STRUCTURE OF RESIDUAL INTERACTION IN SPHERICAL NUCLEI

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The effect of residual interaction between nucleons (quasiparticles) on shell oscillations of the masses of spherical nuclei is considered. The singularity of the ground state energy of the system in the vicinity of nucleon magic numbers is analyzed for various types of the dependence of residual interaction on orbital momentum of the quasiparticle. It is shown that only the perturbation band width of the Fermi distribution due to residual interaction which is proportional to the square of the angular momentum vector is consistent with the character of the magic cusps. The coupling constants between the quasiparticles are determined on the basis of the available data. The constant decreases rapidly with increase of nuclear radius. Possible consequences pertaining to the energy spectrum of infinite nuclear matter are discussed.

1. INTRODUCTION

In a preceding paper[1] we considered the influence of the sharpness of the boundary of quasiparticle Fermi distribution on the characteristics of spherical nuclei. It turned out that the number $\tilde{N}(\rho_f)$ of single-quasiparticle states located below the boundary $\rho_f = k_f R \gg 1$ (k_f is the corresponding value of the wave number and R is the radius of the nucleus) is strictly speaking not an analytic function of its argument. In the ρ scale, the singularities of the function are equidistant, with an interval $\pi/2$; a qualitative idea of their character is gained from Fig. 1a. These are the magic cusp points (jumps of the first derivative); a consistent analysis shows that singularities of this type are possessed also by the energy of the ground state of the body, i.e., the mass of the nucleus. Figure 2 shows schematically a typical experimental plot of the mass in the vicinity of a magic nucleus[1)]. We have in mind relatively "weak" singularities; they appear in the higher-order terms of the expansion in the reciprocal powers of ρ_f. We note that in this approximation, which is of interest in nuclear physics, there are no grounds whatever for identifying the function $\tilde{N}(\rho_f)$ with the total number of true particles $N(\rho_f)$. Furthermore, we can advance the following considerations: The roughly intuitive concept of the quasiparticle as moving in the average field produced by practically all the particles of the body accounts nevertheless quite well for the main gist of the phenomenon. Since the dimensionless parameter $\rho_f = k_f R$ is determined, roughly speaking, from the characteristics of this average field, it is natural to assume this field to have a smooth dependence on the total number of particles N, as is indeed assumed (see Fig. 1b). It is easy in practice to make the transition to the physically most significant N scale, for example, by expanding in reciprocal powers of ρ_f (see[1], formulas (3), (26), and (27) and the explanations pertaining to them).

We shall now comment on the nature of the shell and magic oscillations in spherical nuclei from a point of view that reveals clearly the role of the quantum num-

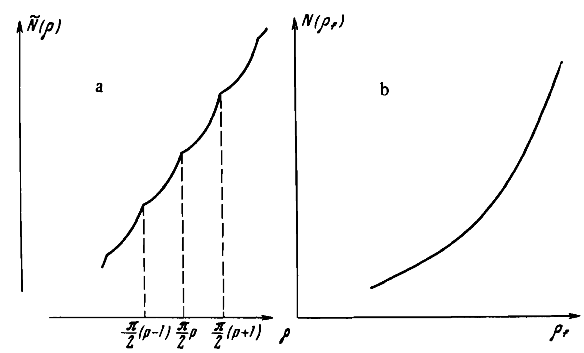

FIG. 1

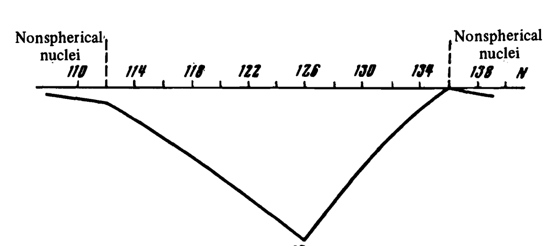

FIG. 2. Schematic diagram of the shell oscillations of the nuclear mass in the vicinity of the magic number N = 126.

[1)]We emphasize the physically unrealistic character of the "submagic" nucleon numbers, which can be formally set in correspondence with the filling of each j-level in a certain spherically-symmetrical potential well. No phenomenon pertaining to the nucleus as a whole is observed experimentally in this case; nor does a consistent theoretical analysis seem to predict any phenomenon. Moreover, not all the characteristics of the referred-to "well" admit of a clear-cut physical definition. The question is: what takes place at the point where the magic nucleus is located? Does the Fermi boundary rise because the occupation of the last j-level becomes filled (the most frequently advanced point of view) or, to the contrary, does the edge of the "well" drop jumpwise in a direction opposite to the Fermi boundary? This question has apparently no sufficiently distinct physical meaning, all the more since the position of the bottom of the "well" is not a rigorous quantitative concept, owing to the strong damping of the deep quasiparticles. These jumps of the chemical potential occur by far not after each filling of the next j-level. There are no submagic phenomena in spherical nuclei at all, there are only true magic nuclei. Their position is given theoretically by formulas (19) and (3) of the preceding paper [1].

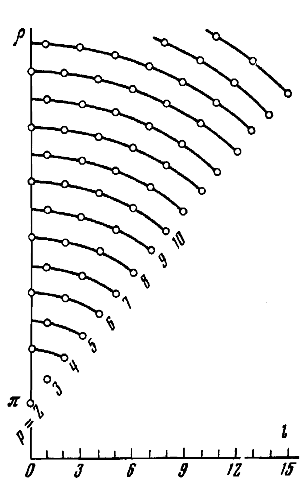

FIG. 3. Roots of the Bessel functions with half-integer index on the (l, ρ) diagram.

bers of the individual quasiparticles. Of particular importance is the conservation of the orbital angular momentum l; the eigenvalues $\rho = kR$ can then be graphically represented by points on the (l, ρ) plane. It is easily seen from Fig. 3 how they are grouped in the region of relatively small orbital angular momenta. The "Regge trajectories" drawn in accordance with the rule $2n + l = p$ (n is the principal quantum number and p is the number of the trajectory) lie near their maximum. More concretely, their form is given by the equation

$$\Delta\rho = -(l + 1/2)^2 / 2\rho \quad (1)$$

(see also Appendix 1). The vertical distance between the curves is equal to $\pi/2$, corresponding precisely to the aforementioned interval between the neighboring shells in the ρ scale[2)]. We can now easily visualize the situation that arises if the Fermi boundary

$$\rho = \rho_f \quad (2)$$

is assumed to move, say, upward. After the levels of the last Regge trajectory are exhausted at the tangency point (see Fig. 3), the density of states $d\tilde{N}/d\rho$ decreases jumpwise. A more formal natural interpretation is that the function $\tilde{N}(\rho)$ has an oscillating component $\tilde{N}_1(\varphi)$, the period of which is determined by the distance between the trajectories in Fig. 3.

