The possibility of creating a nuclear γ laser

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The possibility is analyzed in principle of creating a nuclear γ laser with a population inversion produced as the result of pulsed radiative neutron capture. It is found that stimulated radiation can occur only if the Mossbauer effect is used. This imposes severe limitations on the allowable heating of the laser material, which arises unavoidably as the result of cascade γ rays and recoil energy from capture and scattering of neutrons. It is shown that such heating can be avoided if the working isotope is introduced in the form of low-concentration impurities in a light matrix and the laser itself is prepared in the form of thin rods with a diameter to a length ratio of the order 10^{-3}—10^{-4}. In addition it is necessary that most of the pumping neutrons have an energy of no more than a few tens of eV. Estimates are obtained for the critical concentration of excited nuclei and the corresponding value of critical integrated neutron flux. Whether a high-power neutron pulse with the definite energy and time characteristics necessary for pumping can actually be achieved remains an open question.

1. INTRODUCTION

A very alluring problem is the substantial reduction of the wavelength of radiation for which it is possible to achieve the laser effect—amplification by means of stimulated radiation. In this connection the problem of creating lasers based on nuclear γ transitions takes on special interest.

Below we develop reasoning which indicates the possibility in principle of creating conditions for stimulated emission of γ rays with energies of the order of keV or tens of keV. Here it is proposed to use γ transitions of the Mossbauer type (with a line width close to the natural width) with population of the excited level by capture of a neutron by an isotope neighboring in mass number. For such a neutron pump an extremely powerful pulsed neutron source must be used—such as, for example, a nuclear explosion.

As will be evident from the subsequent analysis, the creation of a nuclear γ laser imposes very severe and sometimes contradictory requirements on the parameters of the system. The purpose of the present article is to prove the possibility in principle of achieving the conditions for occurrence of stimulated emission of γ rays which are compatible with real physical parameters. Therefore such questions as the integrated power, the angular directivity and kinetics of nuclear laser radiation, and so forth, will be discussed here either not at all or only very superficially.

2. GENERAL CONSIDERATIONS

Let us consider a transition between two nuclear levels which corresponds to a γ-ray energy E_0 and an actual total line width in matter Γ. The number of normal oscillators of the electromagnetic field in a volume of material V which in principle can be excited in such a transition and which radiate in an energy interval Γ is given by

$$g \approx \frac{8\pi E_0^2 \Gamma}{(2\pi\hbar c)^3} V \qquad (1)$$

(for simplicity we have omitted the spin factors in Eq. (1)).

The general condition for emission of stimulated radiation has the form

$$n^* VW / g = c / l(E_0). \qquad (2)$$

Here $W = W_0 f$, where $W_0 = 1/\tau_{rad} = \Gamma_\gamma/\hbar = \Gamma_0/\hbar(1+\alpha)$ is the probability of radiation of a γ ray by the excited nucleus, (Γ_0 is the natural line width of the radiation, and α is the internal-conversion coefficient), and f is the fraction of γ rays emitted from the upper level which fall in the energy interval Γ. In addition, n is the excess concentration (and n^* is the critical excess) of nuclei in the upper excited level which for neutron pumping is equal to $n = n_0\xi$, where n_0 is the number of nuclei per unit volume which have captured neutrons in the pumping pulse, and ξ is the population difference in the upper and lower levels per captured neutron (in the upper level there turn out to be $n_0(1+\xi)/2$, and in the lower level $n_0(1-\xi)/2$ nuclei per cm^3). We emphasize that since we are discussing the radiative capture of neutrons it is possible to avoid the presence in the initial sample of working nuclei in the lower level. Finally $l(E_0)$ is the mean free path of resonance γ rays in matter, and c is the velocity of light, so that the right-hand side of Eq. (2) is the probability of absorption of a γ ray.

