Longitudinal susceptibility and correlations in degenerate systems

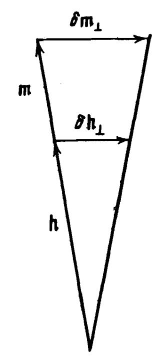
A. Z. Patashinskiĭ and V. L. Pokrovskiĭ
L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
(Submitted November 27, 1972)
Zh. Eksp. Teor. Fiz. 64, 1445-1451 (April 1973)

It is shown that, in physical systems which have a vector order parameter $\boldsymbol{\varphi}$ and are invariant under uniform rotations of φ, a hydrodynamic instability causes the longitudinal susceptibility to become infinite. Strongly developed fluctuations arise, in which the modulus of the vector $\boldsymbol{\varphi}$ is conserved at each point in space. In principle, these fluctuations prevent us from using the self-consistent field method to describe phenomena below the phase-transition point when φ is sufficiently close to its spontaneous value φ_S in the uniform medium. A general expression for the longitudinal susceptibility is found. A relation is established between the longitudinal and transverse correlators. It is shown that the thermodynamic potential Φ is a non-analytic function of φ at the point $\varphi = \varphi_S$, and its form is found for φ close to φ_S. The region of applicability of the Ginzburg-Landau and Ginzburg-Pitaevskiĭ equations is indicated.

In isotropic ferromagnets at temperatures below the transition point, the transverse magnetic susceptibility $\chi_\perp$ becomes infinite in zero magnetic field:

$$\chi_\perp = m/h. \tag{1}$$

Here h is the absolute value of the magnetic field and m is the magnetization. The relation (1) follows from the fact that m depends only on the magnitude h of the magnetic field and not on its direction, while the direction of **m** coincides with the direction of **h** (see the figure). The fact that $\chi_\perp$ becomes infinite when h = 0 means that a state with h = 0 is unstable against infinitesimal transverse spatially uniform fluctuations $\delta m_\perp$ of the moment.



In the work of Vaks, Larkin and Pikin[1], it was shown that the longitudinal susceptibility $\chi_\parallel = \partial m/\partial h$ also tends to infinity as $h \to 0$, in accordance with the law $h^{-1/2}$. The proof presented in[1] was given for the Heisenberg model with long-range interaction. We shall show that the divergence of $\chi_\parallel$ to infinity is not associated with a specific model and is a very general consequence of a hydrodynamic instability intrinsic to a whole class of physical systems. We are referring to systems describable by a multi-component ordering field $\varphi_\alpha(\mathbf{x})$ ($\alpha = 1, 2, \ldots, n$). It is assumed that the Hamiltonian of the system is invariant under rotations in the n-dimensional space (which are independent of the coordinates **x** in real space). An example is the Heisenberg ferromagnet, where the spin **S** plays the role of φ. Degenerate systems include superconductors, where the ordering field is the wavefunction ψ of the superconducting electrons that appears in the Ginzburg-Landau equation, and superfluid helium. In the latter case, the ordering field is a two-component field: $\psi = \psi_1 + i\psi_2$.

We introduce the n-component field **h**, assuming the energy density of the interaction of the system with the field to be $-\mathbf{h}\cdot\boldsymbol{\varphi}$. In the case of a ferromagnet, this is an ordinary magnetic field, while in other cases (superfluid liquid, superconductor) it is not physically realizable. Such a "field" was introduced by Bogolyubov[2] as an auxiliary research tool. For visualizability, in the following we shall speak of the moment and magnetic field.

If the field **h** is equal to zero, the thermodynamic potential Φ of the system depends only on the absolute magnitude φ of the vector $\boldsymbol{\varphi}$, and not on its direction. A change from one direction of $\boldsymbol{\varphi}$ to another does not require the surmounting of an energy barrier, as it does in the case of a non-degenerate system, e.g., the Ising ferromagnet. In degenerate systems, long-wavelength transverse fluctuations of the moment develop and lead to the instability described by formula (1). We shall show that such fluctuations are accompanied by a change of the longitudinal component of the moment. Although the longitudinal fluctuations are considerably weaker than the transverse ones, they are sufficiently strong for $\chi_\parallel$ to become infinite.