This specific grouping of the single-quasiparticle levels, due to the quantum number l, is reflected also in the energy E of the nucleus. Its oscillating component is equal to

$$E_1(\rho_f) = -\varepsilon \tilde{N}_1(\rho_f), \quad (3)$$

where $\tilde{N}_1(\rho_f)$ is the oscillating part of the number of the states filled with fermions (quasiparticles). The

[2)]In connection with Fig. 3, there is a curious, simple, and experimentally confirmed consequence: each nuclear shell contains either one s or one p state, and the corresponding single-particle levels lie close in energy to the end of the filling of this shell. For the atom, there is no such theorem. The absence of any far-reaching analogy with the shell structure of the atom was already indicated in the preceding paper (see footnote 2 of [1]). Certain features of the nuclear structure, which are weakly pronounced in the case of the atom, were mentioned also in footnote 1 above.

proportionality coefficient $-\epsilon$ is the first functional derivative with respect to the variation of the distribution function near the Fermi boundary; in other words, it is the value assumed here by the energy of one quasiparticle. Since, on the other hand, $\epsilon = -dE/dN$ is the binding energy of the nucleon in the nucleus, we are thus dealing with a quantity that can be directly determined from experiment. For case (2) of an absolutely sharp Fermi boundary, the form of the function $\tilde{N}_1(\rho_f)$ was determined in[1]. Of particular interest is that decrease $\Delta\epsilon$ in the binding energy of the nucleon which is observed in the vicinity of the magic nucleus. In the ideal case (2) we have

$$(\Delta\varepsilon)_0 = \bar{\varepsilon}\, d\rho_f^2 / dN, \quad (4)$$

where $\bar{\epsilon}$ is the arithmetic mean of the values of ϵ on both sides of the magic nucleus.

To attempt to make even clearer the correspondence with the level distribution with respect to the momenta, shown in Fig. 3, let us consider the following example, which incidentally is somewhat artificial: let the quasiparticles fill only all the vacancies that lie below the boundary, the equation for which is

$$\rho(l) = \rho_f - g(l + 1/2)^2 / 2\rho_f. \quad (5)$$

Then calculations similar to those in[1] yield

$$\Delta\varepsilon = \bar{\varepsilon}\frac{d\rho_f^2}{dN}\frac{1}{1-g}. \quad (6)$$

Thus, as $g \to 1$, when the curvilinear boundary (5) of the statistical distribution merges with the successive trajectories in Fig. 3 (see also Eq. (1)), the oscillations tend to infinity in the considered approximation.

The example (5) illustrates only the large sensitivity of the oscillations to redistributions of the quasiparticles with respect to the quantum states; it is has, of course, little in common with the real situation in spherical nuclei. Finite nuclear dimensions mean apparently that it is impossible to have a canonical transformation to quasiparticles with respect to which the case (2) of an ideally abrupt "stepwise" Fermi distribution would be realized. This circumstance is frequently called in nuclear physics the "residual interaction" between the nucleons. It is felt intuitively that the "smearing" of the Fermi boundary, due to this interaction, suppresses, generally speaking, the oscillations. It is seen already from Fig. 3 that as the "smeared" Fermi boundary moves forward the Regge trajectories, roughly speaking, are gradually captured. Indeed, in the experiment the magic jumps $\Delta\epsilon$ are always smaller than the ideal value $(\Delta\epsilon)_0$ calculated from formula (4). However, by far not any smearing of the Fermi boundary is capable of corresponding, even qualitatively, to the character of the experimental data. For example, a statistical distribution of the temperature type

$$w_T(\rho) = [\exp\{(\rho - \rho_f)/\tau\} + 1]^{-1} \quad (7)$$

yields for $\tilde{N}_1(\rho_f)$ an expression that is everywhere differentiable an arbitrary number of times. Accordingly, in this case the function $N_1(\rho_f)$ is analytic and there are no magic singularities[3)].

Let us touch also on the question of the most advantageous way of specifying and investigating the residual interaction. One can imagine, of course, a situation wherein the residual interaction between quasiparticles is specified in the form of a corresponding Hamiltonian; one such model example will be calculated in the next section. It is important, however, not to lose sight of the following circumstance: although the presence of the interaction causes, strictly speaking, the energy of the individual quasiparticle to be no longer a definite quantity, this does not influence the applicability of formula (3) in the approximation of interest to us. Indeed, let the Fermi-distribution smearing, due to the residual interaction have a width $\delta\epsilon$; this can be naturally interpreted as an uncertainty, of the same order of magnitude, in the quasiparticle energy. On the other hand, in the ρ scale (see Fig. 3, and also[1]), the quantity $\delta\rho$ characterizing the oscillations is of the order of unity. Therefore in the region of importance for the oscillations we have $\delta\epsilon \sim (d\epsilon/d\rho)\delta\rho \sim \epsilon/\rho_f$. Recognizing that $\rho_f \gg 1$, we have

$$\delta\varepsilon \ll \varepsilon, \tag{8}$$

i.e., the first factor in the right-hand side of (3) remains definite with sufficient degree of accuracy. In other words, the redistribution of the quasiparticles over the states, due to the residual interaction, should be determined in principle from the condition of the minimum of the energy of the nucleus as a whole. But then the oscillations can be calculated without taking into account the additional energy of the interaction between the quasiparticles. This curious feature of the theory will be illustrated by a concrete example in the next section.

A "dynamic," so to speak, treatment (meaning that the interaction Hamiltonian between the quasiparticles is explicitly specified) would be highly ambiguous. Since, in any case, there is no canonical transformation that leads to an absolutely abrupt Fermi boundary(2), we apparently have no sufficiently reasonable criterion for a unique choice of the transformation to a new quasistatic Hamiltonian. In accordance with (8), the residual interaction can be more adequately described by directly specifying the quasiparticle distribution function with respect to the states. By regarding this function $w(\rho, l)$ in a certain sense as a primary concept, we can apparently hope to use successfully its simple single-parameter approximations. Such a characteristic feature of the phenomenon as the tendency of the residual interaction to decrease with increasing dimensions of the nucleus is likewise fairly well accounted for in this case. We shall return to the pertinent questions in the last two sections.

