Anisotropic solution of the gravitation equations with an isotropic singular hypersurface

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A broad class of anisotropic solutions of the gravitation equations possessing an isotropic singular hypersurface is found in the case of the extremely rigid equation of state $p = \epsilon$. The solution contains seven physically arbitrary functions of the space coordinates. Matter exerts a considerable influence right up to the point of collapse. It is noted that such a class of solutions is contained only in space filled with matter. The energy density increases without limit in the course of the collapse.

1. Broad classes of solutions of equations with a singularity are at present known. For the case when the singular hypersurface is spacelike in nature, Lifshitz and Khalatnikov have constructed a class of solutions containing seven physically different functions.[1] In this paper[2] by Frenkel' and the present author, solutions are found for which the singular hypersurface is isotropic. These solutions are found in the case of central symmetry and contain only one arbitrary function, which is one less than the number necessary for the general centrally symmetric case.

The object of the present paper is to investigate the broadest possible class of solutions with an isotropic singular hypersurface. In contrast to the previous work,[2] we consider the case of the extremely rigid equation of state $p = \epsilon$ (p is the pressure, ϵ is the energy density). This case leads to a simpler mathematical situation. The equation of state $p = \epsilon$ was first investigated by Zel'dovich.[3] As shown in Belinskiĭ and Khalatnikov's paper,[4] this equation of state also arises when, for example, the source of the gravitational field (matter) is a scalar field. The present work differs from Belinskiĭ and Khalatnikov's work[4] in that we are considering the case of an isotropic singular hypersurface, whereas a spacelike singular surface was studied in [4].

The most distinctive property of the solutions, with an isotropic singularity is the fact that the solutions in question exist only in a matter-filled space (in this case the system is not necessarily described by the equation of state $p = \epsilon$), whereas the solutions with a spacelike singularity can exist in free space.[1] Let us note in connection with this property that our solutions are not contained in Lifshitz and Khalatnikov's special class of solutions (Appendices B and F in [1]), since the indicated special class of solutions is also possible in free space (this question is considered in greater detail in Sec. 2). We emphasize that the singularity in [1,2,4] and in the present paper is physical (the corresponding invariants of the Riemann curvature tensor and the matter density become infinite on the singular hypersurface).

In Sec. 2 we construct the anisotropic solution of the Einstein equations with an isotropic singularity (relegating the unnecessary mathematical details to the Appendix) correct to two principal orders in the variable that measures the proximity to the singularity. In Sec. 3 we investigate the more particular case of a centrally symmetric collapse with an isotropic singularity, but in all orders in the variable that indicates the proximity to the collapse. It turns out in this case that the constructed solutions are just as general as the corresponding solutions with the spacelike singularity. Thus, the anisotropic solution contains seven physically different functions of the space coordinates (compare with[1]), while the solution of Sec. 3 contains two physically different functions, i.e., are general for a centrally symmetric collpase.

2. The gravitation equations in a synchronous reference frame take at $p = \epsilon$ the form[5] (1)

$$R_0^0 = -\frac{1}{2}\frac{\partial}{\partial t}(\varkappa_\alpha^\alpha) - \frac{1}{4}\varkappa_\beta^\alpha\varkappa_\alpha^\beta = 2\varepsilon u_0 u^0, \tag{2.1}$$

$$R_\alpha^0 = \frac{1}{2}(\varkappa_{\alpha;\beta}^\beta - \varkappa_{\beta;\alpha}^\beta) = 2\varepsilon u_\alpha u^0, \tag{2.2}$$

$$R_\alpha^\beta = -P_\alpha^\beta - \frac{1}{2\gamma^{1/2}}\frac{\partial}{\partial t}(\gamma^{1/2}\varkappa_\alpha^\beta) = 2\varepsilon u_\alpha u^\beta. \tag{2.3}$$

Since we assume beforehand the isotropic nature of the singular hypersurface, let us choose a system of coordinates (t, x_1, x_2, x_3) such that the singular hypersurface has the simple form $\nu = x_1 - t = 0$.

