulated by the spin frequency Ω.

Using the Poisson bracket formalism, we can write down the expression for ω_{cl} as a function of the induced solutions of the field equations (2.5) in the form

$$\omega_{cl} = \left\{ \int_{-\infty} V_{t'}\, dt', \omega_t \right\}, \tag{4.3}$$

where the Poisson brackets affect only the field variables c_k and c_k^*, and the integral is taken over the unperturbed particle and spin trajectory. The spin modulation of the radiation is caused by the part of the action Hamiltonian V_ψ which depends on the precession phase ψ (in terms of the variables of the actions and the phases). Instead of a direct transformation of V in accordance with formulas (3.3) and (3.4), we can resort to a more brilliant device, which is analogous to that used to obtain the Hamiltonian (2.3). Equation (4.1) in canonical form is equivalent to $\dot{s}_n = \{V, s_n\}$. Since the components $\mathbf{s} \cdot \mathbf{n}$, $\mathbf{s} \cdot \mathbf{l}_1$, and $\mathbf{s} \cdot \mathbf{l}_2$ satisfy (in the linear approximation in the spin) the usual commutation relations ($\{s_\alpha, s_\beta\} = \epsilon_{\alpha\gamma\beta} s_\gamma$), Eq. (4.1) determines V_ψ uniquely: $V_\psi = \boldsymbol{\omega} \cdot \mathbf{s}$. Substituting (4.3) in (4.1) and averaging over the time t, we obtain

$$\overline{\dot{s}_n} = \alpha_-(\mathbf{s}^2 - s_n^2) / \hbar, \tag{4.4}$$

$$\frac{\alpha_-}{\hbar} = \frac{1}{2}\operatorname{Im}\overline{\left\{ (\omega\eta)_t, \int_{-\infty}^{t} (\omega\eta^*)_{t'}\, dt' \right\}} = \frac{i}{4}\int_{-\infty}^{\infty} d\tau\, \overline{\{(\omega\eta)_{t+\tau/2}, (\omega\eta^*)_{t-\tau/2}\}}.$$

With the spin-orbit coupling taken into account, the radiation field can thus act on the polarization not only directly, but also via the trajectory, perturbing the quantization axis. In turn, the spin dependence of the radiation field consists of the spin part of the radiation field of a particle moving along the average trajectory not modulated by the spin motion, and the field of the radiation of the charge on the modulated trajectory (this field is obtained neglecting the spin). Solving (4.4), we obtain the variation of the degree of polarization $\zeta \equiv s_n/s$ with time:

$$\zeta_t = \left[\zeta_0 + \operatorname{th}\left(\frac{\alpha_-}{\hbar} st\right)\right] \Big/ \left[1 + \zeta_0 \operatorname{th}\left(\frac{\alpha_-}{\hbar} st\right)\right].$$

As should be the case, in the classical theory the radiation leads to total polarization (along $\mathbf{n}_S$) within a time on the order $(|\alpha_-|s/\hbar)^{-1}$.

In quantum theory, in addition to the radiation field induced by the particle motion, there is produced a "fluctuating" free field (field in the state of photon vacuum), which leads at $\alpha_- = 0$ to complete depolarization. Since the average value of the field in the vacuum state is equal to zero, its average action appears only in second order in the interaction:

$$(\dot{s}_\eta)_{fl} = -\overline{\langle 0 | \operatorname{Im}(\omega\eta)_t \int_{-\infty}^{t} (\dot{s}_\eta)_{t'}\, dt' \,|\, 0 \rangle} \tag{4.5}$$

(the correction to the frequency Ω can be neglected). Here $\langle 0|\ldots|0\rangle$ denotes averaging over the state of the photon vacuum; the averaging over the quantum state of the particle, in view of the classical character of the orbital motion, reduces to replacement of $\boldsymbol{\omega} \cdot \boldsymbol{\eta}$ as function of the operators $\mathbf{p}_t$ and $\mathbf{r}_t$ by the classical expression.

