Stochastic heating of plasma during the development of Langmuir turbulence instability

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The nonlinear theory of absorption of Langmuir oscillations during developed Langmuir turbulence is reported. The spectra of strong turbulence of low-frequency pulsations excited by the Langmuir turbulence are found. The heating of electrons produced by the low-frequency oscillations and generated by the Langmuir turbulence is estimated. The calculations are performed on the assumption of one-dimensional motion (magnetization) of electrons, $\omega_{he} \gg \omega_{pe}$.

INTRODUCTION

High-intensity Langmuir oscillations can produce a universal time-independent turbulence spectrum independently of the method used to generate them (see [1-4]). If nonlinear energy transformation plays the leading role in forming the spectrum, then in the region analogous to the inertial region for liquids the spectrum is generated as a result of the balance of nonlinear transfers.

In contrast to liquids, the transformation of turbulent energy occurs from small to large scales (i.e., from large wave numbers k to smaller wave numbers). The oscillations are thus shifted from the region of efficient Landau absorption, and the problem arises of the dispersal of energy or an effective mechanism for the absorption of the oscillations in the small-scale region. This mechanism would appear to be nonlinear and comes into play automatically at a definite stage of the development of turbulence, since the oscillations grow continuously (in the presence of constant sources) at large scales until their energy becomes sufficient to exceed the nonlinear absorption threshold. Of course, if there are sufficiently intensive two-body collisions, the linear absorption may damp the oscillations which build up in the large-scale region, and these may not then exceed the threshold for nonlinear absorption. We shall be interested here in the opposite case.

Effective excitation of low-frequency oscillations, i.e., nonlinear Langmuir turbulence instability, may be a mechanism leading to nonlinear absorption. The simplest form of this kind of instability is the decay instability[5] which involves, in particular, the decay into ion-acoustic oscillations (for small k, i.e., large scales, decay instability is occasionally, and quite inappropriately, referred to as parametric [6]). Langmuir turbulence instability of this kind, with correct allowance for the difference between the acting field and the mean field in the low-frequency region of the excited oscillations, was first obtained by Vedenov et al.[7,8]

$$W / nT > (v_{Te} / v_{ph})^2, \qquad (1)$$

where W is the energy of the Langmuir turbulent oscillations per unit volume of the plasma, nT is the thermal energy of the particles, and v_{ph} is the characteristic phase velocity of the oscillations.

A general theory of turbulence instability in plasma was developed in [9], where the range of validity of the npn-linear growth ratesfound in [7] was obtained. In particular, it was shown that when $v_{ph} < v_{Te}\sqrt{9m_i/m_e}$, the specific expressions for the growth rates found in [7] were, in fact valid only in nonisothermal plasma with $T_e \gg T_i$ and in a relatively narrow interval of W. It was also shown in [9] that when these criteria were not satisfied, the instability did not disappear but substantially modified the growth rates. The various necessary expressions are given in [9]. As a rule, low-frequency instability of the Langmuir oscillations is aperiodic, and this gave rise to definite difficulties when attempts were made to develop a theory of the low-frequency instabilities and to determine their effect on the Langmuir oscillations, which leads to the nonlinear absorption of these oscillations. The aim of the present work was to overcome these difficulties.

In spite of the aperiodic nature of the instability, it is found that the nonlinear stage is reached when the energy of the low-frequency oscillations is much less than the mean thermal energy of the plasma particles:

$$w / nT \ll 1; \qquad (2)$$

where w is the energy of the low-frequency oscillations per unit volume of the plasma. Therefore, in the sense of Eq. (2), the low-frequency turbulence may be referred to as weak. However, the aperiodic character of its excitation ensures that the frequencies ω and wave numbers k of these oscillations are not uniquely related i.e., they do not correspond to any linear plasma oscillation mode. In this respect, the low-frequency turbulence may be referred to as strong. The derivation of Eq. (2) involves the following important small parameters:

$$\omega / \omega_L \ll 1, \; k / k_L \ll 1, \qquad (3)$$

where ω_L is the frequency and k_L the wave number of the Langmuir oscillations, and ω and k are the same quantities for the excited low-frequency oscillations.

The fact that the parameters in Eqs. (2) and (3) are small enables us to overcome the difficulties mentioned previously in the development of the nonlinear theory of the instability which we are considering. In this development we shall start with the statistical description of the low-frequency pulsations. Assuming that the system is ergodic, we may suppose that the results of averaging are the same as those obtained by time and space averaging of scales considerably exceeding the periods and wavelengths of the low-frequency oscillations. Several workers [10,11] have suggested that the development of the above low-frequency instability should lead to the excitation of nonlinear solitones. On the other hand, it is suggested in [7] that the development of the instability leads to the splitting of the plasma into bunches or layers, so that regions of high plasma density alternate with regions with high density of Langmuir oscillations.