From Eqs. (1) and (2) we obtain the critical number of neutron-capture events by the nuclei of the initial isotope per cm^3;

$$n_0^* = \left(\frac{E_0}{\pi\hbar c}\right)^2 \frac{\Gamma}{\Gamma_0} \frac{1+\alpha}{f\xi} \frac{1}{l(E_0)}, \qquad (3)$$

or, expressing E_0 in keV,

$$n_0^* \approx 2{,}6\cdot 10^{14} E_0^2 \frac{\Gamma}{\Gamma_0} \frac{1+\alpha}{f\xi} \frac{1}{l(E_0)}. \qquad (4)$$

This very simple relation already permits a number of important conclusions to be drawn. In a condensed phase for $E_0 \sim 10$—100 keV $l(E_0)$ is a quantity of order no more than centimeters. On the other hand, it is known from studies of the Mossbauer effect that it is possible to obtain the natural width of lines in practice only for excited-state lifetimes $\tau_0 = \hbar/\Gamma_0 \lesssim 10^{-6}$—$10^{-5}$ sec, and for longer times the ratio Γ/Γ_0 increases roughly in proportion to τ_0. Here even in the absence of conversion ($\alpha = 0$) and for $f \approx \xi \approx 1$ the values of n_0^* reach N, the total number of nuclei per cm^3, for $\tau_0 > 1$—10^{-2} sec (for $E_0 = 10$—100 keV).

Thus, long-lived isomeric states apparently cannot be used to create a γ laser; the idea of separate preparation of a long-lived isomer and subsequent use of crystals containing a high concentration of it with ordinary densities of matter is unrealistic, at least without the appearance of radically new ideas.

An important limitation in the lifetime of the excited state exists actually also on the low end. It is dictated by the necessity of capture in 1 cm^3 of material in a pulse time $\tau \ll \tau_0$ of a very large number n_0^* of neutrons, approaching in the limit N. From this point of view it is desirable to increase τ_0 as much as possible, not going too far below 10^{-7}—10^{-6} sec, which to all appearances are the minimum times for the high-power pulsed neutron pumping.

In view of the above, the optimal procedure for creation of nuclear γ lasers is to use transitions of the Mossbauer type with a lifetime $\tau_0 \sim 10^{-4}$—10^{-8} sec, with the largest possible probability of the Mossbauer effect and the smallest possible internal-conversion coefficient α. The quantity f in Eq. (3) now takes on the meaning of the Mossbauer effect probability.

For E_0 values close to 10 keV, the probability is $f \approx 1$ and falls off comparatively weakly with increasing temperature; for $E_0 \sim 100$ keV, f is appreciably less than unity for $T \ll \Theta$ (Θ is the Debye temperature) and drops rapidly with increasing temperature. However, since the conversion coefficient α also decreases with increase of E_0, the entire region of the order of keV turns out to be reasonable from the point of view of n_0^* values, at least for low temperatures.

The drop in f with increasing temperature imposes limitations on the permissible heating of the material and turns out to be a critical factor in the consistent choice of the system parameters. The existence of these limitations—the requirements of retaining not only a solid but even a cold working medium of the γ-resonance laser—leads us to ask if it is impossible to abandon the use of the Mossbauer effect and work in the entire energy interval of the line dictated by the Doppler broadening. In this case ($T > \Theta$) in Eqs. (3) and (4) $f \sim 1$, $\Gamma \approx \sqrt{R_0 kT}$, where $R_0 = 5.4 \times 10^{-4} A_0^{-1} E_0^2$ (R_0 in eV, E_0 in keV) is the recoil energy in emission of a γ ray by a free nucleus with mass number A_0.

If we consider excited levels with the same lifetimes (the same Γ_0) as above, then, as the result of the large Γ/Γ_0, the value of n_0^* rapidly goes beyond the physical limits of the ordinary density of matter under terrestrial conditions. The selection of transitions with substantially smaller τ_0 (increase of Γ_0) leads to requirements of unrealistically large integrated neutron fluxes in a very short time. Therefore in what follows we will not consider the idea of a non-Mossbauer resonance with recoil.

From the point of view of limiting the heating of the working medium of the laser, which is extremely dangerous if the Mossbauer effect is used, and reducing the value of the critical neutron pumping fluxes, it is necessary to have the minimum possible value of n_0^* as given by Eq. (3). For a fixed resonance transition, only the one parameter $l(E_0)$ is at our disposal. Hence the idea arises of introducing nuclei of the initial working isotope in the form of impurities in an extremely light matrix which only weakly absorbs γ rays with energy E_0 and neutrons. It is well known that the Mossbauer effect is completely preserved in impurity nuclei, and the freedom in choice of the matrix permits achievement of optimal conditions from the point of view of the value of f, the heat capacity, and other parameters.