The term $(\dot{s}_n)_f$ thus describes the influence exerted on the polarization by the quantum fluctuations of the radiation. In the radiation act, simultaneous changes take place in the direction of the spin and of the quantization axis. Formula (4.5) takes into account also the correlation of these effects. The average rate of change of s_n under the influence of the stimulated part of the wave field is obviously of the form (4.4), and the coefficient $\alpha_-/\hbar$, by virtue of the linearity of Maxwell's equation, coincides with the classical value. We thus obtain ultimately

$$\overline{\dot{s}_n} = \hbar^{-1}\alpha_-(\mathbf{s}^2 - s_n^2) - \alpha_+ s_n, \tag{4.6}$$

$$\alpha_\pm = \frac{1}{4}\int_{-\infty}^{\infty} d\tau' \langle 0 | \overline{[(\omega\eta)_{t+\tau/2}, (\omega\eta^*)_{t-\tau/2}]_\pm} | 0 \rangle. \tag{4.7}$$

The brackets $[\,,\,]_\pm$ denote here the anticommutator and the commutator. Equation (4.6) was obtained under the assumption of a relatively small radiation-induced change of the integrals of the spin-orbit motion within the characteristic times of the dynamic motion[4)]

$$|\Omega - m_\lambda \Omega^\lambda| \gg T_r^{-1},$$

where T_r are the times of relaxation (T_r^{-1} are of the order of the decrements of radiative damping) over the spin and orbital degrees of freedom:

$$T_r^{-1} \sim (\dot{\gamma})_{rad} / \gamma \sim \gamma^3 r_0 / R^2$$

($r_0 = e^2/m$, and R is the radius of curvature of the particle trajectory).

We note that in the derivation of (4.6) we actually did not use the particular nature of the perturbing electromagnetic field. Therefore (4.6) can be used to describe the action exerted on the polarization by any classical source of the stochastic field (for example, the residual gas, various types of noise in the storage ring). The brackets $\langle 0|\ldots|0\rangle$ then denote averaging over the source state unperturbed by the particle.

It is easy to establish the quantum-mechanical meaning of the constant coefficients $\alpha_\pm$, by comparing (4.6) with the elementary balance equation for the occupation numbers in the case of spin 1/2: the coefficients $\alpha_\pm$ are the sum and difference of the flip probabilities per unit time, $p_\uparrow \equiv p_{1/2,-1/2}$, $p_\downarrow \equiv p_{-1/2,1/2}$, i.e., $\alpha_\pm = p_\uparrow \pm p_\downarrow$. From (4.7) we see that we always have $\alpha_+ \geq |\alpha_-|$.

For a particle with spin 1/2, Eq. (4.6) gives a complete description of the kinetics of the polarization:

$$\zeta_t = \frac{2}{\hbar} s_n = \frac{\alpha_-}{\alpha_+} + \left[\zeta_0 - \frac{\alpha_-}{\alpha_+}\right] e^{-\alpha_+ t} \to \frac{\alpha_-}{\alpha_+}.$$

For a particle with arbitrary spin $J(s^2 = \hbar^2 J(J+1))$ we can extract from (4.6) an estimate from the spin relaxation time:

$$T \sim [\max(|\alpha_-|J, \alpha_+)]^{-1}.$$

The degree of equilibrium polarization can be obtained from the following simple considerations. The nonzero of the transition probabilities per unit time are obviously expressed in terms of the transition probabilities for a spin 1/2

$$p_{m,m'} \sim |\langle m|\omega s|m'\rangle|^2, \quad s_n|m\rangle = \hbar m|m\rangle,$$
$$p_{m,m-1} = (J+m)(J-m+1)p_\uparrow, \quad p_{m-1,m} = (J+m)(J-m+1)p_\downarrow.$$