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General considerations do not directly suggest that, as assumed in [12], such bunches should become fully isolated from one another. In the statistical description which we are using the averaging is carried out over a range of scales substantially exceeding the scales of the above solitones and bunches which cannot, strictly speaking, be introduced when there is a large number of them and their interaction is sufficiently strong.

We thus have the problem of finding the average effect of the above inhomogeneities which develop as a result of the low-frequency instability. In this sense, we carry out double averaging. The distribution is first divided into two parts

$$f = f^R + f^T, \quad \langle f^T\rangle_H = 0, \quad f^R = \langle f\rangle_H \tag{4}$$

where $\tilde{f}^T$ describes the Langmuir turbulence and $\langle\ \rangle_H$ represents averaging over high frequencies. Secondly, we subdivide

$$f^R = \Phi + \tilde{f}^R, \quad \langle \tilde{f}^R\rangle_L = 0, \quad \Phi = \langle f^R\rangle_L \tag{5}$$

where $\tilde{f}^R$ describes the low-frequency oscillations and $\langle\ \rangle_L$ indicates averaging over low frequencies. The function

$$\Phi = \langle f^R\rangle_L = \langle f\rangle_{H,L} \tag{6}$$

takes into account the average effect of both low- and high-frequency oscillations.

Our aim will be to derive the equation for Φ which is quasilinear in the low-frequency oscillations, i.e., it contains only terms which are linear in w, and there is no expansion over the energy W of the low-frequency oscillations. It is precisely this approach (in which expansion in terms of W) is not used, which has enabled us to develop a general nonlinear theory of low-frequency instability.[9] The resulting equation for Φ is quasilinear in the low-frequency oscillations, and enables us to investigate the time dependence of the average distribution function and to obtain information on the possible heating of plasma due to the above effects.

1. GENERAL EQUATIONS FOR AVERAGING THE DISTRIBUTION FUNCTION

Without loss of generality, and to simplify specific calculations, we can illustrate the general method by considering the simple model of magnetized plasma, which can be described by the drift equations.[9] The present calculations are therefore restricted by the assumption of a strong magnetic field $\omega_{He} = eH/mc \gg \omega_{pe}$, small phase velocities of the low-frequency oscillations $\omega/k \ll \omega_L/k_L$, and nonresonant nature of the high-frequency oscillations $\omega/k \gg v_{Te}$.

To find the equations which are quasilinear in the low-frequency pulsations, it is sufficient to know the response of the distribution $\tilde{f}^R$, which is linear in $\tilde{E}^R$. It is sufficient for this purpose to use the equations given in[9] with $f^{R(1)}$ replaced by $\tilde{f}^{R(1)}$ (in the notation of [9])[1)] and take Φ in the form of Eq. (6). The equation for Φ is obtained in the form

$$\frac{\partial\Phi}{\partial t} + v\frac{\partial\Phi}{\partial z} = I_1 + I_2, \tag{7}$$

$$\begin{aligned} I_1 &= \frac{e}{m_e}\left\langle E^R \frac{\partial \tilde{f}^{R(1)}}{\partial v}\right\rangle_L \\ &= \frac{e}{m_e}\frac{\partial}{\partial v}\int dk\,dk'\,d\omega\,d\omega'\,\langle E^R_{k',\omega'} f^{R(1)}_{k,\omega}\rangle_L \end{aligned} \tag{8}$$

$$\begin{aligned} I_2 &= \frac{e}{m_e}\left\langle E^{T(1)}\frac{\partial f^{T(1)}}{\partial v}\right\rangle_{HL} \\ &= \frac{e}{m_e}\frac{\partial}{\partial v}\left\langle \int dk_1'\,dk_1''\,d\omega_1'\,d\omega_1''\,\langle E^{T(1)}_{k_1'',\omega_1''} f^{T(1)}_{k_1',\omega_1'}\rangle_H\right\rangle_L. \end{aligned} \tag{9}$$

The first quasilinear integral I_1 is analogous to the usual integral, whilst the second I_2 is due to the perturbation of the high-frequency fields by the low-frequency fields, and is usually absent. We shall henceforth omit the subscripts z on E and v; ω, k, ω', k' and ω'', k'' refer to low frequencies and small wave numbers, while ω_1, k_1, ω_1', k_1' and ω_1'', k_1'' refer, correspondingly, to high frequencies and their wave numbers of the order of those which correspond to Langmuir pulsations.

