Quantum fluctuations in the photon flux in vacuum and in the diffraction pattern

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Fluctuations in the number $\nu_{\Sigma T}$ of photons passing through an area Σ during a long interval of time T are considered. The mean $\langle\nu_{\Sigma T}\rangle$, the mean square $\langle(\Delta\nu_{\Sigma T})^2\rangle$, and the space correlation functions for $\nu_{\Sigma T}$ are found for a coherent monochromatic source and the for radiation from a thermal source transmitted through a narrow-band filter. For light propagating in free space and when $\Sigma \gg \lambda^2$, the predicted values are in agreement with photon counting. The transverse correlation length for a plane wave is of the order of λ when $\Sigma \ll \lambda^2$ and of the order of $\sqrt{\Sigma}$ when $\Sigma \gg \lambda^2$. The quantities $\langle\nu_{\Sigma T}\rangle$ and $\langle(\Delta\nu_{\Sigma T})^2\rangle$ for a diffraction pattern due to an aperture are calculated in the scalar approximation. The transverse correlation length for $\nu_{\Sigma T}$ turns out to be of the same order as the size of the diffraction lobe and $\langle(\Delta\nu_{\Sigma T})^2\rangle/\langle\nu_{\Sigma T}\rangle < 1$ if Σ is small in comparison with the size of the central diffraction peak.

1. INTRODUCTION

The statistics of photon counting has been discussed in a large number of papers (see, for example, the monographs [1,2]). Most of these are based on the so-called quantum correlation functions which are the mean values of normally ordered operators. This definition of the correlation functions is connected with the fact that it is precisely in this form that they appear when the interaction of radiation with the photodetector medium is considered. By virtue of their definition, therefore, the quantum correlation functions are very convenient for the description of the statistics of photoelectrons or, in other words, the statistics of photon counting.

There is, however, another method suitable for the description of the free electromagnetic field that does not interact with a medium. This is based on studying the operator representing the number of photons which have passed through a given area.

It is well known (see, for example, [3]) that the particle current-density vector cannot be introduced for the electromagnetic field photons. However, it is possible to introduce an operator corresponding to the number of photons which have passed through a given area Σ in an infinite period of time (or a sufficiently long period of time T).[4] This operator is of the form

$$\nu_{\Sigma T}(\mathbf{r},t)=\iint\limits_{\Sigma} d^2\rho \int\limits_{-T/2}^{T/2} d\tau\, \mathbf{nJ}(\mathbf{r}+\boldsymbol{\rho},t+\tau), \qquad \mathbf{n}\boldsymbol{\rho}=0, \tag{1}$$

where

$$\mathbf{J}(\mathbf{r},t)=\frac{c}{8\pi^3}\sum_{\lambda,\lambda'=1,2}\iint d^3p\, d^3q\,(\mathbf{e}_\lambda(\mathbf{p})\mathbf{e}_{\lambda'}(\mathbf{q}))\frac{\mathbf{p}+\mathbf{q}}{p+q} \times \exp[-i(\mathbf{p}-\mathbf{q})\mathbf{r}+ic(p-q)t]a_\lambda^+(\mathbf{p})a_{\lambda'}(\mathbf{q}). \tag{2}$$

In these expressions $p=|\mathbf{p}|$, $\mathbf{e}_\lambda(\mathbf{p})$ ($\lambda=1, 2$) are unit vectors satisfying the conditions $\mathbf{p}\cdot\mathbf{e}_\lambda(\mathbf{p})=0$, $\mathbf{e}_\lambda(\mathbf{p})\mathbf{e}_{\lambda'}(\mathbf{p}) = \delta_{\lambda\lambda'}$, and the creation and annihilation operators $a_\lambda^+(\mathbf{p})$ and $a_\lambda(\mathbf{p})$ satisfy the commutation relations

$$[a_\lambda(\mathbf{p}),\ a_{\lambda'}^+(\mathbf{q})]=\delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{q}). \tag{3}$$

The time T in Eq. (1) must be long enough to enable us to replace the factor $\exp[ic(p-q)t]$ in Eq. (2) with the delta function $\delta(p-q)$ when the integral is evaluated.

When this is so, then, as shown in [4], the integral $\int_{-T/2}^{T/2}\mathbf{J}d\tau$ becomes a time integral of the space part of a 4-fector, although $\mathbf{J}$ itself is not the component of the 4-vector and cannot, therefore, be the photon current density ($\mathbf{J}$ can be referred to as the photon current quasidensity). We note in this connection that the fact that operators such as $\mathbf{J}$ cannot be regarded as the photon current-density operators was noted by Pauli.[5] However, space averages of operators of this kind have been used by Mandel.[6] In contrast to the latter, the operator $\nu_{\Sigma T}$ can be used to investigate in greater detail not the time but the space structure of the photon current.