In the equilibrium state, the probability flux between neighboring levels is equal to zero (N_m are the occupation numbers):

$$p_{m,m-1}N_{m-1} - p_{m-1,m}N_m = 0,$$

from which we obtain the equilibrium distribution and the degree of equilibrium polarization:

$$N_m \sim (p_\uparrow / p_\downarrow)^m,$$

$$\zeta_{st} = \frac{\Sigma m N_m}{\Sigma N_m} = \left(1 + \frac{1}{2J}\right) \frac{p_\uparrow^{2J+1} + p_\downarrow^{2J+1}}{p_\uparrow^{2J+1} - p_\downarrow^{2J+1}} - \frac{1}{2J} \frac{p_\uparrow + p_\downarrow}{p_\uparrow - p_\downarrow}.$$

Thus, the kinetics of particle polarization with an arbitrary spin is fully determined by coefficients $\alpha_\pm$ that do not depend on the spin.

We note that the direction of the equilibrium polarization (parallel or antiparallel to $\mathbf{n}_S$) is determined by the classical radiation fields (by the sign of α_-). At $|\alpha_-|J \gg \alpha_+$, a degree of equilibrium polarization close to unity is established within a time $|\alpha_- J|^{-1}$. In the opposite case ($|\alpha_-|J \ll \alpha_+$), depolarization takes place within a time α_+^{-1}:

$$|\zeta_{st}| \approx \frac{2}{3}(J+1)\frac{|\alpha_-|}{\alpha_+} \ll 1.$$

In the calculation of the radiative coefficients $\alpha_\pm = p_\uparrow \pm p_\downarrow$ it is convenient to use the formulas for the vacuum mean values of the products of the components of the electromagnetic-field potential $\hat{A}^i$ (see [12]):

$$\langle 0|\hat{A}^i(\mathbf{r}', t')\hat{A}^k(\mathbf{r}t)|0\rangle = \frac{\hbar}{\pi} \frac{g_{ik}}{(t'-t-i0)^2 - |\mathbf{r}'-\mathbf{r}|^2}$$
$$g_{ik} = 0,\ i \neq k; \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1. \qquad (4.8)$$

The mean values of the products of the field components in (4.7) can be obtained by differentiating (4.8) ($\mathbf{H} = \operatorname{curl} \hat{\mathbf{A}}$, $\mathbf{E} = -\partial\hat{\mathbf{A}}/\partial t - \nabla\hat{A}^0$).

As is should be, the classical part of $\dot{s}_n$, which is dissipative in character, is determined by the local characteristics of the trajectory of the particle and of the spin (the residue at $\tau = 0$):

$$\alpha_- = \frac{\pi i}{12} \frac{d^3}{d\tau^3} \tau^4 \overline{\langle 0|(\omega\eta)_{t+\tau/2}(\omega\eta^*)_{t-\tau/2}|0\rangle}_{\tau=0}. \qquad (4.9)$$

The diffusion term $\alpha_+ s_n$ is of pure quantum origin and, generally speaking, is a nonlocal function of the trajectory. An expression for α_+ in terms of elementary functions can be obtained only in several limiting situations. In the case of practical importance, that of ultrarelativistic motion, when the change of the acceleration is relatively small over the length $\sim|\gamma\dot{\mathbf{v}}|^{-1}$ in which the radiation is formed, the change of the acceleration is relatively small, and the integral in α_+ is concentrated in the region $|\tau| \sim |\gamma\dot{\mathbf{v}}|^{-1}$ and it can be calculated by using the expansion (see [7,8])

$$\mathbf{r}_{t+\tau} \approx \mathbf{r}_t + \mathbf{v}\tau + \dot{\mathbf{v}}\tau^2/2 + \ddot{\mathbf{v}}\tau^3/6.$$

5. RADIATIVE POLARIZATION OF PARTICLES WITH ARBITRARY MAGNETIC MOMENTS IN A HOMOGENEOUS MAGNETIC FIELD

The polarization of ultrarelativistic electrons in a homogeneous magnetic field was considered in [1-3]. It is of physical interest to investigate radiative polarization for a particle with an arbitrary value of the anomalous magnetic moment.