For $\tilde{f}^{R(1)}$ we can use the solution found in [9] by transforming it to the form

$$f^{R(1)}_{k,\omega} = i\frac{e}{m_e}\frac{E^R_{k,\omega}}{1+\beta_{k,\omega}}\left(\frac{1}{\omega - kv}\frac{\partial\Phi}{\partial v} + \frac{\beta_{k,\omega}}{k}\frac{\partial^2\Phi}{\partial v^2}\right), \quad \omega < k_i v_{Ti} \tag{10}$$

where

$$\beta_{k,\omega} = \alpha\frac{kC_{k,\omega}}{n_0}, \quad C_{k,\omega} = \int\frac{1}{\omega - kv}\frac{\partial\Phi}{\partial v}dv, \quad \alpha = d_1 n_0\Big/\left(1 + \frac{n_0 m_e}{T_e + T_i}d_2\right); \tag{11}$$

$$\begin{aligned} d_1 &= \frac{\omega_{pe}^2}{n_0}\left(\frac{e}{m_e}\right)^2\int\frac{I_{k_1\omega_1}dk_1\,d\omega_1}{\tilde{\Pi}(k-k_1, \omega-\omega_1)\,\omega_1(\omega_1-\omega)^3} \\ d_2 &= \frac{\omega_{pe}^2}{n_0}\left(\frac{e}{m_e}\right)^2\int\frac{I_{k_1,\omega_1}dk_1\,d\omega_1}{\tilde{\Pi}(k-k_1, \omega-\omega_1)\,\omega_1^2(\omega_1-\omega)^2}; \end{aligned} \tag{12}$$

I_{k_1,ω_1}, $\tilde{\Pi}^{-1}$ are, respectively, the correlator and renormalized propagator for the high-frequency fields (see [9]).

$$\langle E^{T(0)}_{k_1,\omega_1} E^{T(0)}_{k_1',\omega_1'}\rangle_H = I_{k_1,\omega_1}\delta(k_1' + k_1)\,\delta(\omega_1' + \omega_1),$$

and the effective temperature T_e is given by

$$T_e = -n_0 m_e/2\int\frac{\partial\Phi}{\partial v^2}dv. \tag{13}$$

In deriving Eq. (10) we did not use the expansion in I_{k_1,ω_1}, and in terms of the type $\Pi(k_1, \omega_1) \approx 0$ we took into account small corrections of the order of I_{k_1,ω_1}. If we introduce the spectral density of low-frequency oscillations defined by

$$\langle E^R_{k,\omega} E^R_{k',\omega'}\rangle_L = 4\pi w_{k,\omega}\delta(k+k')\,\delta(\omega+\omega'), \tag{14}$$

$$w = \int w_{k,\omega}dk\,d\omega \tag{15}$$

and use Eq. (10), we obtain the specific expression for I_1:

$$\begin{aligned} I_1 &= \frac{4\pi e^2}{m_e^2}\frac{\partial}{\partial v}\int i\frac{w_{k,\omega}}{1+\beta_{k,\omega}}\frac{d\omega\,dk}{\omega - kv}\frac{\partial\Phi}{\partial v} \\ &+ \frac{4\pi e^2}{m_e^2}i\int\frac{w_{k,\omega}\beta_{k,\omega}}{k(1+\beta_{k,\omega})}dk\,d\omega\frac{\partial^3\Phi}{\partial v^3}. \end{aligned} \tag{16}$$

It is interesting to note that this expression contains derivatives of Φ of order higher than two. This is the basic difference between this equation and the previous quasilinear equations which in the one-dimensional case do not lead to stationary solutions other than the so-called plateau $\partial\Phi/\partial v = 0$. We emphasize that, in the present case, the main mass of particles—and not merely a small fraction of them—may be in resonance with the low-frequency oscillations. Moreover, the first term in Eq. (16) contains, in addition to the resonance term

$$I_1^R = \frac{4\pi e^2}{m_e^2}\frac{\partial}{\partial v}\int w_{k,\omega}\delta(\omega - kv)\frac{1 + \mathrm{Re}\,\beta_{k,\omega}}{|1+\beta_{k,\omega}|^2}d\omega\,dk\frac{\partial\Phi}{\partial v}, \tag{17}$$

the following nonresonance term:

$$\frac{4\pi e^2}{m_e^2}\frac{\partial}{\partial v}\int\frac{w_{k,\omega}}{\omega - kv}\frac{\mathrm{Im}\,\beta_{k,\omega}}{|1+\beta_{k,\omega}|^2}\frac{\partial\Phi}{\partial v}dk\,d\omega. \tag{18}$$

If low-frequency oscillations have phase velocities ω/k much smaller than the mean electron velocities v_{Te}, then for most of electrons with velocities $v \gg \omega/k$ there remains only the nonresonant interaction described by the sum of Eq. (18) and the second term in Eq. (16):

$$I_1^N \approx -\frac{4\pi e^2}{m_e^2}\frac{\partial}{\partial v}\int \frac{w_{k,\omega}\,\mathrm{Im}\,\beta_{k,\omega}}{k|1+\beta_{k,\omega}|^2}dk\,d\omega\left(\frac{1}{v}\frac{\partial\Phi}{\partial v}+\frac{\partial^2\Phi}{\partial v^2}\right). \qquad (19)$$