In the ensuing analysis we shall need the following expression for the vector potential operator:

$$\mathbf{A}(\mathbf{r},t)=\frac{\sqrt{\hbar c}}{2\pi}\sum_\lambda\int\frac{d^3p}{\sqrt{p}}\mathbf{e}_\lambda(\mathbf{p})\,[\exp(i\mathbf{pr}-icpt)a_\lambda(\mathbf{p}) + \exp(-i\mathbf{pr}+icpt)a_\lambda^+(\mathbf{p})]. \tag{4}$$

Equation (1) refers to the electromagnetic field in empty space. If, on the other hand, we consider a field in a medium with a given refractive index (this is meaningful when the wavelength is much greater than the interatomic distances) then in the derivation of the expression for $\mathbf{J}$ we must use solutions of Maxwell's equations for a medium with given $n(\mathbf{r})$ instead of the plane-wave expansions. This was done in our previous paper[7] in the approximation where the analysis can be restricted to a single polarization. We note, moreover, that, in the case of the scalar field in free space, the same approach was developed by Holliday and Sage.[8] In the present paper we shall consider photon statistics in the diffraction pattern, and will again restrict our analysis to the scalar approximation. We shall analyze the problem in which the boundary conditions for the field are given on certain curfaces ("screens") whilst outside the screens $n=1$. Let $u(\mathbf{k}, \mathbf{r})$ be a solution of

$$\Delta u(\mathbf{k},\mathbf{r})+k^2u(\mathbf{k},\mathbf{r})=0, \tag{5}$$

which satisfies the necessary boundary conditions and corresponds to the incidence on the screen of the wave $u_0(\mathbf{k},\mathbf{r})=(2\pi)^{-3/2}\exp(i\mathbf{k}\cdot\mathbf{r})$. We note that the quantity $\mathbf{k}$ in the function $u(\mathbf{k}, \mathbf{r})$ is not the wave vector for the mode $u(\mathbf{k}, \mathbf{r})$, but a parameter which has the significance of the wave vector of that incident wave which excites this particular mode. The expression for $\mathbf{J}$ in the case of arbitrary $u(\mathbf{k}, \mathbf{r})$ is[7]

$$\mathbf{J}(\mathbf{r},t)=2c\int d^3p\int d^3q\,\frac{\mathbf{m}(\mathbf{p},\mathbf{q},\mathbf{r})}{p+q}\exp[i(q-p)ct]a^+(\mathbf{q})a(\mathbf{p}), \tag{6}$$

where

$$m(p,q,r)=\frac{1}{2i}[u^*(q,r)\nabla u(p,r)-u(p,r)\nabla u^*(q,r)], \quad (7)$$

$$[a(p),a^+(q)]=\delta(p-q). \quad (8)$$

We note that in Eq. (6) the operators a, a^+ are the photon creation and annihilation operators for the mode $u(\mathbf{k}, \mathbf{r})$ and not for photons with wave number $\mathbf{k}$, as in Eq. (3).

In the next section we shall consider the statistical properties of the quantity $\nu_{\Sigma T}$ in free space for coherent and thermal radiations, and will show that they are identical with the photon counting characteristics. In the third section we shall consider the diffraction of light by an aperture, and will determine the statistical characteristics $\nu_{\Sigma T}$ for the diffraction pattern, which differ from the corresponding quantities in free space.

2. STATISTICS OF FLUCTUATIONS IN THE NUMBER OF PHOTONS IN FREE SPACE

We begin by considering the mean

$$\langle\nu_{\Sigma T}\rangle\equiv\langle z|\nu_{\Sigma T}|z\rangle \quad (9)$$

and the correlation function

$$B(\mathbf{r},t;\mathbf{r}',t')=\langle z|\nu_{\Sigma T}(\mathbf{r},t)\nu_{\Sigma T}(\mathbf{r}',t')|z\rangle-\langle z|\nu_{\Sigma T}(\mathbf{r},t)|z\rangle\langle z|\nu_{\Sigma T}(\mathbf{r}',t')|z\rangle \quad (10)$$

for the quantity $\nu_{\Sigma T}$ in the coherent state $|z\rangle$.