At $\gamma \gg 1$ the radiation, as is well known, is concentrated in a cone with angle $\sim\gamma^{-1}$ about the velocity. Taking this circumstance into account, expression (4.2) can be replaced by the approximate one:

$$\omega = q'[\mathbf{v}(\hat{\mathbf{E}} + [\mathbf{v}\hat{\mathbf{H}}])] - \frac{q}{\gamma}\hat{\mathbf{H}}_v - \frac{q}{\gamma^2}\hat{\mathbf{H}}_{tr}.$$

Although the gradient $\partial\mathbf{n}/\partial p_\alpha$ is different from zero also in a homogeneous field (see (3.6)), the corrections that must be introduced in $\boldsymbol{\omega}$ because of the gradient are relativistically small.

In the case of motion across the field we have

$$\mathbf{n}_s = \frac{[\mathbf{v}\dot{\mathbf{v}}]}{v|\dot{\mathbf{v}}|}, \quad \eta = \left(-\frac{\dot{\mathbf{v}}}{|\dot{\mathbf{v}}|} + i\frac{\mathbf{v}}{v}\right)\exp\left(-i\frac{q'}{q_0}|\dot{\mathbf{v}}|t\right).$$

From (4.9) we obtain

$$\alpha_-(x) = -\hbar q_0^2\gamma^5|\dot{\mathbf{v}}|^3(1 + {}^{14}/_3 x + 8x^2 + {}^{23}/_3 x^3 + {}^{10}/_3 x^4 + {}^2/_3 x^5),$$

where $x \equiv q'/q_0$. The expression for α_+, after expanding the trajectory in powers of τ, can be calculated by taking the residues at $\tau = 0$ and $\tau = -i\sqrt{12}|x|/\gamma|\dot{\mathbf{v}}|x$:

$$\alpha_+ = -\alpha_-|x|/x + R,$$
$$R = \hbar q_0^2\gamma^5|\dot{\mathbf{v}}|^3 e^{-\sqrt{12}|x|}\left[\left(-1 - \frac{11}{12}x + \frac{17}{12}x^2 + \frac{13}{24}x^3 - x^4\right)\frac{|x|}{x} + \frac{1}{\sqrt{3}}\left(\frac{15}{8} + \frac{41}{24}x - \frac{115}{48}x^2 - x^3 + \frac{7}{4}x^4\right)\right].$$

The obtained expressions for the coefficients $\alpha_\pm$ in a homogeneous magnetic field are exact if the radiation is quasiclassical. We see that the degree of equilibrium polarization is determined by the ratio q'/q_0 and does not depend on the energy. Let us discuss the main feature of the radiative polarization as a function of the anomalous magnetic moment, confining ourselves to the case of spin 1/2. At $|q'| \ll |q_0|$ we obtain the known result (see [1-3]):

$$T^{-1} = \alpha_+(0) = \frac{5\sqrt{3}}{8}\hbar q_0^2\gamma^5|\dot{\mathbf{v}}|^3, \quad \zeta_{st}(0) = -\frac{8}{5\sqrt{3}} \approx -0.92$$

(the particle is polarized in a direction opposite to $\mathbf{v} \times \dot{\mathbf{v}}$). Further, the polarization does not vanish in the absence of a magnetic moment (q = 0):

$$T^{-1}(-1) = \hbar q_0^2\gamma^5|\dot{\mathbf{v}}|^3\left[\frac{2}{3} + \frac{5}{24}\left(1 + \frac{5}{\sqrt{12}}\right)e^{-\sqrt{12}}\right],$$

$$\zeta_{st}(-1) = \left[1 + \frac{5}{16}\left(1 + \frac{5}{\sqrt{12}}\right)e^{-\sqrt{12}}\right]^{-1} \approx 0.98$$

(the equilibrium polarization is directed along $\mathbf{v}\times\dot{\mathbf{v}}$). Here the effect is due to the spin dependence of the radiation of the charge (the acceleration of the charged particle depend on the proper mechanical angular momentum (spin), even if the magnetic moment is equal to zero).