In the region of resonant interaction, Eq. (17) can be rewritten in the form

$$I_1^R = \frac{\partial}{\partial v}D(v)\frac{\partial\Phi}{\partial v}, \qquad (20)$$

$$D(v) = \frac{4\pi e^2}{m_e^2}\int w_{kv,k}\frac{1+\mathrm{Re}\,\beta_{kv,k}}{|1+\beta_{kv,k}|^2}dk. \qquad (21)$$

It follows from Eq. (11) that, in general, $\mathrm{Re}\beta$ is greater than $\mathrm{Im}\beta$:

$$\frac{\mathrm{Im}\,\beta}{\mathrm{Re}\,\beta} = \pi\int\delta(\omega-kv)\frac{\partial\Phi}{\partial v}dv \Big/ \frac{2}{k}\int\frac{\partial\Phi}{\partial v^2}dv, \qquad (22)$$

which for the Maxwell distribution is of the order of ω/kv_{Te}. We emphasize, however, that the diffusion coefficients in Eqs. (19) and (20) depend substantially on Φ itself (or, more precisely, the integrals of Φ), and the general quasilinear equation is a relatively complicated integrodifferential equation.

The general form of the second part of the quasilinear collision integral I_2 can be found from the equation for the linear perturbation of the turbulent high-frequency distribution function by the low-frequency field $f^{T(1)}$.[9] After a number of calculations we obtain

$$I_2 = i\frac{e^2}{m_e^2}\frac{\omega_{pe}^4}{n_0(T_e+T_i)}\frac{\partial}{\partial v}\int\frac{w_{k,\omega}dk\,d\omega}{k|1+\beta_{k,\omega}|^2}\omega$$
$$\times\left\{2\beta_{k,\omega}\int\frac{I_{k_1,\omega_1}dk_1\,d\omega_1}{\tilde\Pi(k_1+k,\omega_1+\omega)(\omega_1^2-\omega^2)^2}-\frac{k(T_e+T_i)}{n_0m_e}C_{k,\omega}\right. \qquad (23)$$
$$\left.\times\int\frac{I_{k_1\omega_1}dk_1\,d\omega_1}{\tilde\Pi(k+k_1,\omega+\omega_1)}\frac{3\omega_1^2}{(\omega_1^2-\omega^2)^3}\right\}\frac{\partial}{\partial v}\left(\frac{1}{\omega-kv}\frac{\partial\Phi}{\partial v}+\frac{\beta_{k,\omega}}{k}\frac{\partial^2\Phi}{\partial v^2}\right)$$

When $kv_{Ti} \ll \omega \ll kv_{Te}$ we may substitute

$$\int\frac{I_{k_1\omega_1}d\omega_1\,dk_1}{\Pi(k+k_1,\omega+\omega_1)\omega_1^4} \approx \frac{3\pi}{\omega_{pe}^4}\frac{k^2v_{Te}^2}{\omega^2}W$$

and, correspondingly,

$$I_2 = -i\frac{\omega_{pe}^2T_e}{4n_0^2(T_e+T_i)}\frac{3W}{m_e^2}\frac{\partial}{\partial v}\int\frac{w_{k,\omega}k\,dk\,d\omega}{\omega(1+\beta_{k,\omega})^2}\left\{2\beta_{k,\omega}\right.$$
$$\left.-\frac{3k(T_e+T_i)}{n_0m_e}C_{k,\omega}\right\}\frac{\partial}{\partial v}\left(\frac{1}{\omega-kv}\frac{\partial\Phi}{\partial v}+\frac{\beta_{k,\omega}}{k}\frac{\partial^2\Phi}{\partial v^2}\right). \qquad (24)$$

This result enables us to compare I_1 with I_2. Taking into account the imaginary components of all the terms in Eq. (24), we find that I_2 in Eq. (24) differs from I_1 by the factor $Wkv_{Te}/nT\omega$. Although kv_{Te}/ω cannot be greater than unity, the peak of the spectral distribution $w_{k,\omega}$ occurs at $\omega/kv_{Te} \sim 1$, i.e., $I_2/I_1 \sim W/nT \ll 1$.

The solution of the equation for the resonant particles [Eq. (20] is possible only if we know the correlation functions for the low-frequency oscillations. On the other hand, for the nonresonant particles Eq. (19) gives

$$\Phi^N = \frac{n^N}{v_{max}}\ln\frac{v_{max}}{v}, \qquad v \gg \frac{\omega}{k}, \qquad (25)$$

where v_{max} is the maximum velocity in the particle distribution. Calculation of T_e with the aid of Eqs. (13) and (25) for $\omega/k \ll v_{Te}$ results in a divergence and does not enable us to relate v_{max}^2 with T_e if we do not know the minimum value of velocity which can be found from Φ for resonant particles, i.e., from the correlator $w_{k,\omega}$. In the case of the distribution given by Eq. (25), and sufficiently large $v_{max} \sim v_{Te}$, the fraction of nonresonant particles is small in comparison with the resonant fraction. The problem of the heating of plasma particles thus reduces to the determination of $w_{k,\omega}$.