First we shall consider a system in a field **h**, with an average moment φ that is not equal to its equilibrium value. The thermodynamic potential Φ of such a system has the form

$$\Phi = f(\boldsymbol{\varphi}^2) - \mathbf{h}\boldsymbol{\varphi}. \tag{2}$$

The equilibrium value of $\boldsymbol{\varphi}$ is found by minimizing Φ:

$$\mathbf{h} = 2\boldsymbol{\varphi} f'(\boldsymbol{\varphi}^2). \tag{3}$$

Small deviations $\delta\boldsymbol{\varphi}$ from the equilibrium value of $\boldsymbol{\varphi}$ lead to a change $\delta\Phi$ of the thermodynamic potential; this change can be represented by the positive-definite quadratic form

$$\delta\Phi = {}^1/_2 \chi_{\alpha\beta}^{-1}\delta\varphi_\alpha\delta\varphi_\beta. \tag{4}$$

Here $\chi_{\alpha\beta}^{-1}$ is the tensor inverse to the susceptibility tensor

$$\chi_{\alpha\beta} = \partial\varphi_\alpha/\partial h_\beta. \tag{5}$$

We separate the longitudinal and transverse components of the tensors:

$$\chi_{\alpha\beta} = \chi_\parallel n_\alpha n_\beta + \chi_\perp(\delta_{\alpha\beta} - n_\alpha n_\beta),$$
$$\chi_{\alpha\beta}^{-1} = \frac{1}{\chi_\parallel} n_\alpha n_\beta + \frac{1}{\chi_\perp}(\delta_{\alpha\beta} - n_\alpha n_\beta), \qquad \mathbf{n} = \frac{\mathbf{h}}{h}. \tag{6}$$

We thereby reduce the quadratic form $\delta\Phi$ to diagonal form:

$$\delta\Phi = \frac{1}{2}\left(\frac{1}{\chi_\perp}\delta\varphi_\perp^2 + \frac{1}{\chi_\parallel}\delta\varphi_\parallel^2\right). \tag{7}$$

The condition for stability of the state is that the quantities $\chi_\perp^{-1}$ and $\chi_\parallel^{-1}$ be positive. It is not difficult to relate the susceptibility to the function f appearing in (2):

$$\chi_\perp^{-1} = 2f'(\varphi^2), \tag{8}$$

$$\chi_\parallel^{-1} = 4\varphi^2 f''(\varphi^2) + 2f'(\varphi^2). \tag{9}$$

We have already cited arguments which entail that $\chi_\perp >> \chi_\parallel$ at small h. This means that $4\varphi^2 f'' >> 2f'$ and, in place of (9), we can write approximately

$$\chi_\parallel^{-1} = 4\varphi^2 f''(\varphi^2). \tag{10}$$

The inequality $\chi_\perp >> \chi_\parallel$, which lies at the basis of the calculation, is confirmed by the result.

We shall consider the change $\delta\Phi$ of the thermodynamic potential (2) to greater accuracy: we shall retain powers of $\delta\varphi_\perp$ up to the fourth:

$$\delta\Phi = \frac{1}{2\chi_\perp}\delta\varphi_\perp^2 + \frac{1}{2\chi_\parallel}\left(\delta\varphi_\parallel + \frac{\delta\varphi_\perp^2}{2\varphi}\right)^2. \tag{11}$$

In the derivation of (11) from (2), the equalities (3), (8) and (10) were used. Formula (11) shows that the smallest increment $\delta\Phi$ for a given magnitude $\delta\varphi_\perp$ is given by a fluctuation in which φ does not change in magnitude, i.e.,

$$\delta\varphi^2 = 2\varphi\delta\varphi_\parallel + \delta\varphi_\perp^2 = 0. \tag{12}$$

Obviously, the equality (12) will also be fulfilled locally in non-uniform fluctuations with long wavelengths. The relation (12) is central to our discussions. Its physical content consists in the fact that, in long "spin" waves, the moment rotates as a whole without changing its length, as was assumed in the derivation of the equation of Landau and Lifshitz (cf., e.g.,[3]). Such a description has been called a hydrodynamic description[4].