If we denote the density of the working atoms by N_0 and that of the matrix atoms by N_1 (assuming for simplicity a monatomic matrix), then

$$l(E_0) = 1 / [N_0\sigma_0(E_0) + N_1\sigma_1(E_0)], \qquad (5)$$

where σ_0 and σ_1 are the cross sections for absorption of resonance γ rays by the working atoms and the matrix atoms. It follows from this expression that the working material can be added only up to relative densities

$$N_0 / N_1 \approx \sigma_1(E_0) / \sigma_0(E_0). \qquad (6)$$

If we consider, for example, a matrix of beryllium, then the mean free path of the γ rays is $l(10 \text{ keV}) \approx 0.5$ cm and $l(100 \text{ keV}) \approx 2$ cm and in this case the concentration of working nuclei with mass number $A_0 \approx 150$ lies in the range 2×10^{-4}—2×10^{-3}.

In neutron pumping of excited resonance levels, unavoidable sources of heating are the energy of recoil nuclei in scattering and capture of neutrons, the recoil energy in emission of capture γ rays, and also the part of the energy of the capture γ rays which remains in the working medium of the laser. The last source of heating could present the greatest danger, for each capture event is accompanied by a γ-ray cascade with a total energy of about 8 MeV. However, this heating can be limited by creation of conditions under which the overwhelming fraction of the γ rays will leave the laser material without being absorbed. Hence we come to the idea of preparing the laser in the form of thin rods—a needle with a length of the order of several $l(E_0)$ and diameter $d \ll l(E_0)$, $d \ll l_{\text{compt}}$, where l_{compt} is the mean free path in the laser material of cascade γ rays with energy ~1 MeV, which is determined by Compton scattering. Under these conditions the average path of a γ ray to the surface of the needle is $\sim\pi d/4$ and the energy which remains per unit volume of the needle is given by

$$Q_\gamma \approx \frac{1}{2}\frac{\pi}{4}\frac{d}{l_{\text{compt}}}\varkappa n_0\left[1+\frac{N_1\sigma_{n\gamma}^1(E_n)}{N_0\sigma_{n\gamma}^0(E_n)}\right]\cdot 8\cdot 10^6 \left[\frac{\text{ev}}{\text{cm}^3}\right], \qquad (7)$$

where $\sigma_{n\gamma}^0(E_n)$ and $\sigma_{n\gamma}^1(E_n)$ are the cross sections for radiative capture of neutrons by the working nuclei and the matrix nuclei, and the factor $1/2$ is introduced as the result of the fact that in Compton scattering only about half of the energy is transferred to the electrons. The factor $\kappa \approx d/l_{el}$ arises in the case where the needle diameter d is less than the range of the secondary (Compton) electrons themselves l_{el}, which has values, for example, for beryllium from 70 μ at 100 keV to 2.4 mm at 1 MeV. Since the matrix is assumed to absorb neutrons very weakly, we can assume that for the optimum admixture of working nuclei

$$N_1\sigma_{n\gamma}^1(E_n) \ll N_0\sigma_{n\gamma}^0(E_n).$$

If we require that the heating not exceed, say, 300°K, then for beryllium this leads to $d \approx 1$—$3\ \mu$ for γ rays with energy 100 keV— 1 MeV (without inclusion of the factor κ) and $d \approx 10$—$100\ \mu$ if we take into account the escape of Compton electrons from the needle.

In the version of a one-pass γ laser (without reflectors), the shape of the needle determines the direction and angular divergence of the beam of stimulated coherent γ radiation. Simultaneous pumping of a large number of needles by the pulsed neutron source is of course possible. It is necessary only that in this case we avoid appreciable absorption of the γ rays emitted from one needle by the other needles.