2. CORRELATION EFFECTS OF LOW-FREQUENCY OSCILLATIONS

Correlation effects determine the specific form of $w_{k,\omega}$ and depend on the nonlinear response of the plasma to the low-frequency perturbations. In the first instance, we must find $\tilde f^{R(2)}$—the change in $\tilde f^R$ which is quadratic in $\tilde E^R$ and is due to the low-frequency pulsations. The general form of this equation is

$$-i(\omega-kv)\tilde f^{R(2)}_{k,\omega}-\frac{e}{m_e}\frac{\partial}{\partial v}\int\left(\tilde E^R_{k_1,\omega_1}\tilde f^{R(1)}_{k_2,\omega_2}-\langle\tilde E^{R(1)}_{k_1,\omega_1}\tilde f^{R(1)}_{k_2,\omega_2}\rangle_{\text{H}}\right)$$
$$\times\delta(\omega-\omega_1-\omega_2)\delta(k-k_1-k_2)dk_1\,dk_2\,d\omega_1\,d\omega_2$$
$$=\frac{e}{m_e}\frac{\partial}{\partial v}\int\left(\langle\tilde E^{T(1)}_{k_1,\omega_1}\tilde f^{T(1)}_{k_2,\omega_2}+E^{T(0)}_{k_1,\omega_1}f^{T(2)}_{k_2,\omega_2}+E^{T(2)}_{k_1,\omega_1}f^{T(0)}_{k_2,\omega_2}\rangle_{\text{B}}\right. \qquad (26)$$
$$\left.-\langle E^{T(1)}_{k_1,\omega_1}f^{T(1)}_{k_2,\omega_2}+E^{T(0)}_{k_1,\omega_1}f^{T(2)}_{k_2,\omega_2}+E^{T(2)}_{k_1,\omega_1}f^{T(0)}_{k_2,\omega_2}\rangle_{\text{B.H}}\right).$$
$$\times\delta(\omega-\omega_1-\omega_2)\delta(k-k_1-k_2)dk_1\,dk_2\,d\omega_1\,d\omega_2.$$

The second term in Eq. (26) corresponds to the standard expression for a nonlinear perturbation. However, the standard expression now leads to a substantially different result because of the modification of the relation between $f^{R(1)}$ and E^R by the high-frequency pulsations. The right-hand side, on the other hand, of Eq. (26) provides a complete description of the effects due to the modulation of high-frequency oscillations by the low-frequency perturbations.

Using the small parameter kv_{Te}/ω and proceeding by analogy with the calculations described in [9], we can set up and solve the equation for

$$\langle E^{T(0)}_{k_1,\omega_1}E^{T(2)}_{k-k_1,\omega-\omega_1}\rangle_{\text{B}}$$

and transform Eq. (26) to the form

$$\tilde f^{R(2)}_{k,\omega} = i\left(\frac{e}{m_e}\right)\frac{1}{\omega-kv}\frac{\partial}{\partial v}\int\left(\tilde E^R_{k_1,\omega_1}\tilde f^{R(1)}_{k_2,\omega_2}-\langle\tilde E^R_{k_1,\omega_1}\tilde f^{R(1)}_{k_2,\omega_2}\rangle_{\text{H}}\right) \qquad (28)$$
$$\times\delta(\omega-\omega_1-\omega_2)\delta(k-k_1-k_2)dk_1\,dk_2\,d\omega_1\,d\omega_2+\frac{\alpha n^{(2)}_{k,\omega}}{\omega-kv}\frac{\partial}{\partial v}(\omega-kv)\frac{\partial\Phi}{\partial v},$$

where α is given by Eq. (11). Using the equation div $E = 4\pi en$ and Eq. (27), we obtain the equation for the correlation functions for the low-frequency pulsations in the form

$$\tilde\Pi(k,\omega)w_{k,\omega} = 2\left(\frac{e}{m_e}\right)^2(1-A_{k,\omega})^2\int\frac{1}{\tilde\Pi(-k,-\omega)}\left|S_{k,\omega;k_1,\omega_1;k-k_1,\omega-\omega_1}\right.$$
$$\left.-\frac{A_{k-k_1;\omega-\omega_1}}{k-k_1}\Sigma_{k,\omega;k_1,\omega_1}\right|^2 w_{k_1,\omega_1}w_{k-k_1,\omega-\omega_1}dk_1\,d\omega_1,$$

where

$$A_{k,\omega} = \beta_{k,\omega}/(1+\beta_{k,\omega}),$$

$$S_{k,\omega;k_1,\omega_1;k-k_1,\omega-\omega_1} = \frac{\omega_{pe}^2}{n_0}\int\frac{dv}{(\omega-kv)(\omega_1-k_1v)}\frac{\partial\Phi/\partial v}{[\omega-\omega_1-(k-k_1)v]},$$

$$\Sigma_{k,\omega;k_1,\omega_1} = \frac{\omega_{pe}^2}{n_0}\int\frac{1}{(\omega-kv)^2}\left[\frac{2k}{\omega-kv}+\frac{k-k_1}{\omega-\omega_1-(k-k_1)v}\right]\frac{\partial\Phi}{\partial v}dv.$$

