Gravitational collapse with a physical singularity on an isotropic hypersurface

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A special class of solutions of the Einstein equation is found for the case of "gravitational collapse" of a central-symmetric distribution of matter with an ultrarelativistic equation of state. The collapse hypersurface in these solutions is isotropic whereas in the "general" solution of the present problem[1] the singularity occurs on a space-like hypersurface.

1. Lifshitz and Khalatnikov (LKh)[1] considered the problem of the collapse of a centrally-symmetrical distribution of matter with ultrarelativistic equation of state of matter $p = \epsilon/3$[1)]. The centrally-symmetrical element of the interval was expressed in the form

$$-ds^2 = -dt^2 + e^{\lambda}dr^2 + e^{\mu}(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (1.1)$$

The solution was sought in the form of series in powers of $t^{2/3}$. The expressions obtained for λ and μ were

$$\lambda = -{}^2/_3 \ln t + \lambda^{(0)} + \lambda^{(1)}t^{2/3} + \ldots, \qquad \mu = {}^4/_3 \ln t + \mu^{(0)} + \mu^{(1)}t^{2/3} + \ldots, \qquad (1.2)$$

where $\lambda^{(0)}$, $\mu^{(0)}$, $\lambda^{(1)}$, $\mu^{(1)}$, ... are functions of r. For the energy density $\epsilon(r)$ and the radial velocity $u_1(r)$, the expressions obtained were $\epsilon = \epsilon^{(0)}t^{-1/4} + \ldots$ and $u_1(r) = u^{(0)}(r)t^{1/3} + \ldots$.

The cited solution contains three arbitrary functions (for example, $\mu^{(0)}(r)$, $\lambda^{(0)}(r)$, $\epsilon^{(0)}(r)$), from which it is possible to construct two physically different arbitrary functions. This number corresponds to the fact that the arbitrary initial centrally-symmetrical distribution of the matter is specified by simultaneous distributions of its density and radial velocity, and this problem has no "degrees of freedom" corresponding to a free gravitational field, since such a field (gravitational waves) cannot have central symmetry. On this basis, LKh called the obtained solution "general." However, since the solution is constructed in the form of series, there is no clear and sufficiently detailed connection between the two methods of sorting out the solutions: with the aid of two physically different arbitrary functions constructed from $\mu^{(0)}$, $\lambda^{(0)}$, and $\epsilon^{(0)}$, or, on the other hand, by specifying the initial distributions of the density of matter and of its radial velocity. One can therefore not exclude beforehand the case in which singular initial distributions of the density and velocity of matter exist and generate solutions that cannot be obtained in the form (1.2) for any choice of the arbitrary functions $\lambda^{(0)}$, $\mu^{(0)}$, and $\epsilon^{(0)}$ (see also[2]). The class of solutions presented in[1], with one physically arbitrary function (quasi-isotropic collapse), is not contained in the class of (1.2). The quasi-isotropic collapse must be regarded as an example of the aforementioned singular solutions.

In the present paper we present a class of singular solutions of the given problem, which differs from the quasi-isotropic collapse and contains no fewer solutions than the latter. It should be noted that for each of the solutions described in[1] there exists a four-coordinate transformation, with the aid of which the collapse hypersurface $t = \varphi(r)$ can be transformed into a hypersurface $t = 0$ without violating the synchronism conditions. There is no such transformation, as will be shown below, for any of the solutions of the new class. In other words, the collapse corresponding to solutions of this class is inhomogeneous in any synchronous system.