It is easy to verify that the solutions of Eqs. (2.1), (2.2) and (2.3) are (the details are given in the Appendix)

$$\begin{gathered} ds^2 = dt^2 - (1 + a\nu^{p_1} + \ldots)dx_1^2 - b\nu^{p_2}(1 + \ldots)dx_2^2 \\ - c\nu^{p_3}(1+\ldots)dx_3^2 - 2k\nu^{p_2}(1+\ldots)dx_1dx_2 - 2m\nu^{p_3}(1+\ldots)dx_1dx_3 \\ - 2n\nu^{p_3}(1+\ldots)dx_2dx_3, \end{gathered} \tag{2.4}$$

where a, b, c, k, m, n, as well as p_1, p_2, and p_3 are functions of the space coordinates; furthermore

$$\begin{gathered} 0 < p_1 < p_2 < 1, \quad p_1 < p_3 - p_2, \\ p_1 = 1 - {}^1/_2(p_2 + p_3), \quad {}^1/_2(p_2 + p_3) - {}^1/_4(p_2^2 + p_3^2) = 2Eu^2 > 0. \end{gathered} \tag{2.5}$$

The energy density ϵ and the velocities u_α (α = 1, 2, 3) can also be represented in the form of a power series in ν:

$$\begin{gathered} \varepsilon = E\nu^\gamma(1 + \ldots), \quad u_1 = u\nu^x(1 + \ldots), \\ u_2 = v\nu^y(1 + \ldots), \quad u_2 = w\nu^z(1 + \ldots), \end{gathered} \tag{2.6}$$

where

$$\begin{gathered} \gamma = p_1 - 2, \quad x = -p_1/2, \quad u^2 = 1/a, \\ y > p_2 - p_1/2, \quad z > p_3 - p_1/2. \end{gathered}$$

To compute the quantities v, w, z, and y, we must expand the metric to terms of higher order in ν than those written out in (2.4). We only point out that they are ultimately expressible in terms of the arbitrary functions which figure in the dominant order and which can be chosen to be the functions

$$a,\ b,\ c,\ k,\ m,\ n,\ p_2,\ p_3. \tag{2.7}$$

The region of definition of the arbitrary functions p_2 and

p_3 is determined by the conditions (2.5) and is shown in the figure. As can be seen from the figure (the domain of definition of p_2 and p_3 is the hatched region ABC),

$$0 < p_1 < {}^2/_5, \quad {}^1/_3 < p_2 < 1, \quad {}^4/_5 < p_3 < 1.$$

The singular hypersurface is defined by the equation $\nu = x_1 - t = 0$. As $\nu \to 0$, the length element $ds^2 \to 0$ and, consequently, the hypersurface of collapse is isotropic in nature. The metric (2.4) does not fix the reference frame finally. It can be verified that the function $n(x_1, x_2, x_3)$, for example, can be made to vanish by a suitable choice of the coordinate system. Thus, the constructed solution contains seven physically different functions of the space coordinates, e.g.,

$$a,\ b,\ c,\ k,\ m,\ p_2,\ p_3\,.$$

Notice that the solution (2.4) exists only in a space filled with matter. Indeed, the admissible domain of definition for p_2 and p_3 does not directly touch the circle shown in the figure, whereas in free space the admissible domain of definition is made up of the points of the circle (see (2.5)). The domain of definition of s_1 and s_2 for the Lifshitz-Khalatnikov special solution can be shown in the same figure (for the notation and the solution itself, see Appendix F in [1]). For this purpose let us formally set

$$2s_1 = p_2, \quad 2s_2 = p_3, \quad 2s_3 = p_1, \quad s_3 = {}^1/_2(1 - s_1 - s_2).$$

However, only two cases are possible for this solution: a) $s_1 < s_3 < s_2$ b) $s_1 < s_2 < s_3$. In the case a) the domain of definition of s_1 and s_2 in free space is the minor arc ML; in the presence of matter ($p = \epsilon$), the domain of s_1 and s_2 are bounded by the straight lines MK and KL and the minor arc ML. In the case b), in free space, the domain of definition of s_1 and s_2 is the minor arc OM; in the presence of matter ($p = \epsilon$) the domain of s_1 and s_2 is bounded by the straight lines OK, KM, and the minor arc OM. Thus, in fact, our solution and the Lifshitz-Khalatnikov special solution are different solutions.