We note that the correlation function introduced in this way differs from the quantum correlation functions constructed from normally ordered field operators because $\nu_{\Sigma T}\nu_{\Sigma T}'$ includes the unordered product a^+aa^+a. If in Eq. (10) we took instead of $\nu_{\Sigma T}\nu_{\Sigma T}'$ the normally ordered quantity, then in the coherent states we would obtain B = 0. The fact that B is not zero is connected with the fact that the operators a, a^+ do not commute, i.e., we are dealing with a purely quantum-mechanical effect. We note that Eq. (10) corresponds to definitions of correlation functions and averaging operations which are standard in probability theory[1)] and in quantum mechanics. We now consider the coherent state $|z\rangle$ for which

$$a_\lambda(\mathbf{p})|z\rangle=z_\lambda(\mathbf{p})|z\rangle,\quad \langle z|a_\lambda^+(\mathbf{p})=z_\lambda^*(\mathbf{p})\langle z|,\quad \langle z|z\rangle=1, \quad (11)$$

and the special case of a monochromatic coherent state with fixed polarization for which

$$z_\lambda(\mathbf{p})=z_0\delta_{\lambda 1}\delta(\mathbf{p}-\mathbf{k}) \quad (12)$$

The constant z_0 can be expressed in terms of the amplitude A of the electric field $\langle z|\mathbf{E}|z\rangle=\mathbf{e}_1 A\sin(\mathbf{k}\cdot\mathbf{r}-kct+\varphi)$ as follows:

$$|z_0|=\pi A/\sqrt{\hbar ck}, \quad (13)$$

which is obtained if we use Eq. (4) to find $\langle z|\mathbf{A}|z\rangle$ and then $\langle z|\mathbf{E}|z\rangle$ for the coherent state described by Eq. (12).

Taking the average of Eq. (1) over the states given by Eq. (11), and using Eqs. (12) and (13), we obtain the following expression (for $\mathbf{n}=\mathbf{k}/k$):

$$\langle\nu_{\Sigma T}\rangle=A^2T\Sigma/8\pi\hbar k. \quad (14)$$

It is readily verified that $\langle\nu_{\Sigma T}\rangle$ is equal to the ratio of energy transmitted through Σ in a time T to the energy $\hbar ck$ of a single photon.

We now determine $B(\mathbf{r}, t; \mathbf{r}', t')$. Using Eqs. (2), (3), (11), (12), and (13), and replacing the double integral of $\exp[-i(p-k)c(\tau_2-\tau_1)]$ by $2\pi c^{-1}T\delta(p-k)$ in accordance with the above remarks about the choice of T, we obtain

$$B(\mathbf{r},t;\mathbf{r}',t')=\frac{A^2T\Sigma^2}{32\pi^3\hbar k}\sum_{\lambda=1,2}\int d^3p\left[\frac{\mathbf{n}(\mathbf{p}+\mathbf{k})}{p+k}\right]^2(\mathbf{e}_\lambda(\mathbf{p})\mathbf{e}_1(\mathbf{k}))^2 \times\delta(p-k)|V(\mathbf{p}_\perp)|^2\exp[i(\mathbf{k}-\mathbf{p})(\mathbf{r}-\mathbf{r}')]. \quad (15)$$

Here $\mathbf{p}_\perp=\mathbf{p}-\mathbf{n}(\mathbf{n}\cdot\mathbf{p})$, $\mathbf{n}=\mathbf{k}/k$,

$$V(\mathbf{p}_\perp)=\frac{1}{\Sigma}\iint_\Sigma\exp(i\mathbf{p}_\perp\boldsymbol{\rho})d^2\rho. \quad (16)$$

The function V describes the Fraunhofer diffraction by the aperture Σ.

If we suppose that Σ is a circle of radius R, then

$$|V(\mathbf{p}_\perp)|^2=[2J_1(p_\perp R)/p_\perp R]^2, \quad (17)$$

where J_1 is the Bessel function.

We note that the presence in Eq. (15) of integration with respect to $\mathbf{p}$ and summation over λ means that virtual photons of other modes and polarizations (i.e., other than the exciting ones) contribute to the fluctuations.

Substituting Eq. (17) in Eq. (15), using spherical polar coordinates with the polar axis along $\mathbf{k}$, and assuming that

$$\mathbf{e}_1(\mathbf{k})=(1,0,0),\quad \mathbf{e}_2(\mathbf{k})=(0,1,0),\quad \mathbf{k}=(0,0,k),$$
$$\mathbf{p}=p(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta),$$
$$\mathbf{e}_1(\mathbf{p})=(\cos\varphi\cos\theta,\ \sin\varphi\cos\theta,\ -\sin\theta),$$
$$\mathbf{e}_2(\mathbf{p})=(-\sin\varphi,\ \cos\varphi,\ 0),$$

we find, after integration with respect to the radial coordinate, the following expression for the transverse ($\rho=\mathbf{r}-\mathbf{r}'\perp\mathbf{k}$) correlation function:

$$B_\perp(\rho)=\frac{A^2kT\Sigma^2}{128\pi^3\hbar}\int_0^\pi\sin\theta\,d\theta\int_{-\pi}^{\pi}d\varphi(1+\cos\theta)^2(\cos^2\varphi\cos^2\theta+\sin^2\varphi)\times[2J_1(kR\sin\theta)/kR\sin\theta]^2\exp[-ik\rho\sin\theta\cos\varphi]. \quad (18)$$