2. The gravitation equations for the given problem are[1]

$$\begin{aligned}
R_0^0 &= {}^1/_4\dot\lambda^2 + {}^1/_2\dot\mu^2 + {}^1/_2\ddot\lambda + \ddot\mu = -{}^1/_3\varepsilon(3 + 4u_1^2e^{-\lambda}),\\
R_1^0 &= \dot\mu' - {}^1/_2\dot\lambda\mu' + {}^1/_2\dot\mu\mu' = {}^4/_3\varepsilon u_1(1 + u_1^2e^{-\lambda})^{1/2},\\
R_1^1 &= e^{-\lambda}({}^1/_2\mu'\lambda' - \mu'' - {}^1/_2\mu'^2) + {}^1/_2(\ddot\lambda + \dot\lambda\dot\mu + {}^1/_2\dot\lambda^2) = {}^1/_3\varepsilon(1 + 4u_1^2e^{-\lambda}),\\
R_2^2 &= R_3^3 = e^{-\mu} + {}^1/_2e^{-\lambda}({}^1/_2\mu'\lambda' - \mu'' - \mu'^2) + {}^1/_2(\ddot\mu + {}^1/_2\dot\mu\dot\lambda + \dot\mu^2) = {}^1/_3\varepsilon.
\end{aligned} \qquad (2.1)$$

Eliminating u_1 and ϵ, we obtain two equations for λ and μ

$$R_i^i = \ddot\lambda + {}^1/_2\dot\lambda^2 + \dot\lambda\dot\mu + 2\ddot\mu + {}^3/_2\dot\mu^2 + e^{-\lambda}(\mu'\lambda' - 2\mu'' - {}^3/_2\mu'^2) + 2e^{-\mu} = 0, \qquad (2.2)$$

$$\begin{aligned}
&(R_1^1 - R_2^2)(R_1^1 + 3R_2^2) - e^{-\lambda}(R_1^0)^2 = -3e^{-2\mu} + e^{-\lambda-\mu}(\mu'' + 2\mu'^2\\
&- {}^1/_2\mu'\lambda') + e^{-\mu}(\ddot\lambda + {}^1/_2\dot\lambda^2 - 3\ddot\mu - 3\dot\mu^2 - {}^1/_2\dot\lambda\dot\mu) + {}^1/_2e^{-2\lambda}({}^5/_8\mu'^2\lambda'^2 - {}^5/_2\mu''\mu'\lambda'\\
&+ {}^5/_2\mu''^2 - \lambda'\mu'^3 + 2\mu''\mu'^2) + {}^1/_4(\ddot\lambda^2 + \ddot\lambda\dot\lambda^2 + {}^1/_4\dot\lambda^4 + 2\ddot\lambda\ddot\mu + 2\ddot\lambda\dot\mu^2\\
&+ 3\ddot\lambda\dot\lambda\dot\mu + \ddot\mu\dot\lambda^2 + {}^9/_4\dot\lambda^2\dot\mu^2 + {}^3/_2\dot\lambda^3\dot\mu - 3\ddot\mu^2 - 6\ddot\mu\dot\mu^2 - \dot\lambda\dot\mu\ddot\mu - 3\dot\mu^4\\
&- \dot\lambda\dot\mu^3) + {}^1/_4e^{-\lambda}(3\mu'\lambda'\ddot\lambda + {}^3/_2\mu'\lambda'\dot\lambda^2 - \mu'\lambda'\ddot\mu - \mu'\lambda'\dot\mu^2 + {}^5/_2\mu'\lambda'\dot\mu\dot\lambda\\
&- 6\mu''\ddot\lambda - 3\mu''\dot\lambda^2 + 2\mu''\ddot\mu + 2\mu''\dot\mu^2 - 5\mu''\dot\lambda\dot\mu - 4\mu'^2\ddot\lambda - 3\mu'^2\dot\lambda^2 + 4\mu'^2\ddot\mu\\
&+ 3\mu'^2\dot\mu^2 - 4\dot\mu'^2 + 4\dot\lambda\mu'\dot\mu' - 4\dot\mu\mu'\dot\mu') = 0.
\end{aligned} \qquad (2.3)$$

The second and fourth equations of the system (2.1) then express respectively the velocity u_1 and the energy density ϵ in terms of the "metric" functions μ and λ. It is easy to verify that the system (2.2) and (2.3) and the aforementioned two equations of (2.1) is equivalent to the complete system (2.1). We seek the solution of the system in the form of the series

$$\begin{aligned}
\lambda &= \Lambda(r) + \sum_{n=2}^{\infty}\lambda_n(r)\nu^{n/3}, \qquad \mu = \frac{2}{3}\ln\nu + \sum_{n=0}^{\infty}\mu_n(r)\nu^{n/3},\\
\varepsilon &= \sum_{n=-4}^{\infty}\varepsilon_n(r)\nu^{n/3}, \qquad u = \sum_{n=-1}^{\infty}u_n(r)\nu^{n/3},
\end{aligned} \qquad (2.4)$$

where $\nu = t_0(r) - t$ (on the collapse hypersurface $t = t_0(r)$ we have $\nu = 0$.)