To the contrary, when $|q| \gg |q_0|$ the particles are fully polarized under the influence of the radiation of the magnetic moment:

$$\alpha_+ = |\alpha_-| = \frac{2}{3}\hbar |q|^5 H^3 \gamma^2, \quad \zeta_{st} = -\frac{q}{|q|}.$$

The non-inertial character of the particle motion is inessential here, and the last result could have been obtained by recalculating the spin-flip probability per unit time from the rest system of the particle[13]:

$$p = \frac{1}{\gamma} p_c = \frac{1}{\gamma}\left(\frac{2}{3}\hbar |q|^5 H_c^3\right) = \alpha_+.$$

At $|q| \sim |q_0|$ the effect is determined by superimposing the action of the spin part of the charge radiation on the magnetic moment.

When x varies from $-\infty$ to $+\infty$, the degree of polarization varies smoothly from $+1$ to -1 (Fig. 1), unlike the nonrelativistic case ($\mathbf{v} = 0$). The equilibrium degree of polarization vanishes at $q' \approx -0.4q_0$. The maximum relaxation time $T_{max} \approx 4.8T(0)$ is reached at $q' \approx -0.5q_0$ (Fig. 2).

6. RADIATIVE POLARIZATION OF ULTRA-RELATIVISTIC ELECTRONS IN HOMOGENEOUS FIELDS

In the case of motion in inhomogeneous fields, the gradients of $\mathbf{n}$ in the longitudinal and transverse directions are generally speaking of the same order. Recognizing that $f_v \sim \gamma f_{tr}$, we obtain the following expression for $\boldsymbol{\omega}$:

$$\omega \approx -\frac{q_0}{\gamma}\hat{\mathbf{H}}_v - \frac{q_0}{\gamma^2}\hat{\mathbf{H}}_{tr} - q_0(\hat{\mathbf{E}}\mathbf{v})\left[\mathbf{n}\frac{\partial \mathbf{n}}{\partial \gamma}\right].$$

In the calculation of the integral with respect to τ in (4.7), in view of the smallness of the anomalous moment of the electron (positron) we can neglect the variation of $\mathbf{n}$ and η over the length of formation of the radiation $|\tau| \sim |\gamma\dot{\mathbf{v}}|^{-1}$ ($|\Delta\mathbf{n}|$, $|\Delta\eta| \ll 1$). Operations that are analogous in all other respects to those of the preceding section yield

$$\alpha_- = -\hbar q_0^2\gamma^5\langle |\dot{\mathbf{v}}|^2[\mathbf{v}\dot{\mathbf{v}}](\mathbf{n} - \gamma\,\partial\mathbf{n}/\partial\gamma)\rangle, \tag{6.1}$$

$$\alpha_+ = \frac{5\sqrt{3}}{8}\hbar q_0^2\gamma^5\left\langle |\dot{\mathbf{v}}|^3\left[1 - \frac{2}{9}(\mathbf{n}\mathbf{v})^2 + \frac{11}{18}\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2\right]\right\rangle. \tag{6.2}$$

The brackets $\langle\ \rangle$ denote here averaging over the azimuth θ and over the ensemble of the particles in the beam.