Let us consider the special cases $kR\ll 1$ and $kR\gg 1$.

When $kR\ll 1$ then for all θ we have $[2J_1(kR\sin\theta)/kR\sin\theta]^2\approx 1$, and we can neglect this factor. As a result, we obtain an integral which contains only one dimensionless parameter, namely, $k\rho$. Hence, it follows that $B_\perp(\rho)=f(k\rho)$, i.e., when $kR\ll 1$ the correlation length for $\nu_{\Sigma T}$ is the wavelength λ.

Assuming, on the other hand, that $\rho=0$, and evaluating the integral of the trigonometric functions, we find that for $kR\ll 1$

$$B_\perp(0)=\langle(\Delta\nu_{\Sigma T})^2\rangle=\frac{7}{240\pi^2}\frac{A^2T\Sigma^2k}{\hbar}. \quad (19)$$

Hence, using Eq. (14), we have

$$\frac{\langle(\Delta\nu_{\Sigma T})^2\rangle}{\langle\nu_{\Sigma T}\rangle}=\frac{7}{30\pi}k^2\Sigma\ll 1, \quad (20a)$$

$$\beta^2\equiv\frac{\langle(\Delta\nu_{\Sigma T})^2\rangle}{\langle\nu_{\Sigma T}\rangle^2}=\frac{56}{30}\frac{\hbar k^2}{A^2T} \quad (20b)$$

It is clear from Eq. (20a) that, when $kR\ll 1$, the ratio $\langle(\Delta\nu_{\Sigma T})^2\rangle/\langle\nu_{\Sigma T}\rangle$ is very different from unity, which is characteristic for the Poisson distribution. It follows from Eq. (20b) that for $R\ll\lambda$ the magnitude of the relative fluctuations β^2 is independent of R and remains finite when $R\to 0$, whereas, for the Poisson distribution $\beta^2=\langle\nu_{\Sigma T}\rangle^{-1}\to\infty$ as $R\to 0$.

We shall now consider the case when $kR\gg 1$. Here, because the factor $[2J_1(kR\sin\theta)/kR\sin\theta]^2$ decreases rapidly as θ increases, the main contribution to the integral is provided by the small-angle region $\theta<(kR)^{-1}\ll 1$. In this region we may suppose that $\sin\theta\approx\theta$,

$\cos\theta \approx 1$, and the integral can be evaluated between 0 and ∞. Evaluating the integral with respect to φ, and substituting $u = kR\theta$, we obtain

$$B_\perp(\rho) = \frac{A^2 T\Sigma}{16\pi\hbar k} \int_0^\infty \left[\frac{2J_1(u)}{u}\right]^2 J_0\left(\frac{\rho}{R}u\right) u\, du.$$

The integral in this expression can be evaluated and the result is

$$B_\perp(\rho) = \begin{cases} \dfrac{A^2 T\Sigma}{8\pi\hbar k}\left\{2\left[\dfrac{1}{\pi}\arccos\left(\dfrac{\rho}{2R}\right) - \dfrac{\rho}{2R}\sqrt{1-\left(\dfrac{\rho}{2R}\right)^2}\right]\right\} & \text{if } \rho \leqslant 2R, \\ 0 & \text{if } \rho > 2R. \end{cases} \tag{21}$$

The function in the curly braces is the common area of two intersecting circles of radii R, whose centers are separated by the distance ρ, divided by πR^2. Thus, when $R \gg \lambda$, the correlation function for $\nu_{\Sigma T}$ is nonzero only when the areas Σ_1 and Σ_2 intersect, and is proportional to the relative intersection area.

Substituting $\rho = 0$ in Eq. (21), we obtain

$$\langle(\Delta\nu_{\Sigma T})^2\rangle = \frac{A^2 T\Sigma}{8\pi\hbar k} = \langle\nu_{\Sigma T}\rangle, \tag{22a}$$

$$\beta^2 = \frac{8\pi\hbar k}{A^2 T\Sigma} = \frac{1}{\langle\nu_{\Sigma T}\rangle} \tag{22b}$$

Therefore, when $R \gg \lambda$ the ratio of $\langle(\Delta\nu_{\Sigma T})^2\rangle$ and $\langle\nu_{\Sigma T}\rangle$ corresponds to the Poisson distribution.