3. In the present section we show that in the case of the extremely rigid equation of state $p = \epsilon$ the solutions with an isotropic singular hypersurface form a general class of solutions of the gravitation equations for a centrally symmetric collapse, i.e., they are described by two physically arbitrary functions of the coordinates (in contrast to the case of the ultrarelativistic equation of state $p = \epsilon/3$, where the analogous class of solutions is described by a single function [2]).

The gravitation equations for the present problem have the form[5] (let us write the square of the 4-length element in the form $ds^2 = d\tau^2 - e^{\lambda}dR^2 - e^{\mu}(d\theta^2 + \sin^2\theta d\varphi^2)$).

$$R_0^0 = -{}^1/_4\dot\lambda^2 - {}^1/_2\dot\mu^2 - {}^1/_2\ddot\lambda - \ddot\mu = 2\varepsilon(1 + e^{-\lambda}u_1^2), \tag{3.1}$$

$$R_1^0 = -\dot\mu' + {}^1/_2\dot\lambda\mu' - {}^1/_2\dot\mu\mu' = -2\varepsilon u_1(1 + e^{-\lambda}u_1^2)^{1/2}, \tag{3.2}$$

$$R_1^1 = e^{-\lambda}({}^1/_2\mu'\lambda' - \mu'' - {}^1/_2\mu'^2) + {}^1/_2(\ddot\lambda + \dot\lambda\dot\mu + {}^1/_2\dot\lambda^2) = -2\varepsilon(1 + e^{-\lambda}u_1^2), \tag{3.3}$$

$$R_2^2 = R_3^3 = e^{-\mu} + {}^1/_2e^{-\lambda}({}^1/_2\mu'\lambda' - \mu'' - \mu'^2) + {}^1/_2(\ddot\mu + {}^1/_2\dot\mu\dot\lambda + \dot\mu^2) = 0. \tag{3.4}$$

It is more convenient for us to transform the system (3.1–3.4), eliminating ϵ and u_1. We then obtain

$$R_0^0R_1^1 + e^{-\lambda}(R_1^0)^2 = 0, \tag{3.5}$$

$$R_2^2 = 0, \tag{3.6}$$

$$\varepsilon = \frac{R}{2}, \quad u_0^2 = \frac{R_0^0}{2\varepsilon}, \quad u_1 = \frac{R_1^0}{2\varepsilon u_0}, \quad R = R_i^i \quad (i = 0, 1, 2, 3). \tag{3.7}$$

We seek the solution to the system (3.5)–(3.6) in the form

$$\lambda = \lambda_0(R) + \sum_{n=1}^{\infty} \lambda_n(R)\nu^{\sigma_n},$$
$$\mu = \beta\ln\nu + \sum_{n=1}^{\infty} \mu_n(R)\nu^{\rho_n}, \tag{3.8}$$
$$\lambda_0(R) = 2\ln\tau_0'(R); \quad \beta,\ \rho_n,\ \sigma_n > 0; \quad \nu = \tau_0(R) - \tau.$$

We regard for the present the exponents (β, σ_n, ρ_n) as numbers, although, as follows from the results of the perturbations of the solution (3.8) under consideration, they (β, σ_n, ρ_n) can be functions of the coordinate R. Substituting (3.8) in (3.5)–(3.6), we obtain

$$\lambda_1^2\rho_n\beta\sigma_1({}^1/_2\beta - 1)\mu_n\nu^{\rho_n + 2\sigma_1 - 4} + \lambda_1\sigma_n\beta({}^1/_2\beta - 1) \times(\sigma_n + \beta - 1)\lambda_n\nu^{\sigma_n + \sigma_1 - 4} + \ldots = 0, \tag{3.5'}$$

$$\lambda_1\rho_n({}^1/_2\rho_n - {}^1/_2 + \beta + {}^1/_2\sigma_1)\mu_n\nu^{\rho_n + \sigma_1 - 2} + {}^1/_2\beta(\sigma_n + \beta - 1)\lambda_n\nu^{\sigma_n - 2} + \ldots = 0. \tag{3.6'}$$

The dots stand for higher-order terms.