3. THE PROBLEM OF HEATING BY NUCLEAR RECOIL

Independent of the shape of the laser working medium, there is an unavoidable heating due to the energy of recoil nuclei in scattering (R_n^S) and capture (R_n^C) of neutrons and in the emission of capture γ rays (the γ recoil $R\gamma$). In a unit volume of the material there remains a recoil energy of the working nuclei (Q_0) and the matrix nuclei (Q_1):

$$Q_0 \approx n_0(E_n / A_0 + S / A_0), \qquad (8)$$

$$Q_1 \approx n_0 \frac{N_1}{N_0}\left[\frac{\sigma_{n\gamma}^1(E_n)}{\sigma_{n\gamma}^0(E_n)}\left(\frac{E_n}{A_1}+\frac{S}{A_1}\right)+\frac{\sigma_S^1(E_n)}{\sigma_{n\gamma}^0(E_n)}\xi_1 E_n\right]. \qquad (9)$$

Here E_n (eV) is the neutron energy, ξ_1 is the average logarithmic energy loss of a neutron in one collision with a matrix nucleus, $\sigma_S^1(E_n)$ is the cross section for scattering of neutrons by matrix nuclei, and $S \approx 2 \times 10^4$ eV is a quantity approximately equal to the nuclear recoil energy in emission of a γ-ray cascade with total energy 8 MeV (a factor $1/2$ has been introduced in comparison with emission of a single γ ray of this energy). For slow neutrons we can neglect R_n^C in comparison with $R\gamma$. For a light matrix, as a rule, the cross section for scattering of slow neutrons is much larger than the cross section for their capture. Therefore the recoil from neutron capture in Eq. (9) can be neglected in comparison with the recoil from neutron scattering.

Let us denote the maximal permissible heating of the laser by ΔT_{max}. Then the problem reduces actually to the need of providing the condition

$$Q_0 + Q_1 < 3N_1 k\Delta T_{max} \qquad (10)$$

for a critical population of the upper resonance level, where $n_0 = n_0^*$ in Eqs. (8) and (9). If $A_0 \approx 150$ and $N_1 \approx 10^{23}$, then in conformity with a heating due to recoil of the working nuclei with $\Delta T_{max} \approx 300^\circ$K, the condition arises

$$n_0 \lesssim 10^{20}. \qquad (11)$$

The value of n_0^* naturally must also satisfy this requirement. However, according to Eqs. (4), (5), and (6), we have

$$n_0^* \approx 5 \cdot 10^{16} \frac{\Gamma}{\Gamma_0}\frac{1+\alpha}{f\xi}, \quad E_0 = 10 \text{ keV},$$

$$n_0^* \approx 10^{18} \frac{\Gamma}{\Gamma_0}\frac{1+\alpha}{f\xi}, \quad E_0 = 100 \text{ keV}.$$

Comparing these values of n_0^* with Eq. (11), we are convinced that there is a rather large reserve in the value of the parameter $\Gamma(1+\alpha)/\Gamma_0 f\xi$, amounting to about 2000 for $E_0 = 10$ keV and about 100 for $E_0 = 100$ keV.

Now let us turn our attention to heating as the result of recoil of matrix nuclei. The only parameter which can actually be varied in Eq. (9) is the cross section for neutron capture by the working nuclei $\sigma_{n\gamma}^0(E_n)$. Therefore condition (10) should give the minimum permissible value of this quantity for the maximum n_0 corresponding to Eq. (11).

Let us analyze the situation in the case of a beryllium matrix. In this case $\sigma_{n\gamma}^1 \approx 10^{-27}$ cm^2 for $E_n = 2.6 \times 10^{-2}$ eV[1] and falls off further in proportion to $E_n^{-1/2}$, $\sigma_S^1 \approx 6 \times 10^{-24}$ cm^2 over a wide interval of E_n up to 10 keV, and $\xi_1 = 0.24$. The ratio corresponding to Eq. (6) is $N_0/N_1 = 2 \times 10^{-4}$ for $E_0 = 10$ keV and $N_0/N_1 = 2 \times 10^{-3}$ for $E_0 = 100$ keV. Again assuming $\Delta T_{max} \approx 300^\circ$, we find for $\sigma_{n\gamma}^0(E_n)$ the following approximate minimum values (in barns):