[3)]The last statement can be regarded also as a consequence of a theorem of more general character. It is easy to verify that any continuous distribution of quasiparticles over the states, which does not depend on l, results in no singularities of the type of a jump in the first derivative. The next two sections of this paper we obtain a relation compatible with the experimentally observed picture of the magic phenomena between the width of the smearing of the Fermi distribution and the quantum number l.

2. SIMPLEST MODEL OF RESIDUAL INTERACTION

When choosing a model example, it is desirable to take into account the experimental fact that the spin of an even-even nucleus in the ground state is equal to zero, and that for odd nuclei the spin always has the single-particle value within the framework of the shell model (see, for example,[2]). This suggests the expression

$$H_{int}^j = -G_j \sum_{m,m'>0} a_{m'}^+ a_{-m'}^+ a_{-m} a_m \tag{9}$$

for the interaction between quasiparticles pertaining to the same j-level. Here a_m^+ and a_m are the creation and annihilation operators of a quasiparticle with a z-projection of the angular momentum equal to m. The Hamiltonian (9) can be diagonalized exactly (see, for example,[3]); the eigenvalues are given by the well-known formula of Racah and Mottelson

$$E_{int} = -G_j b_j [\Omega_j - (b_j - 1) - s_j], \tag{10}$$

where $2\Omega_j = 2j + 1$ is the total number of vacancies, b_j is the number of interacting pairs at the j-level, and s_j is the number of noninteracting quasiparticles (seniority).

It is clear that the smallest energy of the system corresponds to zero seniority, and consequently for the nucleus as a whole the situation reduces to a minimization of the sum

$$E = \sum_j [2\varepsilon_j b_j - G_j b_j (\Omega_j - b_j + 1)] \tag{11}$$

over the j-levels (ϵ_j is the initial value of the quasiparticle energy). The redistribution of the quasiparticles should be visualized as occurring at a zero variation of the quantity

$$\bar{N} = \sum_j 2b_j. \tag{12}$$

This additional condition can be easily taken into account by the method of indeterminate Lagrange coefficients. Taking also the Pauli principle into account (i.e., the condition $0 \le b_j \le \Omega_j$), we get

$$w_j = \frac{b_j}{\Omega_j} = \begin{cases} 1, & \varepsilon_j - \varepsilon_f < \Delta_-, \\ \dfrac{G_j(\Omega_j+1) - 2(\varepsilon_j - \varepsilon_f)}{2G_j\Omega_j}, & \Delta_- < \varepsilon_j - \varepsilon_f < \Delta_+, \\ 0, & \varepsilon_j - \varepsilon_f > \Delta_+, \end{cases} \tag{13}$$

where ϵ_j is the chemical potential and $\Delta_\mp = (1/2) G_j (1 \mp \Omega_j)$ We present also the interaction energy (10) corresponding to the equilibrium distribution:

$$E_{int}^j = \begin{cases} -G_j\Omega_j, & \varepsilon_j - \varepsilon_f < \Delta_-, \\ -\dfrac{G_j^2(\Omega_j+1)^2 - 4(\varepsilon_j - \varepsilon_f)^2}{4G_j}, & \Delta_- < \varepsilon_j - \varepsilon_f < \Delta_+, \\ 0, & \varepsilon_j - \varepsilon_f > \Delta_+, \end{cases} \tag{14}$$

We note that the initial expression (10) was the result of simultaneous diagonalization of the Hamiltonian (9) and the operator of the number of quasiparticles at the j-level; then the quantum numbers b_j are naturally integers. However, in the subsequent derivation of the distribution (13) it becomes necessary to differentiate already with respect to variables b_j. They are consequently determined with an error on the order of unity. This accuracy corresponds precisely to the macroscopic character deduced in[1] for the investigated phenomena. We ultimately replace $\Omega_j \pm 1$ by $\Omega_j = j + 1/2$; with the same accuracy, we can replace the total angular momentum j of the quasiparticle by its orbital momentum l and neglect the spin-orbit interaction[4)].

We now change to more convenient variables, using a notation close to that of[1] (see also[4]):

$$\Omega_j \cong \rho \sin \beta \cong \rho_f \tilde{\beta}, \quad \tilde{\beta} = \arcsin (\tilde{l} / \rho), \quad \tilde{l} = l + 1/2,$$
$$G_j = \frac{d\varepsilon}{d\rho}\bigg|_f g_j. \qquad (15)$$

We have taken into account here the fact that it is the small $\tilde{\beta}$ that are significant for the oscillations. In this limit, we assume a power-law dependence of the coupling constant on $\tilde{\beta}$:

$$g_j = g\, \tilde{\beta}^{k-1}. \qquad (16)$$

In terms of the variables ρ and $\tilde{\beta}$, the distribution (13) takes the form

$$w = \begin{cases} 1, & \rho - \rho_f < -\frac{1}{2} g \rho_f \tilde{\beta}^k, \\ \frac{1}{2} - \frac{\rho - \rho_f}{g \rho_f \tilde{\beta}^k}, & -\frac{g}{2} \rho_f \tilde{\beta}^k < \rho - \rho_f < \frac{g}{2} \rho_f \tilde{\beta}^k, \\ 0, & \rho - \rho_f > \frac{1}{2} g \rho_f \tilde{\beta}^k. \end{cases} \qquad (17)$$

Thus, in terms of the distribution function $w(\rho, \tilde{l})$, the model (9)—(11) turns out to be the simplest of all the conceivable ones. It represents, in fact, a linear interpolation between regions where the occupation numbers of the quasiparticles assume the limiting values zero and unity, see Fig. 4. Instead of analyzing the expression for the second variation, it is easier to verify the stability of the distribution (13), (17) directly, by effecting all the conceivable transfers of the pair of interacting quasiparticles between the states, and calculating the corresponding change of the energy of the nucleus. According to Fig. 4, four types of quasiparticle transfer are permissible: I → II, I → III, II → II, and II → III. In all cases we have for the change of the system energy

$$\Delta E \geqslant G_j^{\text{in}} + G_j^{\text{fin}} > 0, \qquad (18)$$

i.e., the distribution of the quasiparticles over the states is stable.