Thus, within a time α_+^{-1} there is established an equilibrium degree of polarization:

$$\zeta_{st} = -\frac{8}{5\sqrt{3}}\frac{\langle|\dot{\mathbf{v}}|^2[\mathbf{v}\dot{\mathbf{v}}](\mathbf{n} - \gamma\,\partial\mathbf{n}/\partial\gamma)\rangle}{\langle|\dot{\mathbf{v}}|^3[1 - {}^2/_9(\mathbf{n}\mathbf{v})^2 + {}^{11}/_{18}(\gamma\,\partial\mathbf{n}/\partial\gamma)^2]\rangle}. \tag{6.3}$$

Let us discuss the results. In a homogeneous magnetic field

$$\mathbf{n} = [\mathbf{v}\dot{\mathbf{v}}]/|\dot{\mathbf{v}}|,\ \partial\mathbf{n}/\partial\gamma = 0;\ \zeta_{st} = -8/5\sqrt{3}.$$

The influence of the inhomogeneity of the leading field on the polarization process in a storage ring reduces on the whole to a deviation of the direction of the equilibrium polarization $\mathbf{n}$ from the axis $\mathbf{v}\times\dot{\mathbf{v}}$ and to the appearance of effects due to the coupling between the non-equilibrium orbital and spin motions. These effects are taken into account in (6.1)–(6.3) by the terms containing $\partial\mathbf{n}/\partial\gamma$; the remaining terms describe only the direct action of the radiation on the spin. The influence of the deviation of $\mathbf{n}$ on the equilibrium polarization in an arbitrary inhomogeneous field was investigated in[10] on the basis of an expression for the mean averaged rate of change of the spin vector as a result of the radiation; this expression was obtained in[8]. The inclination of $\mathbf{n}$, considered separately, can lead only to a decrease of the equilibrium degree of polarization.

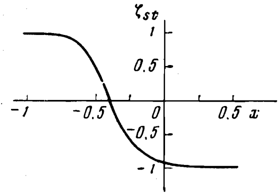

FIG. 1.

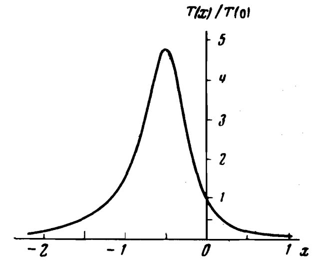

FIG. 2.

FIG. 1. Degree of equillibrium polarization for particles with spin 1/2 in a homogeneous magnetic field as a function of the anomalous moment ($\gamma \gg 1$).

FIG. 2. Dependence of the time of establishment of the equilibrium polarization of particles with spin 1/2 on the anomalous moment in a homogeneous magnetic field ($\gamma \gg 1$).

The term $\sim(\gamma\,\partial\mathbf{n}/\partial\gamma)^2$ in α_+ describes the depolarizing action of the random jumps of the trajectory, which result from quantum fluctuations of synchrotron radiation[5)], an effect considered in[4-6] on the basis of the Bargmann-Michel-Telegdi equation in an external field.

Finally, the term $\sim\partial\mathbf{n}/\partial\gamma$ in α_- takes into account the perturbation of the quantization axis $\mathbf{n}(\mathbf{p}, \mathbf{r})$ by the spin part of the average ("classical") radiation field. This new mechanism whereby radiation acts on the polarization can either decrease or increase the degree of polarization.

A test of (6.3) for its extremum shows that the maximum degree of polarization is attained in an inhomogeneous field when

$$\mathbf{n}_s = -\sqrt{\frac{7}{11}}\frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\dot{\mathbf{v}}|} \pm \sqrt{\frac{4}{11}}\mathbf{v}, \quad \gamma\frac{\partial\mathbf{n}}{\partial\gamma} = \frac{2\sqrt{7}}{11}\left(\sqrt{\frac{4}{11}}\frac{[\mathbf{v}\dot{\mathbf{v}}]}{|\dot{\mathbf{v}}|} \pm \sqrt{\frac{7}{11}}\mathbf{v}\right).$$

Here

$$\zeta_{st}^{max} = \frac{8}{5\sqrt{3}}\frac{9}{\sqrt{77}} \approx 95\%.$$

Such a situation can be realized in practice, for example, in an azimuthally-symmetrical storage ring, by superimposing on the orbit an azimuthally-homogeneous longitudinal magnetic field. A formula that is a simple unification of the results of[1-8] (Eq. (6.3) without the term $\sim\partial\mathbf{n}/\partial\gamma$ in α_-) yields at this point the value $\zeta_{st} \approx 70\%$.