When (2.4) is substituted in (2.2), the terms of order ν^{-2} yield $\Lambda(r) = 2 \ln t_0'$, while the terms of order ν^{-4} in (2.3) yield nothing new. The terms of order $\nu^{(n-5)/3}$ in (2.2) yield

$$\begin{aligned}
&\frac{(n+4)(n-1)}{9}\lambda_{n+1} + \frac{2}{9}(n+1)\mu_1\lambda_n + \frac{2}{9}(n+1)(n-1)\lambda_2\mu_{n-1}\\
&+ (n-2)\left[\frac{1}{3}\mu_1\lambda_2 - \frac{\mu_0'}{t_0'} + \frac{2}{9}(n+1)\lambda_3\right]\mu_{n-2} - \frac{2}{3}(2n-1)\frac{\mu_{n-2}'}{t_0'}\\
&+ \left[\frac{2}{9}\lambda_2 + \frac{4}{9}(n+1)\mu_2 + \frac{1}{6}\mu_1^2\right]\lambda_{n-1} - \frac{\mu_1}{t_0'}\mu_{n-3}'
\end{aligned}$$

$$+(n-3)\left[\frac{2}{9}(n+1)\lambda_4+\frac{1}{3}\mu_1\lambda_3+\frac{2}{3}\mu_2\lambda_2-\frac{(n-1)}{9}\lambda_2^2-\frac{\mu_1'}{t_0'}-\frac{2e^{-\mu_0}}{(n-3)}\right]\mu_{n-3}+\ldots=0. \quad (2.5)$$

The terms of order $\nu^{(n-11)/3}$ of (2.3) (when multiplied by $^9/_4$) yield

$$\frac{(n+4)(n-1)}{9}\lambda_{n+1}+\frac{2}{9}(n+1)\mu_1\lambda_n+\frac{2}{9}(n+1)(n-1)\lambda_2\mu_{n-1}$$
$$+(n-2)\left[\frac{(n-5)}{2}\frac{\mu_0'}{t_0}+\frac{(n+13)}{18}\lambda_3-\frac{(n-5)}{6}\mu_1\lambda_2\right]\mu_{n-2}$$
$$-\frac{2}{3}(2n-1)\frac{\mu_{n-2}'}{t_0'}-\frac{\mu_1}{t_0'}\mu_{n-3}'+(n-3)\left[-\frac{(5n-2)}{72}\lambda_2\mu_1^2\right.$$
$$-\frac{(11n-16)}{36}\lambda_2^2-\frac{(5n-23)}{9}\lambda_2\mu_2-\frac{(4n-19)}{18}\mu_1\lambda_3-\frac{(3n-32)}{18}\lambda_4$$
$$\left.+\frac{5n-26}{6t_0'}\mu_1'+\frac{(n-2)\mu_0'}{4t_0'}\mu_1-\frac{1}{2}\frac{(n^2-7n+16)e^{-\mu_0}}{(n-3)}\right]\mu_{n-3}$$
$$-\left[\frac{2}{9}(n-4)\lambda_2+\frac{n^2-9n-14}{18}\mu_2+\frac{n^2-n-18}{72}\mu_1^2\right]\lambda_{n-1}+\ldots=0. \quad (2.6)$$