The nonlinear equation (28) determines both the structure and form of the correlation functions. It is important that the instability itself is aperiodic and the unstable frequencies purely imaginary. At the same time, Eq. (28) must obviously contain real ω and k. Therefore, strictly speaking, it describes pulsations in which there is no unambiguous relationship between ω and k. In this sense, the low-frequency oscillations are strong although, as we shall see, the energy stored in them is less than nT. This energy must be estimated before we can determine the rate of heating of the plasma.

We shall now write down the approximate equation for the correlation functions when the phase velocities of all the low-frequency pulsations satisfy the inequality $v_{Ti} < \omega/k \lesssim v_{Te}$. Specific calculations of the nonlinear currents, $\tilde{\Pi}(k, \omega)$, and the coefficient $A_{k,\,\omega}$ will be carried out for the Maxwellian distribution function substituting

$$\lambda = \frac{\omega}{kv_{Te}}, \qquad \lambda_0^2 = \frac{3W}{4n_0T_e}, \qquad v_{\lambda,k} = w_{\omega,k}\frac{\omega_{pe}^2}{2\pi n_0T_ev_{Te}}. \tag{29}$$

The result is

$$v_{\lambda,k} = \frac{[1+2(\lambda_0/\lambda)^2]^4}{[1+(\lambda_0/\lambda)^2]^2}\frac{1}{\{(m_e/m_i)[1+2(\lambda_0/\lambda)^2]+\lambda^2[1+(\lambda_0/\lambda)^2]\}^2}$$
$$\times\int_0^\infty d\lambda_1\int_\lambda \frac{(\lambda_1-\lambda_2)\lambda^4}{(\lambda-\lambda_2)(\lambda-\lambda_1)}\left\{1+\frac{3}{2}\right.$$
$$\left.\times\frac{\lambda_0^2}{\lambda_1^2\lambda_2^2}\frac{(\lambda_1\lambda_2-\lambda\lambda_1-\lambda\lambda_2-2\lambda_0^2)}{[1+2(\lambda_0/\lambda_1)^2][1+2(\lambda_0/\lambda_2)^2]}\right\}^2 v_{\lambda_1k_1}v_{\lambda_2k_2}\,d\lambda_2; \tag{30}$$
$$k_1 = k\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}, \qquad k_2 = k\frac{\lambda_1-\lambda}{\lambda_1-\lambda_2}.$$

Although Eq. (30) looks complicated at first sight, it can be investigated in special cases which are of particular physical interest.

Since the coefficient of the equation depends on λ, it is readily seen that the most important solution of this equation corresponds to $v_{\lambda,\,k}$ which is independent of k. At the same time, there is one singularity which must be taken into account and, in particular, there is the logarithmic divergence of the integral kernel at $\lambda = \lambda_1$, $\lambda = \lambda_2$ when k_1 and k_2 tend to zero. However, this divergence can readily be removed if we recall that there are no turbulent pulsations with small k and, in particular, $v_{\lambda,\,k} \to 0$ for $k \ll k^*$, where k^* plays the role of the wave number characterizing the main turbulence scale. The main contribution of the integral expression is determined by $k \gg k^*$ and, therefore, the specific form of k^* is of no particular importance.

Now consider the case $\lambda_0 \ll (m_e/m_i)^{1/2}$. When $\lambda \ll \lambda_0$ the main contribution to the integral is due to $\lambda_1 \sim \lambda_2 \sim 1 \gg \lambda_0$. To within the numerical coefficient ξ, which is of the order of unity, we then have

$$v_\lambda \approx 4\lambda^4(m_i/m_e)^2\xi. \tag{31}$$

Equation (31) remains valid when $(m_e/m_i)^{1/2} \gg \lambda \gg \lambda_0$, but the numerical coefficient is reduced by a factor of four. When $\lambda \gg (m_e/m_i)^{1/2}$ we have

$$v_\lambda \approx \text{const} \approx \xi.$$

Consider now the other limiting case, namely, $\lambda_0 \gg (m_e/m_i)^{1/2}$. Then for $\lambda \ll (m_e/m_i)^{1/2}$ we again obtain Eq. (31), and when $\lambda_0 \gg \lambda \gg (m_e/m_i)^{1/2}$ we have $v_\lambda = 16\xi$. Finally, when $\lambda \gg \lambda_0 \gg (m_e/m_i)^{1/2}$ we have $v_\lambda = \xi$.