To calculate the oscillating part of the nuclear energy, we use the result obtained in[1]:

$$E_1 = \sum_{\nu=1}^{\infty} (E^\nu + E^{\nu *}), \quad E^\nu = \iint_0^\infty e^{2\pi i \nu (2n+l)} E(n, l)\, dn\, dl, \qquad (19)$$

where

$$E(n, l) = 4\varepsilon_f \tilde{l} w = -4\varepsilon \rho w \sin \beta \cong -4\varepsilon \rho_f \tilde{\beta} w \qquad (20)$$

is the summary energy of the quasiparticles having the corresponding values of the quantum numbers; the additional doubling is due to the spin of the individual quasiparticles, to which an energy value $\epsilon_f = -\epsilon$ is assigned (we shall discuss the validity of the last assumption at the end of the section). According to[1] (see also[4])

$$2\pi (2n + l) \cong -4\pi + 4\rho + 2\rho \tilde{\beta}^2,$$
$$dn\, dl = \frac{\rho\, d\rho}{\pi} \cos^2 \beta\, d\beta \cong \frac{\rho_f}{\pi} d\rho\, d\tilde{\beta}. \qquad (21)$$

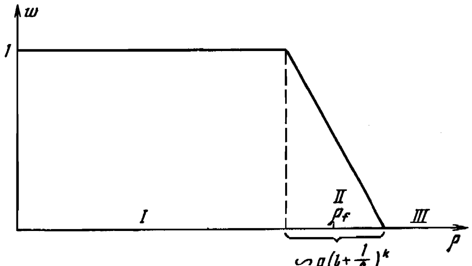

FIG. 4

Taking also (17) into account, substituting in (19), and performing the simple integration with respect to ρ, we obtain

$$E^\nu = \frac{\varepsilon \rho_f \exp\{4i\nu\rho_f\}}{4\pi g \nu^2} \int_0^\infty \exp\{2i\nu\rho_f \tilde{\beta}^2\} \times (\exp\{2i\nu g \rho_f \tilde{\beta}^k\} - \exp\{-2i\nu g \rho_f \tilde{\beta}^k\}) \frac{d\tilde{\beta}}{\tilde{\beta}^{k-1}}. \qquad (22)$$

To determine which values of the exponent k are compatible with the character of the experimental data, it is easiest to turn to the limiting case when the convergence of the integral (22) is due principally to that term of the argument of the exponential which is proportional to g[5)]. We expand $\exp\{2i\nu\rho_f\tilde{\beta}^2\}$ in a series, confine ourselves to the first two terms, and substitute in (19); this yields

$$E_1 \cong -\frac{\varepsilon \rho_f}{\pi k g} \Bigg\{ \frac{(2g\rho_f)^{2-4/k}}{g} \int_0^\infty \frac{\sin y\, dy}{y^{2-4/k}} \sum_{\nu=1}^\infty \frac{\cos 4\nu\rho_f}{\nu^{4/k}} + (2g\rho_f)^{1-2/k} \int_0^\infty \frac{\sin y\, dy}{y^{2-2/k}} \sum_{\nu=1}^\infty \frac{\sin 4\nu\rho_f}{\nu^{1+2/k}} \Bigg\}, \quad g \gg \rho_f^{k/2-1}. \qquad (23)$$

The singularities of (23) are governed by trigonometric series that can be easily investigated. A singularity of the type of a finite jump in the derivative $dE_1/d\rho_f$ (see Figs. 2 and 1b) can be obtained only from an even series in $t = 4\rho_f - 2\pi p$ with cosines. When $k < 2$ we have $4/k > 2$, and the series of the derivatives diverges uniformly in accordance with the Weierstrass criterion. Thus, there are no singularities of this type if $k < 2$.[6)] To the contrary, if $k > 2$ the sum of the series exhibits stronger singularities. To verify this,

4)The latter is confirmed by an analysis of the dependence of the spin-orbit interaction on nuclear dimensions (see, for example, [2], i.e., in final analysis, on $\rho_f = k_f R$. It must be stipulated, however, that this pertains for the time being to the calculation of the effects in the ρ_f scale. Subsequently, the spin-orbit interaction is actually taken into account in a most substantial manner when the final transition is made to the physical N-scale; see [1].

5)Strictly speaking, however, the coupling constant can likewise not be exceedingly large, for then the situation reduces to effects due to individual nucleons at the s or p levels; see Figs. 3 and 4. This would actually be manifest by a complete leveling of the magic numbers, and in any case the macroscopic theory of shell and magic phenomena developed in [1] and in the present paper would no longer hold. As a result we obtain the requirement $g \ll \rho_f^{k-1}$.

6)The case k = 0 calls for a special analysis. Although the expression under the summation sign becomes somewhat more complicated in this case, the absence of a jump in the first derivative of E_1 can be easily proved (see Appendix 2). We note that the variant k = 0 is, in a certain sense, of special interest, since it corresponds to a Fermi-distribution smearing band whose width is independent of the orbital momentum we consider the derivative of the series of interest to us; this derivative is given by

$$\sum_{\nu=1}^{\infty}\frac{\sin\nu t}{\nu^{\alpha}},\quad \alpha<1.$$

Near the singular point $t = 0$ we have $\sin\nu t \sim \nu t$ up to a certain limiting value $\nu = \tilde{\nu} \sim 1/|t|$. Replacing the summation by integration, we obtain the estimate

$$\sum_{\nu=1}^{\infty}\frac{\sin\nu t}{\nu^{\alpha}}\sim t\int_0^{\tilde{\nu}}\nu^{1-\alpha}d\nu\sim t\tilde{\nu}^{2-\alpha}\sim\frac{t}{|t|^{2-\alpha}}. \qquad (24)$$

Consequently, at $k > 2$ we have $\alpha = 4/k - 1 < 1$ and the derivative of the first term of (23) tends to infinity as $t \to 0$ (the result (24) agrees with rigorous mathematical theorems; see, for example,[5]). Thus, only

$$k=2 \qquad (25)$$

is comparable with the experimental data.