Let us discuss also the qualitative dependence of the degree of polarization on the parameter $|\gamma\partial\mathbf{n}/\partial\gamma|$.[6)] In practice at $|\gamma\partial\mathbf{n}/\partial\gamma| \ll 1$ the direction of $\mathbf{n}_s$ coincides with $\mathbf{v}\times\dot{\mathbf{v}}$ (with the exception of special cases). Then ζ_{st} differs little from the value in a homogeneous magnetic field ($\zeta_{st} \approx 92\%$). If $|\gamma\partial\mathbf{n}/\partial\gamma| \sim 1$ and $\mathbf{n}_s \approx \mathbf{v}\times\dot{\mathbf{v}}/|\dot{\mathbf{v}}|$, then

$$\zeta_{st} \approx \frac{8}{5\sqrt{3}}\left[1 + \frac{11}{18}\left\langle|\dot{\mathbf{v}}|^3\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2\right\rangle\Big/\langle|\dot{\mathbf{v}}|^3\rangle\right]^{-1}.$$

The direction of $\mathbf{n}_s$ may differ from $\mathbf{v}\times\dot{\mathbf{v}}$ when the direction of $\mathbf{W}$ varies significantly along the orbit ($|\Delta(\mathbf{W}/W)|$

~ 1). In this case it is necessary to use the general expressions (6.2) and (6.3). The quantity $|\gamma\partial\mathbf{n}/\partial\gamma|$ increases when the spin resonances are approached. At $|\gamma\partial\mathbf{n}/\partial\gamma| \gg 1$ we have

$$\alpha_+ = \frac{1}{2}\left\langle\left(\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2 \overline{\frac{d}{dt}(\Delta\gamma)^2}\right\rangle, \qquad (6.4)$$

where

$$\frac{1}{2}\overline{\frac{d}{dt}(\Delta\gamma)^2} = \frac{55}{48\sqrt{3}}\hbar q_0^2\gamma^7|\dot{\mathbf{v}}|^3$$

gives the diffusion coefficient γ, and depolarization of the beam takes place within a time α_+^{-1}. An expression for the damping decrement of the beam polarization in the form (6.4) was obtained in[5].

We take the opportunity to thank A. N. Skrinskiĭ for numerous useful discussions during the course of the work and D. V. Pestrikov for help with the numerical calculations.

*$[\mathbf{v}\,\mathbf{H}] \equiv \mathbf{v}\times\mathbf{H}$.

1) For spin 1/2, the Hamiltonian (2.3) can be obtained from the Dirac equation for a particle with an anomalous magnetic moment by means of the Foldy-Wouthuysen transformation with linear accuracy in Planck's constant $\hbar$.

2) The phase mixing usually takes place rapidly (in comparison with the relaxation time), owing to the scatter of the frequencies of the motion. In the absence of a scatter, the phase distribution tends to become uniform in the diffusion process.

3) When the high-frequency field needed to compensate for the radiation losses is turned on, the entire formalism remains unchanged if one adds the phase of the high-frequency field to the variables ($\mathbf{p}, \mathbf{r}$). The resultant explicit dependence of ξ_α on the phase of the high-frequency field can in practice be always neglected.

4) An estimate of the influence of the harmonics for which this condition is violated is given in [5].

5) We note that for electrons ($|q'| \ll |q_0|$) the correlation of the jumps of the spin vector on the quantization axis n during the radiation act is relativistically small (there is no term linear in $\delta n/\delta\gamma$ in α_+).

6) A formula for $\partial n/\partial\gamma$ in an approximation linear in the deviations from the equilibrium trajectory is given in [5]. It is meaningful to take into account [5] terms of higher order in $n - n_s$ only near the resonances $\Omega = m_\lambda\Omega^\lambda$, $\Sigma_\lambda|m_\lambda| > 1$.

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Translated by J. G. Adashko
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