These results lead to the following estimate for the total energy carried by the low-frequency pulsations:

$$\frac{w}{n_0T_e} = \frac{1}{n_0T_e}\int w_{k,\omega}\,dk\,d\omega = \frac{2\pi v_{Te}^2}{\omega_{pe}^2}\int v_{\lambda,k}\,d\lambda\,k\,dk.$$

Since the maximum value of λ is of the order of unity, we have

$$w/n_0T_e \approx \xi(k_{max}v_{Te}/\omega_{pe})^2. \tag{32}$$

This result can be readily shown to prove the above statement, namely, that the total energy of the low-frequency oscillations is much less than the energy n_0T_e of the plasma particles. We emphasize that it is precisely this fact that enables us to construct a regular theory. In the presence of strong turbulence [no unambiguous relation between ω and k; white noise for $\lambda \gg (m_e/m_i)^{1/2}$]. In fact, since k_{max} for the low-frequency oscillations is, by hypothesis, much less than the value of k for the high-frequency oscillations (the instability of which is the object of our analysis), and for the latter we have obviously $k_{max} \ll w_{pe}/v_{Te}$, it follows that $w/n_0T_e \ll 1$. A more complete estimate of the low-frequency oscillations can be obtained by setting k_{max} in Eq. (32) equal to the maximum value for which the high-frequency instability sets in: $k_{max} = \zeta' k_1$, where $\zeta' \approx \xi(m_eT_i/m_iT_e)^{1/2}$ ($\omega_{max} \approx k_{max}v_{Te} < k_1v_{Ti}$). From the condition

$$W/n_0T_e > 12(k_1v_{Te}/\omega_{pe})^2$$

and when $\omega_{pe}/k_1v_{Te} > 3\sqrt{m_i/m_*} = \omega_{pe}/k_{1*}v_{Te}$, we obtain for the present case

$$\frac{k_{max}v_{Te}}{\omega_{pe}} = \min\left\{\left(\frac{1}{12}\frac{W}{nT}\right)^{1/2};\ \frac{1}{3}\left(\frac{m_e}{m_i}\right)^{1/2}\right\}\zeta'. \tag{33}$$

Hence, it follows that

$$\frac{w}{nT} \approx \min\left(\frac{W}{nT}\frac{1}{10}, \frac{1}{9}\frac{m_e}{m_i}\right)\zeta'^2.$$

Therefore, when the energy of the high-frequency turbulence is sufficiently high, $W/nT > m_e/m_i$, the energy of low-frequency turbulent pulsations is lower by a factor of m_e/m_i. The fact that w is small indicates that there is no "deep" density modulation in the course of stability development.

Let us now estimate the rate of stochastic heating. Since the phase velocities of the low-frequency oscillations cover a broad spectrum of values (white noise up to velocities of the order of the mean thermal velocity of electrons), the efficiency of heating of resonant and nonresonant particles is roughly the same although nonresonant particles are heated to a lesser extent.[2)] Let us therefore estimate, to begin with, the diffusion coefficient for resonant particles:

$$D(v) \approx 2\pi v_{Te}^3k_{max}\left(\frac{\lambda_0^2v_{Te}^2}{v^2}\right)\left(1+\frac{\lambda_0^2v_{Te}^2}{v^2}\right)\bigg/\left(1+\frac{2\lambda_0^2v_{Te}^2}{v^2}\right)^2. \tag{34}$$

The diffusion coefficient for $v \ll \lambda_0v_{Te}$ is constant and proportional to $\lambda_0^2v_{Te}^2/v^2$ for $w \gg \lambda_0v_{Te}$. Therefore, the most effective take-up of energy is found to occur for those electrons whose velocity is less than $v_\sim = \sqrt{3W/4m_e}$, i.e., the amplitude of the electron oscillations in the field of the high-frequency pulsations. The rate of heating of the particles is found to be

$$\frac{d(n_0T_e)}{dt} = \frac{4\pi e^2}{m_e}\int\frac{\omega w_{k,\omega}\,\mathrm{Im}\,C_{k,\omega}}{k|1+\beta_{k,\omega}|^2}dk\,d\omega \approx n_0T_ev_{Te}\int\lambda^2v_\lambda\,d\lambda\,dk \approx$$
$$\approx n_0T_ek_{max}v_{Te}\lambda_{max}^3.$$

If we substitute for k_{max} from Eq. (33), we obtain

$$\frac{1}{nT}\frac{d}{dt}(nT) = \zeta\omega_{pe}\left(\frac{W}{nT}\right)^{1/2}, \qquad \zeta = \zeta'\lambda_{max}^3 \tag{35}$$

where ζ' is the coefficient introduced above, which is much less than unity ($\lambda_{max}^3 \ll 1$). After a long enough period of time (substantially exceeding the instability development time) we have the following expression for the final plasma temperature:

$$nT = \zeta\omega_{pe}tW. \tag{36}$$

3. DISCUSSION OF RESULTS

The results obtained in the present work can be used directly in the interpretation of plasma heating data obtained with Q-machines since the one-dimensional distribution of electrons in plasma is realized under these experimental conditions. The plasma heating effect has been observed, for example, in the experiments of Demirkhanov et al.[13] when the current frequency was equal to the Langmuir frequency and, consequently, magnetized Langmuir oscillations with frequency $\omega_{pe}k_z/k$ should have been observed. The plasma heating effect has also been observed by Astrelin et al.[14] in the course of the plasma-beam interaction in Q-machines.