Equating the exponents, we find that $\rho_n = \sigma_n - \sigma_1$. The determinant of the system (if μ_n and λ_n are considered as the unknowns)

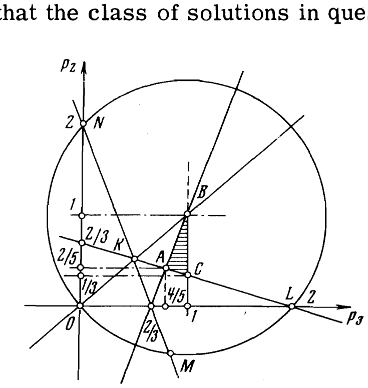
$$D = {}^1/_2\beta(1 - {}^1/_2\beta)(\sigma_n + \beta)(\sigma_n + \beta - 1)^3\lambda_1^2 \tag{3.9}$$

vanishes when $\sigma_n = 1 - \beta$, $\sigma_n = -\beta$, $\beta = 0$, $\beta = 2$; therefore, it is necessary to examine the given values in greater detail. The values $\beta = 0$, $\beta = 2$, and $\sigma_n = -\beta$ can be immediately discarded. Only $\sigma_n = 1 - \beta$ remains. Notice that since in this case $\sigma_n > 0$, then $0 < \beta < 1$. For the values $\sigma_n = 1 - \beta$, we have (for definiteness, we set $n = 1$); $\rho_1 = 0$, $\sigma_1 = 1 - \beta$. Further, μ_1 and λ_1 are arbitrary functions of R. Moreover, $\tau_0(R)$ is also an arbitrary function, while β is an arbitrary number from the interval (0, 1).

Since the determinant D does not vanish when n equals 2, 3, 4, etc, the higher orders (e.g., λ_2, μ_2) are normally expressed in terms of the arbitrary functions μ_1, λ_1, and τ_0. Expressing the energy density and the radial velocity from (3.7) in terms of these functions, we find

$$\varepsilon = {}^1/_2\beta(2 - \beta)\lambda_1\nu^{\sigma_1 - 2} + \ldots, \quad u_1^2 = \frac{\tau_0'^2}{\lambda_1}\nu^{-\sigma_1} + \ldots, \quad \varepsilon \to \infty, \quad u_1 \to \infty \quad \text{as} \quad \nu \to 0. \tag{3.10}$$

It turns out that the class of solutions in question can be



broadened, namely, we can regard the quantity β in the solution

$$\lambda = 2\ln\tau_0' + \lambda_1 \nu^{1-\beta} + \dots,$$
$$\mu = \beta \ln\nu + \mu_1 \nu^0 + \dots \quad (3.11)$$

not as a number, but as an arbitrary function whose values lie in the interval (0, 1).

This is most easily verified by introducing small radial perturbations into the solutions (3.10)—(3.11). Thus, $g_{ik} = g_{ik}^{(0)} + h_{ik}$, where h_{ik} are small perturbations and $g_{ik}^{(0)}$ is the unperturbed metric (3.8). Let us set $h_1^1 = \varphi$ and $h_2^2 + h_3^3 = \chi$ (the raising of indices is accomplished with the aid of the unperturbed metric (3.8)). We indicate right away the solution to the corresponding equations for φ and χ without citing the equations:

$$\varphi = (\varphi_\sigma \nu^\sigma + \dots)\ln\nu + (\tilde{\varphi}_\sigma \nu^\sigma + \dots),$$
$$\chi = (\chi_0 \nu^0 + \dots)\ln\nu + \dots, \quad (3.12)$$

$$\sigma = 1-\beta, \quad \varphi_\sigma = -\tfrac{1}{2}\lambda_2\chi_0, \quad (3.13)$$

where $\tilde{\varphi}_\sigma$ and χ_0 are arbitrary functions of R.