Subtracting (2.5) from (2.6) and making the substitution $n \to n+2$, we obtain

$$(n-1)\left[\frac{2}{9}\lambda_2+\frac{(n+4)}{18}\mu_2+\frac{(n+4)}{72}\mu_1^2\right]\lambda_{n+1}+(n-1)\left[\frac{7}{18}(n-2)\lambda_4\right.$$
$$+\frac{(4n-5)}{18}\mu_1\lambda_3+\frac{(5n-7)}{9}\mu_2\lambda_2+\frac{(7n+2)}{36}\lambda_2^2+\frac{(5n+8)}{72}\mu_1^2\lambda_2$$
$$\left.+\frac{(n-2)}{2}e^{-\mu_0}-\frac{n\mu_0'}{4t_0'}\mu_1-\frac{5}{6}(n-2)\frac{\mu_1'}{t_0'}\right]\mu_{n-1}+\ldots=0. \quad (2.7)$$

The dots denote here terms containing those coefficients λ_i and μ_i of the series for λ and μ, which are contained in (2.4) ahead of those written out in (2.6) and (2.7) (for example, we have omitted from (2.7) the terms containing λ_n, λ_{n-1}, λ_{n-2}, etc.).

Equations (2.5) and (2.7) are recurrence relations between λ_n and μ_n. At $n < 4$, direct calculations yield:

a) $n = 1$, $\lambda_2(r)$ is an arbitrary function;

b) $n=2,\ \mu_1(r)=3\mu_0'/\lambda_2t_0'+3\lambda_2'/2\lambda_2^2t_0',\ \lambda_3(r)=-3\lambda_2'/2t_0'\lambda_2.$ (2.8)

c) $n = 3$, μ_2 and λ_4 are expressed regularly in terms of μ_0, λ_0, and λ_2, with

$$-\frac{7}{2}\lambda_4=\frac{1}{4}\lambda_2^2+4\lambda_2\mu_2+2\mu_1\lambda_3-\frac{9}{4}\frac{\mu_0'}{t_0'}\mu_1-\frac{15}{2}\frac{\mu_1'}{t_0'}+\frac{3}{8}\mu_1^2\lambda_2+\frac{9}{2}e^{-\mu_0}.$$

Using these expressions, we find that the determinant of the system (2.7) and (2.5) is equal to $D = -\frac{1}{54}(n+2)n(n-1)^2\lambda_2^2 \neq 0$ for $n \geq 4$, so that λ_{n+1} and μ_{n+1} are regularly expressed through the preceding terms. Thus, for example, the system (2.5) and (2.7) at $n = 4$ determines λ_5 and μ_3 in terms of $\lambda_4, \lambda_3, \ldots, \mu_2, \ldots, \mu_0$, and in final analysis in terms of the arbitrary functions t_0', μ_0, and λ_2.

Thus,

$$\lambda=2\ln t_0'+\lambda_2\nu^{2/3}+\lambda_3\nu+\ldots,\quad \mu={}^2/_3\ln\nu+\mu_0+\mu_1\nu^{1/3}+\ldots, \quad (2.9)$$

where μ_1 and λ_3 are given by (2.8). Substituting (2.9) in (2.1), we obtain

$$\varepsilon=\varepsilon_{-4}(r)\nu^{-4/3}+\ldots,\quad u=u_{-1}(r)\nu^{-1/3}+\ldots, \quad (2.10)$$

where $\epsilon_{-4} = \frac{1}{3}\lambda_2$, $u_{-1}^2 = t_0'^2/\lambda_2$.

The constructed solution (2.9) and (2.10) contains three arbitrary functions of r, namely $t_0(r)$, $\lambda_2(r)$, and $\mu_0(r)$.

3. Let us consider an arbitrary solution from (2.9)—(2.10). What coordinate transformations

$$\tau=\varphi(t,r),\quad R=\psi(t,r) \quad (3.1)$$

leave it synchronous?