Let us first note some general points about the mechanism responsible for the sufficiently intensive and time-independent Langmuir turbulence. This time-independent turbulence is established as a result of the generation of waves with high k and their transformation toward smaller k, where the build-up of the oscillations takes place, and if the absorption due to two-body collisions is small the instability is found to appear. According to the present work, this instability leads to plasma heating and, consequently, to the absorption of the oscillations. If we use the estimate given by Eq. (33) for k_{max}, the nonlinear absorption rate becomes

$$\gamma=-\omega_{pe}\zeta\sqrt{W/nT}. \quad (37)$$

This absorption is responsible for the balance of the energy flux and the onset of time-independent turbulence. If Q is the rate of turbulence generation, then the balance is achieved when

$$dT/dt=Q \quad (38)$$

and, consequently,

$$W/nT=(Q/\zeta\omega_{pe}nT)^2.$$

The quantity Q is given in[15] (see Sec. 4.3) under different conditions including, in particular, the two-stream instability. The quantity W in Eq. (37) corresponds either to the energy at the maximum of the spectrum (if such a maximum is present), or to the total energy in the unstable region of high phase velocities when the maximum is absent because of the nonlinear absorption. Since $W/nT > m_e/9m_i$, nonlinear absorption is important for $Q/\omega_{pe}nT > \zeta\sqrt{m_e/m_i}$. Finally, nonlinear absorption ensures that the Langmuir oscillations acquire a correlation width $\Delta\omega$ which is proportional to $\sqrt{W}$.

As far as the experiments of Demirkhanov et al.[13] are concerned, we can carry out the following estimates. Assuming that the heating time cannot exceed the electron flight time over the length L of the installation, we find for $n \sim 10^8 - 10^9$ cm^3, $L \approx 25$ cm, and final temperature $T_e \sim 2$ eV that the plasma heating up to this temperature can be fully described by Eq. (35). To verify this proposition we would have to carry out more detailed measurements on the oscillation level and an accurate calculation of the parameter ζ.

Comparison of the above theory with the experiments of Astrelin et al.[14] on the interaction between beams and calcium plasmas is possible both through checking the connection between the plasma temperature and the level of turbulence, and through the dependence of correlation broadening on W. Equating the heating given by Eq. (35) and the heat flux through the ends of the plasma, as in [14], we find that

$$\frac{W}{2nT}=\frac{E^2}{8\pi nT}=\frac{10}{\zeta}\frac{T[\mathrm{eV}]}{n[\mathrm{cm}^{-3}]} \quad (39)$$

For the two regimes a and b described in [14] the use of Eq. (39) for the same value $\zeta = 10^{-4}$ yields:

a) for $T_e \approx 8$ eV, $n \sim 7\times10^8$ cm^{-3} the energy is $W/2nT \sim 10^{-3}$.

b) for $T_e \approx 20$ eV, $n \sim 2\times10^8$ cm^{-3}, which corresponds to the observed values of $E^2/8\pi nT$, the value $\zeta \sim 10^{-4}$ is quite reasonalbe because ζ contains a number of factors much less than unity (λ^3_{max}, v_{Ti}/v_{Te}, $k_{1\,max}/k_{1*}$). Next, the correlation broadening is also satisfactorily explained by the relation $\Delta\omega \propto \sqrt{W}$. Finally, Astrelin et al.[14] have established the relationship $T_e \propto \overline{\varphi}^{4/3} \propto \varphi^{1,4}$ on the basis of the theory developed in [16]. Similar calculations based on Eq. (39) yield $T_e \propto \overline{\varphi}$. This is in exact agreement with observations [14], which suggests that the interpretation of the experimental data referred to in[14] in terms of the nonlinear absorption mechanism discussed here is to be preferred. However, the final conclusion with regard to the agreement of the data given in[14] with the theory developed here can only be established through an analysis of correlations between the low-frequency oscillations.

1) Superscripts (1) and (2) of the regular and turbulent components correspond to the linear and quadratic coefficients in the expansion in terms of $\tilde{E}^R$.

2) According to Eqs. (19) and (21), the oscillations for phase velocities up to v_{Te} are resonant.

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