In the solution (3.12)—(3.13) we have eliminated the unphysical perturbations (the "killing integrals") connected with the leeway in the choice of the reference frame.

The result (3.12)—(3.13) can be obtained from (3.8) by perturbing β through the addition of a small arbitrary function $\chi_0(R)/2$. In fact, φ and χ are determined from the expressions

$$e^\lambda \to e^\lambda(1+\varphi), \quad e^\mu \to e^\mu(1+\chi/2).$$

For $\beta \to \beta + \chi_0/2$, from (3.8) follows the solution (3.12)—(3.13). Consequently, β in fact turns out to be an arbitrary function of R with values from the interval $0 < \beta < 1$.

Thus, the constructed class of solutions (3.10)—(3.11) contains four arbitrary functions of R: they are

$$\tau_0(R), \quad \mu_1(R), \quad \lambda_1(R), \quad \beta(R).$$

It is easy to verify that two of them are connected with the arbitrariness in the choice of the coordinate system, while the remaining two are physically arbitrary, i.e., they correspond to the possibility of an arbitrary choice of the energy density and the radial velocity of the matter. For $\nu = \tau_0(R) - \tau \to 0$, the length element $ds^2 \to 0$ and, consequently, the singular hypersurface is isotropic.

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APPENDIX

It is also convenient for us to have a transformed system of equations (2.1)—(2.3), eliminating the energy density ϵ and the velocity of the matter u_α:

$$R_0^0 R_\beta^\alpha - g^{\alpha\gamma} R_\gamma^0 R_\beta^0 = 0, \quad \alpha = 1, 2, 3. \quad (A.1)$$

$$\varepsilon = \frac{R_0^0 + R_\alpha^\alpha}{2}, \quad u_0^2 = \frac{R_0^0}{2\varepsilon}, \quad u_\alpha = \frac{R_\alpha^0}{2\varepsilon u_0}. \quad (A.2)$$

To substantially shorten the computations, let us note the following: let

$$N = \sum c_n(x_1, x_2, x_3)\nu^{p_n},$$

then the derivatives with respect to x_1 can be expressed in terms of the derivatives with respect to t and the quantities

$$\tilde{N} = \sum c_n' \nu^{p_n}, \quad \tilde{\tilde{N}} = \sum c_n'' \nu^{p_n}$$

(here the prime denotes differentiation with respect to x_1). For example, for N″ we have: $N'' = \ddot{N} - 2\dot{\tilde{N}} + N$ (a point denotes differentiation with respect to t).

Noting that $p_2 < 1$ and $p_3 < 1$, and also taking account of the remark made above, we obtain the dominant orders of the quantities p_β^α:

$$P_1^1 = \left(\frac{p_2+p_3}{2} - \frac{p_2^2+p_3^2}{4}\right)\nu^{-2} + \dots, \quad P_2^1 = 0\cdot\nu^{p_2-2} + \dots,$$
$$P_2^2 = \frac{p_2}{2}\left(1 - \frac{p_2+p_3}{2}\right)\nu^{-2} + \dots, \quad P_3^1 = 0\cdot\nu^{p_3-2} + \dots,$$
$$P_3^3 = \frac{p_3}{2}\left(1 - \frac{p_2+p_3}{2}\right)\nu^{-2} + \dots, \quad P_3^2 = 0\cdot\nu^{p_3-p_2-2} + \dots$$

Similarly for the quantities

$$L_\beta^\alpha = -\frac{1}{2\gamma^{1/2}}\frac{\partial}{\partial t}(\gamma^{1/2}\varkappa_\beta^\alpha)$$