If we now consider the dependence of the relative fluctuations $\beta^2 = \beta^2(R)$ on R, we find that as R decreases from its initial value, which is large in comparison with λ, the quantity β^2 increases as R^{-2}, in accordance with Eq. (22b), but this increase terminates when $R \sim \lambda$ [see Eq. (20b)].

Holliday and Sage[8] have calculated $\langle(\Delta\nu_{\Sigma T})^2\rangle$ for $kR \gg 1$ in the case of a scalar field, but their result does not agree with the formula $\langle(\Delta\nu_{\Sigma T})^2\rangle = \langle\nu_{\Sigma T}\rangle$. This is so because the sensitivity factor was introduced by them into the operator $\nu_{\Sigma T}$ in a fashion which was not wholly correct.

The longitudinal correlation functions can be considered in a similar way, and it turns out that the longitudinal correlation length for $\nu_{\Sigma T}$ is of the order of cT.

Let us now consider the statistical properties of random (thermal) radiation. Here, it is well known (see[1,2]) that we may assume that the eigenvalue $z_\lambda(\mathbf{p})$ in Eq. (11) is a random Gaussian function with the following property (the bar over the symbols represents averaging over the set of possible realizations of z_λ):

$$\overline{z_\lambda(\mathbf{p})} = 0, \quad \overline{z_\lambda(\mathbf{p}) z_{\lambda'}(\mathbf{q})} = 0, \quad \overline{z_\lambda(\mathbf{p}) z_{\lambda'}^*(\mathbf{q})} = \delta_{\lambda\lambda'}\delta(\mathbf{p}-\mathbf{q}) F(\mathbf{p}). \tag{23}$$

The function $F(\mathbf{p})$ defines the spectral composition of the radiation. Since z_λ is normal, we have from Eq. (23)

$$\begin{aligned} \overline{z_{\lambda_1}^*(\mathbf{k}_1) z_{\lambda_2}^*(\mathbf{k}_2) z_{\lambda_3}(\mathbf{k}_3) z_{\lambda_4}(\mathbf{k}_4)} &= \overline{z_{\lambda_1}^*(\mathbf{k}_1) z_{\lambda_3}(\mathbf{k}_3)}\,\overline{z_{\lambda_2}^*(\mathbf{k}_2) z_{\lambda_4}(\mathbf{k}_4)} \\ &+ \overline{z_{\lambda_1}^*(\mathbf{k}_1) z_{\lambda_4}(\mathbf{k}_4)}\,\overline{z_{\lambda_2}^*(\mathbf{k}_2) z_{\lambda_3}(\mathbf{k}_3)}. \end{aligned} \tag{24}$$

For random fields the first and second moments must be additionally averaged over z_λ:

$$\begin{aligned} B(\mathbf{r}_1, \mathbf{r}_2) &= \overline{\langle z|\nu_{\Sigma T}(\mathbf{r}_1, t_1)\nu_{\Sigma T}(\mathbf{r}_2, t_2)|z\rangle} \\ &- \overline{\langle z|\nu_{\Sigma T}(\mathbf{r}_1, t_1)|z\rangle\langle z|\nu_{\Sigma T}(\mathbf{r}_2, t_2)|z\rangle}. \end{aligned}$$

Using Eqs. (3), (11), (23), and (24), we obtain

$$\begin{aligned} B(\mathbf{r}_1, \mathbf{r}_2) &= \left(\frac{c}{8\pi^3}\right)^2 \frac{2\pi T\Sigma^2}{c} \int d^3p \int d^3q \sum_{\lambda,\lambda'} (\mathbf{e}_\lambda(\mathbf{p})\mathbf{e}_{\lambda'}(\mathbf{q}))^2 \delta(p-q) \\ &\times \left[\frac{\mathbf{n}(\mathbf{p}+\mathbf{q})}{p+q}\right]^2 |V(\mathbf{p}_\perp - \mathbf{q}_\perp)|^2 \exp[i(\mathbf{p}-\mathbf{q})(\mathbf{r}_1 - \mathbf{r}_2)][F(\mathbf{p})F(\mathbf{q}) + F(\mathbf{p})]. \end{aligned} \tag{25}$$