The transverse fluctuations in the hydrodynamic approximation are described by the thermodynamic potential

$$\delta\Phi_\perp = \int\left\{\frac{1}{2\chi_\perp}\delta\varphi_\perp^2 + \frac{1}{2}c(\nabla\varphi_\perp)^2\right\}d\mathbf{x}, \tag{13}$$

in which the lowest power of the gradient is taken into account. A correction of higher order in $\delta\varphi_\perp$ contains also higher powers of the gradient. In fact, the modulus φ is assumed unchanged and the energy can depend only on the angles between the vectors $\boldsymbol{\varphi}$ at different points in space. In other words, only the derivatives of the vectors characterizing the direction of $\boldsymbol{\varphi}$, and not these vectors themselves, can appear in the expression for $\delta\Phi_\perp$. Therefore, with asymptotic exactness, the long-wavelength fluctuations do not interact. The constant c has the meaning of the "stiffness" of the system on a non-uniform rotation of φ. This stiffness arises only in the ordered phase. In the disordered phase, the directions of the ordering field φ at points separated by a distance greater than the correlation length are statistically independent. Therefore, fixing certain definite directions at these points does not change the energy of the system. In the case of a superfluid liquid, φc^2 coincides, to within a factor, with the density ρ_S of the superfluid component.

The hydrodynamic approximation (13) makes it possible to obtain the correlator of the transverse fluctuations:

$$\langle\delta\varphi_{\perp\alpha}(\mathbf{x})\delta\varphi_{\perp\beta}(\mathbf{x}')\rangle = (\delta_{\alpha\beta} - n_\alpha n_\beta)G_\perp(\mathbf{x}, \mathbf{x}'), \tag{14}$$

$$G_\perp(\mathbf{x}, \mathbf{x}') = \frac{T}{4\pi c}\frac{\exp(-\varkappa|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}, \quad \varkappa^2 = \frac{1}{c\chi_\perp} = \frac{h}{\varphi c}, \tag{15}$$

or, in the Fourier representation,

$$G_\perp(\mathbf{q}) = \frac{T}{c}\frac{1}{q^2+\varkappa^2}. \tag{16}$$

The hydrodynamic approximation for $G_\perp(\mathbf{q})$ is well known and has been used repeatedly both in concrete calculations[5] and in the study of general questions[4–6].

Using the principle (12) of conservation of the modulus, we can calculate the longitudinal susceptibility. Namely,

$$\chi_\parallel = \frac{\partial(\delta\varphi_\parallel)}{\partial h} = -\frac{1}{2\varphi}\frac{\partial}{\partial h}\langle\delta\varphi_\perp^2\rangle. \tag{17}$$

Differentiating (15) with respect to h and then putting $\mathbf{x} = \mathbf{x}'$, we find

$$\chi_\parallel = T / 8\pi(\varphi c)^{3/2}\sqrt{h}. \tag{18}$$

The result obtained is of a very general character and is valid for any degenerate system in three-dimensional space for sufficiently small h ($\kappa a_0 << 1$, where a_0 is the "interatomic" distance). It follows from (18) and (7) that for h = 0 the degenerate system is in neutral equilibrium not only with respect to transverse but also with respect to longitudinal fluctuations of the moment. From (18), we find by means of a simple integration of equation of state in the hydrodynamic approximation:

$$h = A(\varphi - \varphi_s)^2, \quad A^{-1} = T^2/16\pi^2(\varphi_s c)^3, \quad \varphi \geqslant \varphi_s, \tag{19}$$

where φ_S is the spontaneous moment. One further integration makes it possible to obtain the form of the function $f(\boldsymbol{\varphi}^2)$—the thermodynamic potential in the variables $\boldsymbol{\varphi}$ (cf. formula (2)):

$$f(\varphi^2) = \frac{A}{3}(\varphi - \varphi_s)^3, \qquad \varphi \geqslant \varphi_s. \tag{20}$$

We cannot pass through into the region $\varphi < \varphi_S$ while considering only uniform ordering, since any non-zero field h creates $\varphi > \varphi_S$. On the other hand, the function $f(\varphi^2)$ is defined also for $\varphi < \varphi_S$. It is immediately clear that the function $f(\varphi^2)$ is non-analytic at the point φ_S. In fact, at this point $f(\varphi^2)$ should have a minimum, whereas the analytic continuation of $f(\varphi^2)$ from the side $\varphi > \varphi_S$ does not.