E_n, eV:	$2.6 \cdot 10^{-2}$	1	100
$\sigma_{n\gamma}^0$ $E_0 = 10$ keV:	$\gtrsim 100$	$\gtrsim 100$	$\gtrsim 1000$
$\sigma_{n\gamma}^0$ $E_0 = 100$ keV:	$\gtrsim 10$	$\gtrsim 10$	$\gtrsim 100$

The values given for 1 and 100 eV, on being recalculated according to a law $\sigma \propto E_n^{-1/2}$ to thermal energy (for room temperature), give (for $E_0 = 10$ keV) $\sigma_{therm}^{min} = 600$ and 60 000 barns. Thus, in order to provide the critical population of the upper level, while avoiding heating of the matrix as the result of recoil of beryllium nuclei in scattering, with increasing neutron energy E_n higher and higher cross sections are required for capture of neutrons by the working nuclei. However, up to $E_n \approx 100$ eV the necessary values of $\sigma_{n\gamma}^0$ are realistic.

Thus, it is possible at least in principle to choose a consistent set of all the physical parameters for which the conditions for stimulated emission of γ rays are achieved.

4. THE PROBLEM OF NEUTRON PUMPING

The very high values of n_0^*—not less than 10^{18}–10^{19} cm^{-3} for Mossbauer transitions with $E_0 \approx 10$–100 keV—correspond to very high required neutron pumping fluxes. Even with very high radiative-capture cross sections for the working nuclei ($\sigma_{n\gamma\,therm}^0 \approx 10^{-20}$ cm^2) integrated neutron fluxes of the order $J_\tau \approx 10^{19}$–10^{20} cm^{-2} are necessary for a short time τ much less than the excited level lifetime τ_0. Such fluxes can be obtained at the present time if at all only by use of neutrons from a nuclear explosion[1)]. In this case, of course, a number of specific difficulties must arise.

It is assumed that the flux of γ rays and thermal radiation which precedes the neutrons can be completely filtered out and will not lead to an unacceptable heating of the laser. In addition it is assumed that the stimulated radiation is produced before the shock wave reaches the laser. This places an upper limit on the choice of characteristic moderation times and can hinder the achievement not only of thermal energies (i.e., the largest capture cross sections) but even energies of about $E_n \sim 1$ eV which correspond to the minimum heating from the recoil nuclei of the beryllium matrix. If a hydrogen-containing moderator is used, the neutrons can be slowed down to ~100 eV in 10^{-7} sec at room temperature; for slowing down to ~1 eV, about 10^{-6} sec is already required. In this case the picture of a monoenergetic neutron pulse after the moderator is a crude idealization. In reality the energy spectrum will be extremely spread out, and the tail of this spectrum (the head from the point of view of arrival at the working medium of the laser) will be dangerous from the point of view of heating due to recoil from neutrino scattering, and therefore must be cut off.

If transitions with an upper-level lifetime comensurate with the neutron pulse duration are used, and if the lower state is stable or longer-lived than the upper state, the existence of a steep rise ($\Delta t \ll \tau_0$) of the neutron pulse is essential. Otherwise the spontaneous radiation from the upper level in the course of the neutron pumping can remove any possibility of stimulated radiation. In addition, it is necessary that the required supercritical integrated neutron pumping flux reach the working medium of the laser in a time $\tau \ll \tau_0$. Otherwise a population inversion cannot be achieved even with a rectangular shape of the pumping pulse[2)]. The ratio of τ and τ_0 is extremely important from the point of view of the questions of achievable supercriticality and the kinetics of deexcitation of the laser (we assume this deexcitation is sufficiently rapid that it is not necessary to take into account the contribution to heating of the laser medium due to losses of the resonance radiation itself).

5. CONCLUDING REMARKS

A very important problem is the determination of parameters which are optimal as an entire set and which could be used in the problem discussed here. Unfortunately, the relative populations of those levels of the cascade in radiative neutron capture between which the distance amounts to tens of keV are, as a rule, poorly known.

Furthermore, even such level parameters as τ_0, α, and so forth are often unknown, particularly if none of the levels is the ground state. In this case it is impossible to draw a conclusion as to the suitability of these transitions, it still remains unclear whether a population inversion ($\xi > 0$) will be achieved in general, and if it is achieved, what is the value of the very important parameter ξ.