We now confine our attention to the mean square of the fluctuations in the number of photons $B(\mathbf{r}, \mathbf{r}) = \overline{\langle(\Delta\nu_{\Sigma T})^2\rangle}$, and consider the special case when $F(\mathbf{p})$ describes a quasimonochromatic plane wave of the form

$$F(\mathbf{p}) = F_0\delta(\mathbf{p}_\perp) f(\mathbf{pn}),$$

$$f(p_x) = \begin{cases} 1 & \text{if } |p_x - k| \leqslant \dfrac{\Delta\omega}{2c}, \\ 0 & \text{if } |p_x - k| > \dfrac{\Delta\omega}{2c}. \end{cases} \tag{26}$$

In these expressions $\Delta\omega$ is the radiation frequency band. We shall consider the case where $k^2\Sigma \gg 1$. The calculations can be readily completed if $T\Delta\omega \gg 1$ or $\omega_0^{-1} \ll T \ll (\Delta\omega)^{-1}$, where $\omega_0 = ck$. In the first case, if we evaluate the integral in Eq. (25), we obtain

$$\overline{\langle(\Delta\nu_{\Sigma T})^2\rangle} = \bar{\nu}_{\Sigma T}\left[1 + \frac{2\pi\bar{\nu}_{\Sigma T}}{T\Delta\omega}\right] \quad (T\Delta\omega \gg 1) \tag{27}$$

where $\bar{\nu}_{\Sigma T} = (4\pi^3)^{-1} cT\Sigma F_0 \Delta\omega$. We note that the numerical coefficient in front of the second term in Eq. (27), which is 2π in our case, depends on the form of the function $f(p_x)$.

The first term in Eq. (27) is due to the fact that the operators a^+ and a do not commute, and gives the quantum fluctuations in the photon number. The second term is due to classical fluctuations in z_λ. When $T\Delta\omega \gg \bar{\nu}_{\Sigma T}$, Eq. (27) becomes identical with the corresponding expression for the Poisson distribution.

When $\omega_0^{-1} \ll T \ll (\Delta\omega)^{-1}$ the function $f(p_x)$ is sharper than the function

$$\frac{c}{2\pi}\int_{-T/2}^{T/2} \exp[i(p-q)c\tau]\, d\tau,$$

which in Eq. (25) is approximated by the delta function $\delta(p-q)$. Using this together with Eq. (25), we can readily show that

$$\overline{\langle(\Delta\nu_{\Sigma T})^2\rangle} = \bar{\nu}_{\Sigma T}\left(1 + \frac{1}{2}\bar{\nu}_{\Sigma T}\right), \quad \omega_0^{-1} \ll T \ll (\Delta\omega)^{-1}, \tag{28}$$

where, as before, $\bar{\nu}_{\Sigma T} = (4\pi^3)^{-1} cT\Sigma F_0 \Delta\omega$. Equation (28) corresponds to the Bose-Einstein distribution, and the factor $1/2$ appears because of the presence of photons with two independent polarizations.[2)]

It is clear from the above example that the main relationships for the statistical properties of the fluctuations in the number of photons in free space, obtained with the aid of the operator $\nu_{\Sigma T}$, are identical with the corresponding results for photon counting statistics reported in [1,2].

3. STATISTICAL PROPERTIES OF $\nu_{\Sigma T}$ IN THE CASE OF DIFFRACTION

The statistical properties of $\nu_{\Sigma T}$ in the case of diffraction differ from the corresponding properties in free space. In this section we shall consider as an example the diffraction of light by a rectangular aperture. To simplify the calculations we shall consider the scalar approximation. The operator $\mathbf{J}$ is then given by Eqs. (6)–(8). We must first find the function $u(\mathbf{k}, \mathbf{r})$ which satisfies Eq. (5), the boundary conditions on the screen, and the radiation conditions corresponding to the incidence of a plane wave $u_0(\mathbf{k}, \mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}\cdot\mathbf{r})$. As the first approximation to the solution of this problem we shall take the solution obtained by the Kirchhoff method in the Fraunhofer diffraction zone. If the aperture lies in the $x = 0$ plane, and the lengths of its edges are 2a and $2l$ along the y and z axes, respectively, then for $x \gg ka^2$, $x \gg kl^2$ we have

$$u(\mathbf{p},\mathbf{r}) = \frac{p}{2\pi i}\frac{S\exp(ipx)}{\sqrt{8\pi^3}\,x}\, s\left[\left(p_3 - \frac{pz}{x}\right)l\right] s\left[\left(p_2 - \frac{py}{x}\right)a\right]. \quad (29)$$

where $S = 4al$ is the area of the aperture and $s(x) = \sin x/x$. Evaluating $\mathbf{m}(\mathbf{p}, \mathbf{q}, \mathbf{r})$ near the axis $ay \ll x^2$, $lz \ll x^2$, we obtain

$$m_x(\mathbf{p},\mathbf{q},\mathbf{r}) = \frac{p+q}{2}\, u(\mathbf{p},\mathbf{r})\, u^*(\mathbf{q},\mathbf{r}). \quad (30)$$