To get rid of the lower bound $g \gg 1$ on the coupling constant, we substitute (25) in (22) and integrate; as a result we obtain ultimately

$$E_1=-\frac{\varepsilon\rho_f}{2\pi}\{f_1(g)\mathfrak{M}(\rho_f)+f_2(g)\mathfrak{N}(\rho_f)\}, \qquad (26)$$

where

$$f_1(g)=\frac{1}{2g}\ln\left|\frac{1+g}{1-g}\right|,\quad f_2(g)=\frac{\pi}{2g}\theta(g-1), \qquad (27)$$

$\theta(x) = 1$ at $x > 0$ and $\theta(x) = 0$ at $x < 0$. Plots of the functions f_1 and f_2 are given in Fig. 5. The trigonometric series

$$\mathfrak{M}(\rho_f)=\sum_{\nu=1}^{\infty}\frac{\cos 4\nu\rho_f}{\nu^2}=\frac{\pi^2}{6}-2\pi\left|\rho_f-\frac{\pi}{2}p\right|+4\left(\rho_f-\frac{\pi}{2}p\right)^2,$$

$$\tfrac{1}{2}\pi(p-1)<\rho_f<\tfrac{1}{2}\pi(p+1);$$

$$\mathfrak{N}(\rho_f)=\sum_{\nu=1}^{\infty}\frac{\sin 4\nu\rho_f}{\nu^2} \qquad (28)$$

describe shell oscillations in spherical nuclei in the presence of a residual interaction[7]. Differentiating (26), we obtain the discontinuity

$$\Delta\varepsilon=\varepsilon\frac{d\rho_f^2}{dN}f_1(g) \qquad (29)$$

of the nucleon binding energy in the nucleus. According to (27), this discontinuity increases without limit as $g \to 1$. The physical qualitative picture was already explained in the introduction and illustrated in Fig. 3. Now, however, unlike in (6), the oscillations tend to infinity much less rapidly, logarithmically. The weakening of the singularity is due to the presence of region II, which is not free of fermions (see Fig. 4); on the boundary between regions I and II, the occupation numbers (17) do not change jumpwise in the considered model.

It is quite doubtful whether such a sharply delineated boundary of the region of intermediate values of occupation numbers actually exists, and actually the coefficient of the function $\mathfrak{M}(\rho_f)$ should not become infinite anywhere. We note in this connection the curious possibility of constructing a simple interpolation formula for the function $f_1(g)$. Since the transition $g \to 0$ to the case (4) where there is no residual interaction yields $f_1 \to 1$, and since in the asymptotic region $g \gg 1$ we have according to (27) $f_1 \cong 1/g^2$, the following interpolation comes to mind:

$$f_1(g)\approx 1/(1+g^2). \qquad (30)$$

We then have, in particular,

$$\omega\equiv\frac{(\Delta\varepsilon)_0}{\Delta\varepsilon}-1=\frac{1}{f_1}-1\approx g^2. \qquad (31)$$

Thus, the characteristic ω of the deviation from the ideal case (2) and (4), which was introduced earlier in[1] from intuitive and empirical considerations, turns out to be directly expressed in terms of the constant of the residual interaction between the quasiparticles. Nevertheless, it seems more natural to start from the very beginning from analytic or nearly-analytic expressions for the distribution function $w(\rho, \tilde{\beta})$. One such likely example will be considered in the next section.

The prospects for choosing the most successful interpolation may become clearer if we turn finally to the assumption (made above in connection with formula (20)) that each quasiparticle has a definite energy $-\epsilon$. There exists also the energy (14) of the interaction between the quasiparticles. When the spectral component E^{ν}_{int} of the energy of this interaction is calculated by means of formula (19), integration over the region I will certainly not yield a macroscopic contribution. Indeed (see formula (14) and the text immediately following it) the only inaccuracy in the determination of b_j corresponds precisely to the error in the interaction energy $\sim G_j\Omega_j$. We therefore confine ourselves to integration over the interaction region II, and simple calculations lead to

$$E^{\nu}_{\text{int}}=\frac{d\varepsilon}{d\rho}\bigg|_f\frac{\rho_f}{16\pi}e^{4i\nu\rho_f}\frac{i}{\nu^3}\left\{\frac{1}{1-g^2}-f_1(g)+if_2(g)\right\}. \qquad (32)$$

Since $d\epsilon/d\rho \sim \epsilon/\rho_f$, the contribution (32) to the energy of the nucleus is not macroscopic; according to the criteria analyzed in[1], it need not be taken into account.

This circumstance is apparently not accidental, and is by no way a distinguishing feature of just this model. From a more general and physically more lucid point of view, this question was already analyzed in the introduction above.

(this case was already mentioned in the introduction; see footnote 3). For example, in the Cooper phenomenon, the angular momentum l of the quasiparticle with respect to the geometric center would not play any special role, and only the angular momenta of the components of the Cooper pair with respect to their common center of gravity would be of importance. We defer additional comments to the last section.

[7]Greatest interest, however, attaches nevertheless to the nearest vicinity of the magic nucleus, in which the function $\mathfrak{N}$ tends to zero like $t \ln|t|$. At the present time it is still not quite clear whether allowance for the small term proportional to $\mathfrak{N}$ is an exaggeration of the macroscopic accuracy with which the entire developed concept holds (see also [1]). This has no effect whatever on the subsequent results pertaining to the magic jump $\Delta\epsilon$ of the nucleon binding energy.

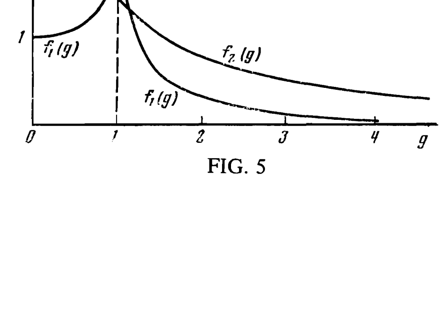

FIG. 5

3. CASE OF ANALYTIC DISTRIBUTION FUNCTION

The fact that the contribution of the interaction energy to the oscillations is of no importance (at a specified quasiparticle distribution function) enables us to reformulate the problem in the manner outlined in the introduction, namely, the oscillating part of the ground-state energy of a spherical nucleus is defined by the function w (which is the diagonal part of the density matrix relative to the basis of the quasiparticles in question). The form of the function $w(\rho, \tilde{\beta})$ is by far not as arbitrary as it might seem at first glance. First, it must tend rapidly to zero and unity in the asymptotic regions; it is not very likely that this function is non-monotonic. Further, the analysis in the preceding section (see in particular (25) and (26)) suggests that only a quadratic dependence of the width of the Fermi-distribution smearing band on the orbital angular momentum can be reconciled with the experimentally observed character of the magic phenomena at more or less arbitrary intensities of the residual interaction. Indeed, it is easy to show that for all distributions of the type $w((\rho - \rho_f)/\tilde{\beta}^2)$ the oscillating part of the energy E_1 always reduces to a linear combination of the expression $\mathfrak{M}(\rho_f)$ and $\mathfrak{N}(\rho_f)$. It is therefore natural to attempt to use a function of the type of the ordinary Fermi distribution (7), but with a modulus that depends quadratically on the angle $\tilde{\beta}$:

$$w(\rho, \beta) = \left(\exp\left\{\frac{\rho-\rho_f}{\tau\beta^2}\right\}+1\right)^{-1}. \tag{33}$$