The requirement that the synchronism be preserved in the new coordinates τ and R is expressed by the equations

$$\dot\varphi^2-e^{-\lambda}\varphi'^2-1=0,\quad \dot\varphi\dot\psi-e^{-\lambda}\varphi'\psi'=0, \quad (3.2)$$

where the derivatives are taken with respect to the old coordinates t and r. The system (3.2) with $e^{-\lambda}$ from (2.9) has a solution in the form of a series in $\nu^{1/3}$

$$\tau=\varphi(t,r)=\varphi_0(r)+\varphi_3(r)\nu+\ldots,\quad R=\psi(t,r)=\psi_0(r)+\psi_3(r)\nu+\ldots. \quad (3.3)$$

Here $\varphi_0(r)$ and $\psi_0(r)$ are arbitrary functions, in terms of which the succeeding coefficients of the series (3.3) are expressed. Thus,

$$\varphi_3=-\frac{1}{2}\left(\frac{t_0'}{\varphi_0'}+\frac{\varphi_0'}{t_0'}\right);\quad \psi_3=\frac{1}{2}\frac{\psi_0'}{\varphi_0'}\left(\frac{t_0'}{\varphi_0'}-\frac{\varphi_0'}{t_0'}\right), \quad (3.4)$$

etc. The solutions (2.9) and (2.10) are meaningful only at $t_0' \neq 0$ (because of $\ln t_0'$). We then have in (3.3)

$$\varphi_0'\neq 0, \quad (3.5)$$

otherwise we obtain for φ_3 in (3.4) a meaningless expression. It follows from (3.5) that $\tau_0(R) \equiv \varphi_0(\psi_0^{-1}(R)) \neq$ const. This means that in terms of the new coordinates the collapse hypersurface (described here, as can be readily seen, by the equation $\tau = \tau_0(r)$) is likewise not a hyperplane.

To the contrary, the solutions obtained in[1] in homogeneous form (with a collapse hyperplane $t = 0$) can be changed by means of the synchronous coordinate transformation

$$\tau=\varphi_0(r)+t+\varphi_5(r)t^{5/3}+\ldots\equiv\varphi(t,r),$$
$$R=\psi_0(r)+\psi_5(r)t^{5/3}+\ldots\equiv\psi(t,r), \quad (3.6)$$

which does not violate the synchronism, into a form that is generally speaking inhomogeneous:

$$\lambda=-{}^2/_3\ln\nu+\lambda^0(R)+\ldots,\quad \mu={}^4/_3\ln\nu+\mu^{(0)}(R)+\ldots,$$
$$\varepsilon=\varepsilon^{(0)}(R)\nu^{4/3}+\ldots,\quad u=u^{(0)}(R)+\ldots,\quad u_0(R)=\tau_0'(R) \quad (3.7)$$

with a collapse-hypersurface equation $\tau = \varphi_0(\psi_0^{-1}(R)) = \tau_0(R)$. Since $\varphi_0(R)$ and $\psi_0(R)$ are arbitrary, they can be chosen such that $\tau_0(R) \neq$ const. In other words, for the solution from (3.7), in certain synchronous systems, the collapse time at different space points is generally speaking different, whereas in the initial synchronous system in which this solution was obtained it has a singularity simultaneously at different space points.

Thus, our solution (2.9), (2.10) is made different from (3.7) by the absolute, so to speak, "inhomogeneity" of the collapse in the synchronous coordinate systems. The class (2.9), (2.10) is closed with respect to the synchronism-conserving coordinate transformations (3.3).

We note that the solutions (2.9) and (2.10) obtained here contain one physically arbitrary function, since we have at our disposal the transformations (3.3), which do not take us out from the given class, with two arbitrary functions $\varphi_0(r)$ and $\psi_0(r)$, the choice of which makes it possible to impose two relations on the three arbitrary functions of the solutions (2.9) and (2.10).

We show in conclusion that the collapse hypersurface $t = t_0(r)$ is isotropic. Indeed, the element of the interval between two infinitesimally close world points on this hypersurface ($\nu = 0$)

$$ds^2 = dt^2 - e^\lambda dr^2 - e^\mu do^2 = (t_0' dr)^2 - t_0'^2 dr^2 - 0 = 0.$$

The isotropic character of the collapse hypersurface confirms once more that the obtained class of solutions is not contained in the general solution[1], where the collapse hypersurface is space-like.

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[1)]In Sec. 1 we follow the notation and the assumptions of LKh.

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