The expression for $J_x(\mathbf{r}, t)$ then assumes the form

$$J_x(\mathbf{r},t) = c\int d^3p \int d^3q\, u(\mathbf{p},\mathbf{r})\, u^*(\mathbf{q},\mathbf{r}) \exp[ic(p-q)t]\, a^+(\mathbf{p})\, a(\mathbf{q}). \quad (31)$$

The mean value $\langle \nu_{\Sigma T}\rangle = \langle z|\nu_{\Sigma T}|z\rangle$ calculated with allowance for Eqs. (29) and (31) for the monochromatic coherent state

$$z(\mathbf{p}) = \sqrt{\frac{2ck}{\hbar}}\, A\delta(\mathbf{p}-\mathbf{k}), \quad \mathbf{k} = (k, 0, 0), \quad (32)$$

where A is the amplitude of the corresponding analytical signal, is given by

$$\langle \nu_{\Sigma T}\rangle = \frac{2c^2kA^2T\Sigma}{(2\pi)^5\hbar}\left(\frac{kS}{x}\right)^2 s^2(\alpha)\, s^2(\beta). \quad (33)$$

In this expression $\alpha = kay/x$, $\beta = -klz/x$, and the area Σ must satisfy the condition $\Sigma \ll (\lambda x)^2/S$, which means that the control area must occupy a small fraction of the diffraction maximum. It is clear from Eq. (33) that $\langle \nu_{\Sigma T}\rangle$ follows the intensity distribution in the classical diffraction pattern.

We must now consider the space correlation function

$$K(\mathbf{r}_1,\mathbf{r}_2) = \iint\limits_{-T/2}^{T/2} dt_1\, dt_2\, \{\langle z|J_x(\mathbf{r}_1,t_1)J_x(\mathbf{r}_2,t_2)|z\rangle - \langle z|J_x(\mathbf{r}_1,t_1)|z\rangle\langle z|J_x(\mathbf{r}_2,t_2)|z\rangle\}$$

(the integration over Σ_1 and Σ_2 will be carried out later). Evaluating the mean values, we obtain

$$K(\mathbf{r}_1,\mathbf{r}_2) = \frac{4\pi c^2kA^2T}{\hbar}\, u^*(\mathbf{k},\mathbf{r}_1)\, u(\mathbf{k},\mathbf{r}_2) \int d^3p\, \delta(p-k)\, u(\mathbf{p},\mathbf{r}_1)\, u^*(\mathbf{p},\mathbf{r}_2) \quad (34)$$

If we now substitute Eq. (29) into this expression, we can readily show that the characteristic scales of the change in K along the y and z axes are $y_0 = \lambda x/2a$ and $z_0 = \lambda x/2l$. Therefore, the correlation length for the fluctuations in the photon flux in the diffraction pattern is of the order of the width of the diffraction lobe.

If the area Σ satisfies the condition $\Sigma \ll y_0 z_0 = (\lambda x)^2/S$, integration with respect to Σ reduces to multiplication by Σ, and hence $B(\mathbf{r}_1, \mathbf{r}_2) = \Sigma^2 K(\mathbf{r}_1, \mathbf{r}_2)$. Let us now consider the mean square of the fluctuations at the center of the diffraction pattern. Assuming that $y = z = 0$, and using the approximate result

$$\int d^3p\, |u(\mathbf{p},\mathbf{r})|^2 \delta(p-k) \approx \frac{k^2S}{(2\pi)^4x^2},$$

we find that

$$\langle(\Delta\nu_{\Sigma T})^2\rangle = \frac{c^2kA^2T\Sigma^2}{(2\pi)^7\hbar}\frac{k^4S^3}{x^4}.$$

If we now use Eq. (33) to express the coefficient in this expression in terms of $\langle \nu_{\Sigma T}\rangle$, we obtain

$$\langle(\Delta\nu_{\Sigma T})^2\rangle = \langle \nu_{\Sigma T}\rangle \frac{S\Sigma}{2(\lambda x)^2}. \quad (35)$$