We calculate the spectral component

$$\mathfrak{N}^\nu = \frac{4}{\pi}\rho_f^2 e^{4i\nu\rho_f}\int_0^\infty d\beta\,\beta\exp\{2i\nu\rho_f\beta^2\}\int w(\rho,\beta)\exp\{4i\nu(\rho-\rho_f)\}\,d\rho \tag{34}$$

of the number of states filled with quasiparticles, and in accord with

$$\mathfrak{N}_1 = \sum_{\nu=1}^{\infty}(\mathfrak{N}^\nu + \mathfrak{N}^{\nu*}) \tag{35}$$

we use formula (3) (taking into account the conclusions drawn above concerning the role of the energy of interaction between quasiparticles, this is equivalent to formulas (19)):

$$E_1 = -\frac{\varepsilon\rho_f}{4\pi^2 g}\sum_{\nu=1}^{\infty}\left\{\frac{e^{4i\nu\rho_f}}{i\nu^2}\zeta\left(2,\frac{1}{2}-\frac{i}{\pi g}\right)+\text{к.с.}\right\}. \tag{36}$$

We have put here $\tau = (1/4)g\rho_f$ (the convenience of such a change of notation will become clear later on). Separating in the Riemann ζ function

$$\zeta(z,q) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{u^{z-1}e^{-qu}}{1-e^{-u}}\,du = \sum_{\mu=0}^{\infty}\frac{1}{(q+\mu)^z} \tag{37}$$

the real and imaginary parts, we arrive at the trigonometric series (28):

$$E_1 = -\frac{\varepsilon\rho_f}{2\pi}\{F_1(g)\mathfrak{M}(\rho_f)+F_2(g)\mathfrak{N}(\rho_f)\},$$

$$F_1(g) = \frac{1}{2\pi g}\int_0^\infty \frac{\sin(u/\pi g)}{\text{sh}(u/2)}u\,du,$$

$$F_2(g) = \frac{1}{2\pi g}\int_0^\infty \frac{\cos(u/\pi g)}{\text{sh}(u/2)}u\,du = \frac{\pi}{2g}\Big/\text{ch}^2\left(\frac{1}{g}\right). \tag{38}$$

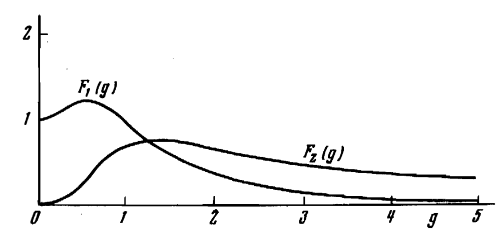

FIG. 6

Plots of the functions $F_1(g)$ and $F_2(g)$ are shown in Fig. 6.

Just as for the model example (17) and (25) of the preceding section, in the region $g \gg 1$ of the strong residual interaction there predominates, generally speaking, the function $\mathfrak{M}(\rho_f)$, which enters here with a coefficient

$$F_2(g) \cong \pi/2g, \qquad g \gg 1. \tag{39}$$

With our choice of the normalization of the coupling constant (see above), Eq. (39) coincides with the set formula of (27). Continuing this analogy, we note that in the non-analytic model (23), (27) there was no term with $\mathfrak{N}(\rho_f)$ at all at $g < 1$. In our case, however, this corresponds to exponential smallness of this term if the coupling between the quasiparticles is weak:

$$F_2(g) \cong \frac{2\pi}{g}e^{-2/g}, \quad g \ll 1. \tag{40}$$

From the more formal point of view, it is curious to note that the limiting behavior of (40) indicates the presence of an essential singularity at $g = 0$. One cannot exclude the possibility that this mathematical property of the limiting case $g \to 0$ of an absolutely sharp Fermi boundary is quite general in character.

The magic singularities are due only to the function $\mathfrak{M}(\rho_f)$ (see also footnote 7). The formulas characterizing these singularities are

$$\Delta\varepsilon = \varepsilon\frac{d\rho_f^2}{dN}F_1(g), \quad \omega \equiv \frac{(\Delta\varepsilon)_n}{\Delta\varepsilon} - 1 = \frac{1}{F_1} - 1. \tag{41}$$

As seen from Fig. 6, the function $F_1(g)$ has a maximum in the region of intermediate values of the coupling constant. Thus, the effect of interest to us does not depend monotonically on the intensity of the residual interaction.[8)]

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8)This non-monotonicity is an interesting feature of the phenomenon, of which we were unaware during the time of writing of the preceding paper [1]; nor is it revealed by rough interpolations of the type (30). The increase of the oscillations at intermediate values of g can be qualitatively understood as a sort of resonant amplification due to the similarity in the form of the lower boundary of the region of intermediate values of occupation numbers and the corresponding Regge trajectory on Fig. 3. In the non-analytic model examples (6) and (29) (see 27)) this resonance was extremely sharply pronounced and caused the effect to become infinite as $g \to 1$. It can be assumed that in the analytic case, too, the presence of a maximum is not accidental, and that if the Fermi boundary is not absolutely sharp, $g = 0$, the oscillations actually reach their maximum swing. For the considered distribution (33), this extremum is given by $g = 0.55$, $F_{1\,max} = 1.20$, and $\omega_{min} = -0.17$.

4. COMPARISON WITH EXPERIMENT

For the 52 magic nuclei considered in the preceding article[1], we obtained the average values pertaining to seven experimentally accessible kinks (four neutron and three proton) on the mass curve. The results of such a reduction are given in Table I. In accordance with the interpolation (38) and (31), $\sqrt{\omega}$ is the crudest characteristic of the residual interaction. The reason why ω (or $\sqrt{\omega}$) may turn out not to be an appropriate characteristic of the phenomenon of interest to us are given in footnote 8. The coupling constant g' was calculated from formulas (27) and (29), and the quantity g corresponding to it in the analytic case (33) was determined from relations (38) and (41).