In order to elucidate how the non-analyticity arises and to find $f(\varphi^2)$ for $\varphi < \varphi_S$, we introduce small anisotropy into the thermodynamic potential (2):

$$\Phi = f(\varphi^2) - \mathbf{h}\boldsymbol{\varphi} + \frac{1}{2}\sum_{\alpha=1}^{3}\lambda_\alpha\varphi_\alpha^2.$$

With no loss of generality we can put $\lambda_1 = 0$. We shall assume that $\lambda \equiv \lambda_2 = \lambda_3 > 0$. We direct the magnetic field along the axis 1. Then a small uniform change $\delta\boldsymbol{\varphi}_\perp$ causes a change of the potential Φ:

$$\delta\Phi_\perp = f'(\varphi^2)\delta\varphi_\perp^2 + \frac{1}{2}\lambda\delta\varphi_\perp^2 = \frac{1}{2}\left(\frac{h}{\varphi} + \lambda\right)\delta\varphi_\perp^2.$$

Going over to the variable φ, according to (19) we find

$$\delta\Phi_\perp = \frac{1}{2}\left[\frac{A(\varphi - \varphi)^2}{\varphi} + \lambda\right]\delta\varphi_\perp^2.$$

Thus, the quantity κ occurring in (15) is an analytic function of φ at the point φ_S for all $\lambda > 0$:

$$\varkappa = \left(\frac{A(\varphi - \varphi_s)^2}{\varphi} + \lambda\right)^{1/2},$$

and only in the limit $\lambda = 0$ does non-analyticity appear. It is precisely this non-analyticity which appeared in the discussion of formula (20). Non-analyticities of this type arise at phase-transition points in the limit of infinite volume. The reason why Φ was found to be a non-analytic function of φ far from the critical point is that the line $\varphi = \varphi_S$ (h = 0) is simultaneously a phase-separation curve (a line of first-order phase transitions) and the boundary of stability of the system (the spinodal). For non-degenerate systems, these curves have only one common point—the critical point. We can now formally determine $\chi_{\parallel}$ when $\varphi < \varphi_S$ too:

$$\chi_{\parallel} = \frac{T}{8\pi(\varphi_s c)^2} \lim_{\lambda\to 0} \frac{1}{\varkappa} = \frac{T}{8\pi(\varphi_s c)^{3/2} A^{1/2} |\varphi - \varphi_s|}. \quad (21)$$

Using the relation (10), we find

$$f(\varphi^2) = {}^1/_3 A |\varphi - \varphi_s|^3. \quad (22)$$

As we should expect, the thermodynamic potential Φ has a minimum at $\varphi = \varphi_S$ and is non-analytic at this point. We stress that values $\varphi < \varphi_S$ do not correspond to stable uniform states. Formally, this is manifested in the fact that the absolute magnitude of h, calculated by means of (22), is found to be negative in this region.

We shall consider the correlator of the longitudinal fluctuations

$$G_{\parallel}(\mathbf{x}, \mathbf{x}') = \langle\langle \delta\varphi_{\parallel}(\mathbf{x}) \delta\varphi_{\parallel}(\mathbf{x}') \rangle\rangle. \quad (23)$$

This quantity is of interest in its own right, inasmuch as it enters in the full correlator:

$$G_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \equiv \langle\langle \delta\varphi_\alpha(\mathbf{x}) \delta\varphi_\beta(\mathbf{x}') \rangle\rangle = (\delta_{\alpha\beta} - n_\alpha n_\beta) G_\perp(\mathbf{x}, \mathbf{x}') + n_\alpha n_\beta G_{\parallel}(\mathbf{x}, \mathbf{x}'). \quad (24)$$