(we cite the dominant orders) we have

$$L_1^1 = 0\cdot\nu^{-2} + \dots, \quad L_2^1 = 0\cdot\nu^{p_2-2} + \dots,$$
$$L_2^2 = \frac{p_2}{2}\left(1 - \frac{p_2+p_3}{2}\right)\nu^{-2} + \dots, \quad L_3^1 = 0\cdot\nu^{p_3-2} + \dots,$$
$$L_3^3 = \frac{p_3}{2}\left(1 - \frac{p_2+p_3}{2}\right)\nu^{-2} + \dots, \quad L_3^2 = 0\cdot\nu^{p_3-p_2-2} + \dots.$$

Thus, in the dominant order $R_\beta^\alpha = -p_\beta^\alpha + L_\beta^\alpha$ are

$$R_1^1 = \left(-\frac{p_2+p_3}{2} + \frac{p_2^2+p_3^2}{4}\right)\nu^{-2} + \dots, \quad R_2^1 = 0\cdot\nu^{p_2-2} + \dots,$$
$$R_2^2 = 0\cdot\nu^{-2} + \dots, \quad R_3^3 = 0\cdot\nu^{-2} + \dots, \quad (A.3)$$
$$R_3^1 = 0\cdot\nu^{p_3-2} + \dots, \quad R_3^2 = 0\cdot\nu^{p_3-p_2-2} + \dots.$$

Let us consider further Eqs. (2.1) and (2.2). Simple computations lead to the expressions

$$R_0^0 = -\frac{1}{2}\frac{\partial}{\partial t}(\varkappa_\alpha^\alpha) - \frac{1}{4}\varkappa_\beta^\alpha\varkappa_\alpha^\beta = \left(\frac{p_2+p_3}{2} - \frac{p_2^2+p_3^2}{4}\right)\nu^{-2} + \dots,$$
$$R_1^0 = \frac{1}{2}(\varkappa_{1;\beta}^\beta - \varkappa_{\beta;1}^\beta) = \left(-\frac{p_2+p_3}{2} + \frac{p_2^2+p_3^2}{4}\right)\nu^{-2} + \dots, \quad (A.4)$$
$$R_2^0 = \tfrac{1}{2}(\varkappa_{2;\beta}^\beta - \varkappa_{\beta;2}^\beta) = 0\cdot\nu^{p_2-2} + \dots, \quad R_3^0 = 0\cdot\nu^{p_3-2} + \dots.$$

Substituting (A.3) and (A.4) into (A.1), we verify that the Eqs. (A.1) are satisfied identically in the dominant orders. Let us set

$$\varepsilon = E\nu^\gamma(1+\dots), \quad u_1 = u\nu^x(1+\dots),$$
$$u_2 = v\nu^y(1+\dots), \quad u_3 = w\nu^z(1+\dots).$$

Then, using simultaneously (A.2), (A.4), (2.1), the first of the Eqs. (2.2) (for $\alpha = 1$), as well as the identity

$$u_0^2 = 1 - u_\beta u^\beta,$$

we find the relations

$$x < 0, \quad y > p_2 + x, \quad z > p_3 + x, \quad \alpha + 2x = -2, \quad (A.5)$$

$$\tfrac{1}{2}(p_2+p_3) - \tfrac{1}{4}(p_2^2+p_3^2) = 2Eu^2. \quad (A.6)$$

In order to compute the exponents γ, y, and z, and the coefficients E, u, v, and w, we must take into account terms of higher order in ν than given in the expansion (2.4). Let us point out, without going through the rather tedious computations, that the following inequalities should be fulfilled:

$$p_1 < p_2 < p_3, \quad p_1 < p_3 - p_2. \qquad (A.7)$$

After this let us write out the quantities R^α_β, R^0_0, and R^0_α corrrect to the second order:

$$\begin{aligned}
R_0^0 &= {}^1/_2(p_2 + p_3 - {}^1/_2(p_2^2 + p_3^2))\nu^{-2} - {}^1/_2 p_1(p_1 - 1)a\nu^{p_1-2} + \ldots,\\
R_1^0 &= {}^1/_2(-p_2 - p_3 + {}^1/_2(p_2^2 + p_3^2))\nu^{-2} - {}^1/_4 p_1(p_2 + p_3)a\nu^{p_1-2} + \ldots,\\
R_1^1 &= {}^1/_2(-p_2 - p_3 + {}^1/_2(p_2^2 + p_3^2))\nu^{-2} - {}^1/_2[p_1(p_2 + p_3) + p_1(p_1 - 1)\\
&\quad - p_2 - p_3 + {}^1/_2(p_2^2 + p_3^2)]a\nu^{p_1-2} + \ldots,\\
R_2^2 &= 0\cdot\nu^{-2} + {}^1/_2[-p_1 + 1 - {}^1/_2(p_2 + p_3)]p_2 a\nu^{p_1-2} + \ldots,\\
R_3^3 &= 0\cdot\nu^{-2} + {}^1/_2[-p_1 + 1 - {}^1/_2(p_2 + p_3)]p_3 a\nu^{p_1-2} + \ldots,\\
R_2^1 &= 0\cdot\nu^{p_2-2} + 0\cdot\nu^{p_2-2+p_1} + \ldots,\\
R_3^2 &= 0\cdot\nu^{p_3-p_2-2} + 0\cdot\nu^{p_3-p_2-2+p_1} + \ldots,\\
R_2^0 &= 0\cdot\nu^{p_2-2} + 0\cdot\nu^{p_2-2+p_1} + \ldots.
\end{aligned} \qquad (A.8)$$

As for the quantities R^1_3 and R^0_3, the terms $\sim \nu p_3 - 2 + p_1$ in them turn out to be quantities of the third order in smallness with respect to ν.

After substituting (A.7) into (A.1), we can verify that the equations

$$R_0^0 R_2^2 - g^{2\gamma} R_\gamma^0 R_2^0 = 0, \quad R_0^0 R_3^3 - g^{3\gamma} R_\gamma^0 R_3^0 = 0$$

yield the following condition:

$$[p_1 - 1 + {}^1/_2(p_2 + p_3)]a = 0, \qquad (A.9)$$

while the remaining equations are identically satisfied. Consequently,

$$p_1 = 1 - {}^1/_2(p_2 + p_3), \qquad (A.10)$$

the function a turning out then to be an arbitrary function of the space coordinates. Finally, substituting (A.8) into (A.2), we find

$$E = {}^1/_2(p_2 + p_3 - {}^1/_2(p_2^2 + p_3^2))a, \; u^2 = 1/a, \qquad (A.11)$$

$$\alpha = p_1 - 2, \quad x = -p_1/2. \qquad (A.12)$$

To compute v, w, y, and z, it is necessary to take still higher-order terms into account in the expansion (2.4). We shall not do this here. Let us only point out that they are in the end usually expressed in terms of the arbitrary functions figuring in the dominant order: a, b, c, k, m, n, p_2, and p_3.

[1)]Our notation coincides with that of Landau and Lifshitz [5].

[1]E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk **80**, 391 (1963) [Sov. Phys.-Uspekhi **6**, 495 (1964)], Adv. in Phys. **12**, 208 (1963).

[2]V. K. Pinus and A. L. Frenkel', Zh. Eksp. Teor. Fiz. **64**, 43 (1973).

[3]Ya. B. Zel'dovich and I. D. Novikov, Relyativistskaya astrofizika (Relativistic Astrophysics), Nauka, 1967 (Eng. Transl., University of Chicago Press, Chicago, 1971).

[4]V. A. Belinskiĭ and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **63**, 1121 (1972).

[5]L. D. Landau and E. M. Lifshitz, Teoriya polya (The Classical Theory of Fields), Nauka, 1972 (Eng. Transl., Pergamon Press, New York, 1971).

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