With respect to the shell oscillations, the dependence of the width of the Fermi-distribution smearing band on $\tilde{\beta}$ extends only to the angles

$$2\rho_f\tilde{\beta}^2 \sim 1, \tag{42}$$

after which the integrals describing them (see (22), (34)), in any case, converge rapidly. According to (33), the corresponding characteristic width in the ρ scale is

$$\overline{\delta\rho} = \tau\tilde{\beta}^2 = {}^1/_4 g\rho_f\tilde{\beta}^2 = {}^1/_8 g. \tag{43}$$

We change over to the energy scale:

$$\overline{\delta\varepsilon} = \frac{d\varepsilon}{d\rho}\bigg|_f \overline{\delta\rho} = \frac{\hbar^2}{R^2}\frac{\rho_f}{m^*}\frac{g}{8}. \tag{44}$$

Here R is the radius of the nucleus. We now assume

$$R = 1{,}2\cdot 10^{-13}\ A^{1/3}\ [\text{cm}] \tag{45}$$

and, as an estimate, set the effective mass m^* of the quasiparticles equal to the mass of the free nucleon. The widths $\overline{\delta\epsilon}$ in MeV are given in Table I.

For the concrete values of the orbital angular momentum, the width $\delta\epsilon$ of the smearing band turns out to be different, since it depends on $\hat{l} = l + {}^1/_2$ quadratically. Taking the stimate (42) into account, we confine ourselves to not too large values of l, which are characteristic of the considered shells. The results are given in Table II.

All the characteristics of the residual attraction listed in the table, regardless of the degree of their accuracy and of the choice of scale, indicate consistently that its intensity decreases rapidly with increasing nuclear dimensions[9].

5. CONCLUSIONS AND DISCUSSION

The most interesting result, in our opinion, is the dependence of the residual interaction on the orbital angular momentum of the quasiparticle:

$$\delta\varepsilon \propto \hat{l}^2 \cong (l + {}^1/_2)^2. \tag{46}$$

On the one hand, this result is quite natural. Since we are dealing with a scalar effect, it should be expressed precisely through the scalar square of the angular-momentum vector.

According to the estimate (42), the angular-momentum values that play an important role are relatively small. From this point of view, (46) can be regarded as the first term of a series expansion in powers of the ratio $\tilde{l}/\rho_f$. Why does the expansion have no zeroth term independent of l? Does this mean that in a nucleus of finite dimensions the residual interaction is due only to the additional integrals of the quasiparticle motion, other than the energy? Unfortunately, we see no possibility of giving a sufficiently clear-cut and definite answer to such questions. Nonetheless, we wish to point out the difficulties that would probably be encountered in attempts to reconcile the obtained picture with the assumption concerning the Cooper phenomenon in nuclear matter. It is precisely such a phenomenon which is characterized by a constant width of the transition region of the statistical distribution, having the same value for all the quasiparticles located near the Fermi boundary[10]. However, the presence of a constant component of the width of this band would lead to the absence of the experimentally observed magic jumps of the binding energy of the nucleons (see Appendix 2, and also footnotes 3 and 6). The available experimental data therefore agree more readily with the simpler and natural hypothesis concerning the character of the energy spectrum of unbounded nuclear matter as such. It seems likely that it can represent a "normal" Fermi liquid with an absolutely sharp Fermi boundary for quasiparticles[7].

Table I.

p	N, Z	Neutron magic nuclei				Proton magic nuclei			
		$\sqrt{\omega}$	g'	g	$\overline{\delta\varepsilon}$, MeV	$\sqrt{\omega}$	g'	g	$\overline{\delta\varepsilon}$, MeV
4	28	2,4	2.7	3.3	5.2	1.8	2.2	2.6	3.8
5	50	1.6	2.0	2.3	3.3	1.5	1.9	2.2	2.6
6	82	1.4	1,8	2.0	2,6	1.0	1.5	1.6	1.6
7	126	1.1	1.6	1.7	1.9				

Table II.

l	p = 4		p = 5		p = 6		p = 7	
	$\delta\varepsilon_N$	$\delta\varepsilon_Z$	$\delta\varepsilon_N$	$\delta\varepsilon_Z$	$\delta\varepsilon_N$	$\delta\varepsilon_Z$	$\delta\varepsilon_N$	$\delta\varepsilon_Z$
0	0,4	0.3			0.14	0.08		
1			1.9	1,5			0,8	
2	10,2	7,5			3.5	2.1		
3			10.2	8.1			4.3	

Note. The values of $\delta\epsilon$ are given in MeV.

[9]The method of coping with the even-odd mass oscillations used in the reduction of the experimental data was already described earlier (see [1], footnote 12). We note in this connection that the change in the number of protons by 2 is a reflection of the Coulomb energy of the nucleus is reflected in the Coulomb energy of the nucleus and in its derivative with respect to the number of charged particles. Since in fact an energy of pure electrostatic origin has no singularity whatever, it is advisable to subtract the "fictitious" effect produced by it from the observed running discontinuities $\Delta\epsilon$. In connection with the introduction of such a correction for the Coulomb interaction, the values of ω used in the present paper for the proton magic nuclei (see Table I) turn out to be somewhat higher than those previously published (see Table II and Fig. 4 of [1]). This correction affects little the conclusions concerning the course and magnitude of the residual interaction of the protons.

[10]A good illustration is the well-known problem of an almost-ideal Fermi gas with weak attraction. Its solution is frequently given in the momentum representation (see, for example, [6]). It is difficult to visualize how the purely formal operation of the transition to the l-representation could change the width of the transtion band of the Fermi distribution, and furthermore make it dependent on l (see also footnote 6).

On the other hand, spherical nuclei of finite radius have a residual interaction of the type characterized by (46). One must not forget, however, that it was actually established only on the basis of data that pertained throughout to the nearest vicinity of some magic nucleus. The experimental data give the impression that the residual interaction of the considered type decreases gradually with increasing distance from the magic nucleus, and becomes somehow restructured in final analysis. The thermodynamics of the transition was considered in[8]. The fact that the phase transition due to weakening and restructuring of the residual interaction causes also the nuclear shape to become nonspherical can hardly be regarded as surprising. It was already shown earlier (see[4]) that in the simplest scheme without interaction the sphere is absolutely unstable in general for any number of particles.