A similar result can be obtained for the more distant diffraction lobes except that the factor 2 will now be absent from the denominator. Equation (35) is obtained on the assumption that $\Sigma \ll (\lambda x)^2/S$. Therefore, the ratio of $\langle(\Delta\nu_{\Sigma T})^2\rangle$ and $\langle \nu_{\Sigma T}\rangle$ in the diffraction pattern is very different from unity, which is characteristic for the Poisson distribution. In this respect, the formula given by Eq. (35) is analogous to Eq. (20a) for free space. In both cases, the ratio $\langle(\Delta_{\Sigma T})^2\rangle/\langle \nu_{\Sigma T}\rangle$ is less than unity in the ratio of Σ to R_{cor}, where R_{cor} is the correlation length for the photon current. The difference lies only in the fact that for a plane wave in free space $R^2_{cor} \sim \lambda^2$, and in the diffraction pattern $R^2_{cor} \sim (\lambda x)^2/S$. The ratio $\langle(\Delta\nu_{\Sigma T})^2\rangle/\langle \nu_{\Sigma T}\rangle$ in the diffraction pattern will approach unity if the size of the control area Σ is much greater than $(\lambda x)^2/S$.

A similar result can be obtained for a random field if the spectral linewidth and the angular width of the beam are so small that the mean diffraction pattern is not sheared out and $T\Delta_\omega \gg 1$.

When the size of the aperture is small in comparison with the wavelength, the scalar and vector Kirchhoff approximations become unacceptable in the diffraction problem. In this case, one can use the approximate solution of the problem of diffraction of electromagnetic waves by a circular aperture of radius $a \ll \lambda$ in a perfectly conducting plane (see, for example, [9]). Using this solution, it can be shown with the aid of Eq. (2) that, instead of Eq. (35), we now have

$$\langle(\Delta\nu_{\Sigma T})^2\rangle \sim \langle \nu_{\Sigma T}\rangle \frac{\Sigma S}{(\lambda x)^2}\left(\frac{a}{\lambda}\right)^4. \quad (36)$$

It is clear from this formula that, in the case of diffraction of electromagnetic waves by a small aperture, the probability distribution for the photon-flux fluctuations is even more different from the Poisson distribution than in the scalar approximation.

It is interesting to note one further property of the fluctuations in $\nu_{\Sigma T}$. The function $z_\lambda(\mathbf{p})$ which determines the form of a coherent state in free space can be chosen so that the mean field and the quantity $\langle z|\nu_{\Sigma T}|z\rangle$ have the same form as in the case of diffraction of a plane coherent wave by an aperture. It then turns out that the ratio $\langle(\Delta\nu_{\Sigma T})^2\rangle/\langle \nu_{\Sigma T}\rangle$ is equal to unity even when the corresponding quantity in the diffraction pattern is very different from unity. Therefore, the statistical characteristics of the fluctuations in the photon flux turn out to be different even when the mean values are equal, depending on the method of producing the field.

4. CONCLUSION

It follows from the above examples that the principal statistical properties of $\nu_{\Sigma T}$ for light propagating in free space are the same as the corresponding quantities for photon counting statistics obtained on the basis of quantum correlation functions. Some differences appear in the case of the statistics of the number of transmitted photons in the diffraction pattern.

It remains unclear, however, as to how $\nu_{\Sigma T}$ could be measured. It can probably be measured by an ideal photodetector which reacts to each photon which reaches it. However, real photodetectors with relatively low quantum efficiency introduce additional fluctuations, and the photon counting statistics may differ from the statistics of the photons reaching the detector. It follows that, as noted above, the quantity $\nu_{\Sigma T}$ is a characteristic of the free electromagnetic field and is not identical with the number of photon counts.

[1)]We note that there is a difference between the Russian and English terminology: in the Russian literature the correlation function is $B = \langle f(M)f(M')\rangle - \langle f(M)\rangle\langle f(M')\rangle$, whereas in English literature it is taken to be $\langle f(M)f(M')\rangle$.

[2)]In fact, we note that if we substitute $\nu = \nu_1 + \nu_2$, where ν_i is the number of photons of i-th polarization, and assume that $\bar{\nu}_1 = \bar{\nu}_2 = \bar{\nu}/2$ and that $\overline{\Delta\nu_1\Delta\nu_2} = 0$, then for photons of each polarization we have $\overline{\Delta\nu_i^2} = \bar{\nu}_i + (\bar{\nu}_i)^2$. This follows from the Bose-Einstein distribution, and hence $\overline{\Delta\nu^2} = \overline{\Delta\nu_1^2} + \overline{\Delta\nu_2^2} = \nu_1 + (\bar{\nu}_1)^2 + \bar{\nu}_2 + (\bar{\nu}_2)^2 = \bar{\nu} + (\bar{\nu})^2/2$, which is identical with Eq. (28).

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