The so-called isomeric ratio $\epsilon = (1+\xi)/(1-\xi)$, the ratio of the yields of the isomeric and ground nuclear states in various reactions (including $n\gamma$), has been determined in a number of experimental studies. However, as a rule, this ratio has been determined for short-lived isomers and we have not found ξ values for the examples of interest to us.

In traditional studies of γ rays in neutron capture, the low-energy transitions have been studied less than the others. Therefore it is desirable to carry out special experiments and calculations of the probabilities of population of various levels involved in low-energy γ transitions in the capture of neutrons by a number of nuclei. Here the initial targets can be not only stable but also sufficiently long-lived radioactive isotopes ($\tau \sim 10^5-10^6$ sec).

As an illustrative example let us consider the 6.2-keV transition to the ground state of Ta^{181} produced in the reaction $Ta^{180}(n\gamma)Ta^{181}$. In this case $\tau_0 \approx 10^{-5}$ sec, $\alpha = 45$[2], and $\sigma_0(E_0 = 6.2\text{ keV}) = 10^{-19}\text{ cm}^2$.[3] For the thermal-neutron radiative-capture cross section we take the value known for Ta^{182}, $\sigma^0_{n\gamma}(300°K = 1.7\times10^{-20}\text{ cm}^2$.[1] The important population-inversion parameter ξ remains unknown. As the matrix material we will take beryllium, for which the neutron cross sections[3] have been given above, and $\sigma_1(E_0) \approx 2.7\times10^{-23}\text{ cm}^2$.[4] The effective Debye temperature for tantalum impurity atoms in the beryllium matrix is given approximately by $\Theta_{eff} \approx \Theta_1\sqrt{A_0/A_1} \approx 270°K$, where $\Theta_1 \approx 1200°K$ is the Debye temperature of the matrix. In view of the extremely low value of the recoil energy of Ta^{181} nuclei in emission of resonance γ rays ($R_0 \approx 1.1\times10^{-4}$ eV), up to many hundreds of degrees we can set $f \approx 1$. The density of working atoms of tantalum in this case, in accordance with Eq. (6), is $N_0 = 2.8\times10^{19}\text{ cm}^{-3}$, and the mean free path of resonance γ rays from Eq. (5) in this case is $l(E_0) \approx 0.16$ cm. Then in accordance with Eq. (3) and for $\Gamma/\Gamma_0 \approx 1$ (which requires in this case special contrivance in view of the large τ_0, but nevertheless apparently can be achieved), we have $n_0^* \approx 3\times10^{18}\xi^{-1}\text{ cm}^{-3}$ and the critical integrated neutron flux for a time $\tau \ll 10^{-5}$ sec will be $J^*_\tau \approx 4\times10^{19}\xi^{-1}\sqrt{E_n}$ [cm^{-2}] (E_n in eV; we assume that $\sigma^0_{n\gamma}(E_n) \propto 1/\sqrt{E_n}$). The condition $n_0^* \le N_0$ requires that the parameter ξ be not less than 0.1, and then

$$J_\tau^* \approx 4\cdot10^{20}\sqrt{E_n}\ [\text{cm}^{-2}].$$

The combination of the high probability of the Mossbauer effect and weak temperature dependence with the long lifetime of the upper Mossbauer level of Ta^{181} would make it possible in this case to ease the requirements on the time and energy characteristics of the neutron pumping pulse. Whether the value $\xi = 0.1$ is realistic and, in general, whether a population inversion of the Ta^{181} levels is achieved in radiative capture of neutrons by Ta^{180} nuclei, remain unknown.

1) The total number of excited nuclei which have captured neutrons in one laser needle, necessary to achieve stimulated radiation, is only $10^{13}-10^{14}$. In principle we are thinking of that form of neutron pumping in which slow neutrons are focused on the working material of the laser and are absorbed in multiple traversals. However, actual means of accomplishing such a pumping arrangement on the necessary scale and in addition for the required short time interval are not yet apparent.

2) For a rectangular pumping pulse of duration τ the quantity $\xi' = (1+\xi)(\tau_0/\tau)(1-\exp\{-\tau/\tau_0\})-1$ enters into Eqs. (3) and (4) instead of ξ.

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