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APPENDIX 1

In the preceding paper[1] (see also[4]) we considered for concreteness a scheme with a wall that is impermeable to the quasiparticles and is located at a distance R from the center of the nucleus. The roots of the wave equation of the free motion of the particle in the spherical region, shown in Fig. 3 above, corresponded to such a boundary condition. We shall show that this does not limit the generality of the results pertaining to shell oscillations of the volume energy of the spheric nucleus.

The Bohr-sommerfeld quantization rule[2] used to determine the eigenvalues is

$$\int_a^R k_l(r)\,dr = \pi(n+\gamma). \tag{A1.1}$$

Here

$$k_l(r) = \sqrt{k^2 - \tilde{l}^2/r^2}, \quad \tilde{l} = l + {}^1/_2, \tag{A1.2}$$

and the lower limit of integration is due to the centrifugal barrier; since the nuclear matter is homogeneous, the wave number k is constant in the interval region. The additional phase γ depends on the properties of the true structure of the transition layer on the surface of the nucleus. We change over to the dimensionless variable $kr = \rho'$:

$$\int_{\tilde{l}}^{\rho} \Re(\rho')\,d\rho' = \pi(n+\gamma), \quad \Re(\rho') \equiv \sqrt{1 - \tilde{l}^2/\rho'^2}. \tag{A1.3}$$

The Regge trajectories responsible for the oscillations are characterized by the relation

$$2n + l = p, \quad p = 2, 3, 4, 5\ldots, \tag{A1.4}$$

between the quantum numbers (see the Introduction and Fig. 3). Therefore

$$dn/d\tilde{l} = -{}^1/_2, \quad d^2n/d\tilde{l}^2 = 0. \tag{A1.5}$$

We now differentiate the entire relation (A1.3) along the trajectory

$$\Re(\rho)\frac{d\rho}{d\tilde{l}} - \tilde{l}\int_{\tilde{l}}^{\rho}\frac{d\rho'}{\rho'^2\Re(\rho')} = \pi\left(\frac{dn}{d\tilde{l}} + \frac{d\gamma}{d\tilde{l}}\right). \tag{A1.6}$$

Elementary integration yields

$$\Re(\rho)\frac{d\rho}{d\tilde{l}} - \arccos\frac{\tilde{l}}{\rho} = -\frac{\pi}{2} + \pi\frac{d\gamma}{d\tilde{l}}. \tag{A1.7}$$

It follows therefore that the derivative $d\rho/dl$ vanishes at $\tilde{l} \equiv 0$ (the last term in the right-hand side, which is due to the differentiation of the phase increment, does not affect the result (see (A1.9) below). In the second differentiation of the Bohr-Sommerfeld formula, we omit the terms that are known to vanish at the extremum:

$$\Re(\rho)\frac{d^2\rho}{d\tilde{l}^2} + \frac{1}{\rho\Re(\rho)} = \pi\frac{d^2\gamma}{d\tilde{l}^2}. \tag{A1.8}$$

We now take into account the fact that the phase γ is determined by the actual details of the nuclear interactions in the nearest vicinity of the upper limit of integration. Any characteristic that influence the results can depend here on the angular momentum $\tilde{l}$ only in the combination $\tilde{l}_2$. Therefore

$$\frac{d\gamma}{d\tilde{l}} = 2\tilde{l}\frac{d\gamma}{d\tilde{l}^2}, \quad \frac{d^2\gamma}{d\tilde{l}^2} = 2\frac{d\gamma}{d\tilde{l}^2} \sim \frac{1}{\rho^2}. \tag{A1.9}$$

Consequently, at the maximum point $l = 0$ we have

$$d^2\rho/d\tilde{l}^2 \cong -1/\rho, \tag{A1.10}$$

neglecting the terms $\sim 1/\rho^2$. Thus, the form of Eq. (1)

$$\Delta\rho \cong -(l + {}^1/_2)^2/2\rho \tag{A1.11}$$

does not depend on the details of the nuclear surface-layer structure. We determined finally the ordinates of the extrema of the successive trajectories. At $\tilde{l} = l + 1/2 = 0$ we have according to (A1.4) $n = p/2 + 1/4$. Substituting in (A1.3), we get

$$\rho_{max} = \pi({}^1/_2 p + \gamma'), \tag{A1.12}$$

where $\gamma' = \gamma + 1/4$. Replacing ρ_{max} by $k_f R = \rho_f$, we see that (A1.12) coincides in fact with the previously published (see[1], formula (19), and footnote 9) "quantization rule" for the magic values of this parameter.

APPENDIX 2

We consider here the special case $k = 0$, for which there is no asymptotic region described by formulas of the type (23). The width of the band II of the smearing of the Fermi distribution (see Fig. 4), which does not depend on l, will be denoted by g_0. Formally, this requires that we put $k = 0$ in (17) and make the substitution $g\rho_f \to g_0$. Substituting then in (34) and (35), we obtain after a simple integration

$$\mathcal{N}_1 = \frac{\rho_f}{4\pi g_0}\sum_{\nu=1}^{\infty}\frac{\sin 2\nu g_0 \cos 4\nu\rho_f}{\nu^3}. \tag{A2.1}$$

In the next application of formula (3), we transform also the trigonometric expressions under the summation sign

$$E_1 = -\frac{\varepsilon\rho_f}{2\pi g_0}\left\{\mathfrak{L}\left(\rho_f + \frac{g_0}{2}\right) - \mathfrak{L}\left(\rho_f - \frac{g_0}{2}\right)\right\}. \tag{A2.2}$$

Here

$$\mathfrak{L}(\rho_f) = \frac{1}{4}\sum_{\nu=1}^{\infty}\frac{\sin 4\nu\rho_f}{\nu^3}, \quad \frac{d\mathfrak{L}}{d\rho_f} = \mathfrak{M}(\rho_f). \tag{A2.3}$$

Since the expression for $\mathfrak{M}(\rho_f)$ (see formula (28) above, and also[1]) is characterized by a jump of the first derivative, only the second derivative of the function $\mathfrak{L}(\rho_f)$ becomes discontinuous. Moreover, these singularities (which are located in this case at the inflection points of the function $E_1(\rho_f)$) are now shifted by a distance $\pm g_0/2$. The physical reason for this is seen directly from Fig. 4. At the very point on the ρ_f scale where the first derivative would experience a discontinuity at $k = 2$, the expression (A2.2) has an ordinary analytic minimum. Thus, the nucleon binding energy ϵ experiences no discontinuity anywhere.

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