We note that, for $\mathbf{h} \to 0$ above the transition point, $G_{\parallel}$ coincides with $G_\perp$, and the correlator $G_{\parallel}(\mathbf{x}, \mathbf{x}')$ tends to a definite limit that is independent of the direction of $\mathbf{h}$. Below the transition point, $G_{\parallel} \neq G_\perp$ even when $\mathbf{h} \to 0$, since the spontaneous moment "remembers" the direction of $\mathbf{h}$. The appearance of two independent correlators in the less symmetric phase is a characteristic feature of the theory of degenerate systems. The description by means of the two Green functions was first introduced by Belyaev[5] in the theory of the Bose liquid and by Gor'kov[7] in the theory of superconductivity.

The principle (12) of the conservation of the modulus φ makes it possible to establish a relation between the correlators of the longitudinal and transverse fluctuations at large distances:

$$G_{\parallel}(\mathbf{x}, \mathbf{x}') = \langle\langle \delta\varphi_{\parallel}(\mathbf{x}) \delta\varphi_{\parallel}(\mathbf{x}') \rangle\rangle = \frac{1}{4\varphi_s^2} \langle\langle \delta\varphi_\perp^2(\mathbf{x}) \delta\varphi_\perp^2(\mathbf{x}') \rangle\rangle. \quad (25)$$

As was remarked in the discussion of formula (13), the transverse fluctuations in the hydrodynamic limit do not interact. This enables us to represent the average of the product of four quantities in the form of a sum of products of pair averages:

$$\langle\langle \delta\varphi_{\perp\alpha}^2(\mathbf{x}) \delta\varphi_{\perp\beta}^2(\mathbf{x}') \rangle\rangle = 2 \langle\langle \delta\varphi_{\perp\alpha}(\mathbf{x}) \delta\varphi_{\perp\beta}(\mathbf{x}') \rangle\rangle^2. \quad (26)$$

Using formula (15) for $G_\perp$ and going over to the Fourier representation, we find

$$G_{\parallel}(\mathbf{q}) = \frac{T^2}{4\pi(\varphi_s c)^2} \frac{1}{q} \operatorname{arc\,tg} \frac{q}{2\varkappa}. \quad (27)$$

We change from the variable h to the variable φ:

$$\varkappa^2 = A(\varphi - \varphi_s)^2 / \varphi_s c. \quad (28)$$

The function $G_{\parallel}(\mathbf{q})$ is not an analytic function of φ at the point $\varphi = \varphi_S$ for any q. It is possible to obtain the expression (21) for the longitudinal susceptibility again, using the relation

$$\chi_{\parallel} = \frac{1}{T} \lim_{q\to 0} G_{\parallel}(q).$$

Thus, in any degenerate system, the susceptibility $\chi_{\parallel}$ should be infinite at h = 0. This requirement is not fulfilled in the Ginzburg-Landau (GL) theory[7]. The GL theory starts from an assumption about the form of the density of the thermodynamic potential Φ:

$$\Phi = {}^1/_2 a|\psi|^2 + {}^1/_4 b|\psi|^4 + {}^1/_2 c|\nabla\psi|^2 - {}^1/_2(h\psi^* + h^*\psi). \quad (29)$$

The last term corresponds to the conventional "magnetic" field, and we shall put h = 0 in the final results.

By the usual method, we find

$$\chi_{\parallel}^{\Gamma\Pi} = \left(\frac{\partial|\psi|}{\partial|h|}\right)_{T,p} = \frac{1}{|h/\psi| + 2b|\psi|^2}. \quad (30)$$

In particular, for h = 0,

$$\chi_{\parallel}^{\Gamma\Pi} = -1/2a \neq \infty. \quad (31)$$

The reason for the contradiction lies in the fundamental premises of the self-consistent field method: the fluctuations are assumed to be negligibly small. At the same time, it is precisely the fluctuations which lead to instability of the state with h = 0, i.e., $\psi = \psi_S$. We emphasize again that the above refers not to a small region about the transition point, but to the whole region $T < T_c$. Since for superconductors h = 0, the fluctuations when $\psi = \psi_S$ cannot be assumed small. Of course, Coulomb forces suppress the fluctuations, as usual. But even without an analysis of the role of the Coulomb forces, it can be shown that the application of the GL equations for superconductors is justified.

The GL theory is applicable for sufficiently large $||\psi| - \psi_S|$, greater than the amplitude of the fluctuations. Comparing $\chi_{\parallel}^{GL}$ with the quantity (18) due to the fluctuations, we find the condition for the applicability of the GL theory:

$$\frac{||\psi| - \psi_s|}{\psi_s} \gg \frac{(Tb)^2}{(4\pi)^2 |a| c^3}. \quad (32)$$

As Ginzburg[8] has shown, the dimensionless parameter which has appeared in the right-hand side of (32) should be small in the region of applicability of the self-consistent field method. It is useful to note that this quantity can be written in the form

$$\frac{(Tb)^2}{|a|c^3} = \frac{\mathrm{Gi}}{\tau}, \qquad \tau = \left|\frac{T - T_c}{T_c}\right|, \quad (33)$$

where Gi is the dimensionless Ginzburg number characterizing the given substance. The condition for the existence of a region of applicability of the Landau theory has the form $\mathrm{Gi} \ll 1$. For superconductors, $\mathrm{Gi} \approx 10^4 (T_c/\epsilon_F)^4 \approx 10^{-12}$.

In real experiments, spatial or temporal non-uniformities of φ always exist. For example, in bulk superconductors in a real magnetic field H, the value of φ is non-uniform in the surface layer because of the Meissner effect. The unusually small value of Gi for a superconductor leads to the result that the criterion (32) is practically always fulfilled. Thus, the GL equations can be applied with confidence to the solution of problems in the theory of superconductivity.

For superfluid helium, Gi is not a small quantity. Therefore, the analog of the GL equations for superfluid helium—the Ginzberg-Pitaevskiĭ equations[9] —have no region of applicability. Attempts have been undertaken[10] to improve the theory in the spirit of the scaling hypothesis—assuming the coefficients a, b and c to be singular functions of τ. As was shown above (formula (14)), such refinements are unsound.

We are grateful to A. I. Larkin, A. M. Polyakov and G. M. Éliashberg for useful discussions.

Note added in proof (March 2, 1973). The expression (18) for the longitudinal susceptibility has been obtained independently by A. A. Migdal and A. M. Polyakov (private communication) by quantum field theory methods.

1V. G. Vaks, A. I. Larkin and S. A. Pikin, Zh. Eksp. Teor. Fiz. **53**, 1089 (1967) [Sov. Phys.-JETP **26**, 647 (1968)].
2N. N. Bogolyubov, Preprint P-1451 (R-1451) Joint Institute for Nuclear Research, Dubna (1961).
3L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Fizmatgiz, M., 1958 [English translation published by Pergamon Press, Oxford, 1960].
4N. N. Bogolyubov, Preprint D-781, Joint Institute for Nuclear Research, Dubna (1961); B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).
5S. T. Belyaev, Zh. Eksp. Teor. Fiz. **34**, 417, 433 (1958) [Sov. Phys.-JETP **7**, 289, 299 (1958)].
6N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).
7V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950) [English translation in Collected Papers of L. D. Landau, Pergamon Press, Oxford, 1965].
8V. L. Ginzburg, Fiz. Tverd. Tela **2**, 2031 (1960) [Sov. Phys.-Solid State **2**, 1824 (1961)].
9V. L. Ginzburg and L. I. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **34**, 1240 (1958) [Sov. Phys.-JETP **7**, 858 (1958)]; L. I. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **35**, 408 (1958) [Sov. Phys.-JETP **8**, 282 (1959)].
10Yu. G. Mamaladze, Zh. Eksp. Teor. Fiz. **52**, 729 (1966) [Sov. Phys.-JETP **25**, 479 (1967)]; D. J. Amit, Phys. Lett. **26A**, 466 (1968); V. A. Slyusarev and M. A. Strzhemechnyĭ, Zh. Eksp. Teor. Fiz. **58**, 1757 (1970) [Sov. Phys.-JETP **31**, 941 (1970)].

Translated by P. J. Shepherd
161