NONLINEAR WAVES OF SECOND SOUND IN SOLIDS

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Nonlinear equations describing the drift velocity and temperature of a quasiparticle gas in the hydrodynamic approximation are considered. A class of exact solutions of these equations, the so-called Riemann waves, is obtained. The possible discontinuities are classified and discontinuities of the shock-wave type are studied in detail.

As the temperature of a solid is reduced, normal collisions (N processes) between quasiparticles become considerably more probable than Umklapp (U processes). The propagation of so-called second sound then becomes possible; from the microscopic point of view this is ordinary sound in a gas of quasiparticles. These oscillations have been studied very thoroughly[1] in the case where the drift velocity u of the quasiparticles is much smaller than their phase velocity s (the linear approximation).

It is obvious, however, that at sufficiently low temperatures the linear theory condition ($u/s \ll 1$) can be violated. For example, Nielsen and Shklovskiĭ[2] have shown that when the temperature difference ΔT between the ends of the sample is sufficiently large the phonon drift velocity can be comparable with the velocity of sound. The linear theory is then, of course, inapplicable and nonlinear equations must be used for the purpose of describing oscillations in a gas of quasiparticles. These equations can be derived by the Chapman-Enskog method; under the given conditions (i.e., when N processes are most probable) the distribution function is

$$N_0(\mathbf{p}) = \{\exp[(\varepsilon(\mathbf{p}) - \mathbf{pu})/T] - 1\}^{-1}, \tag{1}$$

where $\epsilon(\mathbf{p})$ is the energy, $\mathbf{p}$ is the quasimomentum, and T and u are the temperature and drift velocity of the quasiparticle gas.

Neglecting dissipation, the gas-dynamic equations for a gas of quasiparticles are

$$\frac{\partial P_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \quad \frac{\partial E}{\partial t} + \operatorname{div}\mathbf{Q} = 0, \tag{2}$$

where $\mathbf{P}$ and E are the momentum and energy per unit volume, $\mathbf{Q}$ is the energy flux density, and Π_{ik} is the momentum flux tensor. All these quantities are moments of the distribution function $N_0(\mathbf{p})$, i.e., they are expressed in terms of the latter by means of the appropriate integrals,[2] where the integration is carried out, speaking rigorously, over the first Brillouin zone. If for large $|\mathbf{p}|$ the growth of $\epsilon(\mathbf{p})$ is not slower than that of p^α, where $\alpha \geq 1$, then the main contribution to the averages comes from small moments and it is possible to integrate between infinite limits. When this condition is violated the average energy and momentum taken over the entire range $-\infty \leq p \leq \infty$ will become infinite. This means that large momenta play a decisive role and that the given approach has become inapplicable.

We now turn to the case of acoustic phonons $[\epsilon(p) = sp]$ and shall consider, for simplicity, the one-dimensional problem represented by $u = \{u(x, y), 0, 0\}$, $T = T(x, y)$, $y = st$. The equations obtained from (2) for $w = u/s$ and $\theta = \ln(T/T_0)$ (where T_0 is the crystal temperature at thermodynamic equilibrium) are

$$\frac{\partial \mathbf{w}}{\partial y} + \hat{A}\frac{\partial \mathbf{w}}{\partial x} = 0, \quad \mathbf{w} = \{w, \theta\};$$

$$\hat{A} = \begin{pmatrix} \dfrac{3w^4 + 3w^3 - 5w^2 - 5w}{5w^3 + w^2 - 3w - 3} & \dfrac{3(1-w)(1+w)^3}{5w^2 + 6w + 3} \\ \dfrac{-3w^4 + 7w^3 + 4w^2 - w - 1}{5w^3 + w^2 - 3w - 3} & -\dfrac{3w^2 + w}{5w^2 + 6w + 3} \end{pmatrix} \tag{3}$$

Obtaining the left eigenvectors of the matrix $\hat{A}$ from the equation $\mathbf{l}^{(1,2)}\hat{A} = \xi_{1,2}\mathbf{l}^{(1,2)}$, we write (3) as

$$\mathbf{l}^{(1,2)}\left(\frac{\partial \mathbf{w}}{\partial y} + \xi_{1,2}\frac{\partial \mathbf{w}}{\partial x}\right) = 0. \tag{4}$$

Here $\xi_{1,2}$ represents the eigenvalues of $\hat{A}$:

$$\xi_{1,2} = \frac{18w^5 + 6w^4 + 4w^3 - 6w - 6 \pm 4w(P_8(w))^{1/2}}{-3w^6 + 34w^4 + 12w^3 - 15w^2 - 12 \pm (3 + w^2)(P_8(w))^{1/2}}, \tag{5}$$
$$P_8(w) = 9w^8 + 42w^6 + 48w^5 - 35w^4 - 72w^3 - 12w^2 + 24w + 12.$$

The equations

$$dx/dy|_{w=\text{const}} = \xi_{1,2}$$

determine the directions of differentiation, which are the same for w and θ in (4) and which are the characteristic directions for the system (3). Equation (5) shows that for $w < 1$[1)] the quantities ξ_1 and ξ_2 are real and different, i.e., (3) is a system of hyperbolic equations.

The characteristic form (4) of the quasilinear equations (3) enables us to seek a special (i.e., nongeneral) solution of this system. It is easily seen that (4) is satisfied if

$$\frac{\partial \mathbf{w}}{\partial y} + \xi_{1,2}(w)\frac{\partial \mathbf{w}}{\partial x} = 0. \tag{6}$$

It is clear that on the lines

$$dx/dy|_{w=\text{const}} = \xi_{1,2}(w),$$

i.e., on the characteristics, w and θ remain constant, and the equation

$$x = \xi_{1,2}(w)y + \varphi(w) \tag{7}$$

defines implicitly the function w(x, y), which is the so-

1) The phonon drift velocity cannot exceed the sound velocity s, i.e., for phonons $w = u/s < 1$.

lution of the initial system of equations for arbitrary $\varphi(w)$; the explicit form of $\varphi(x, y)$ is determined from the boundary conditions. The solutions of this class are called simple waves or Riemann waves.[3] If $\varphi(w) = 0$, Eq. (7) is the so-called "self-similar" solution of Eq. (3).

It can be shown that in a simple wave the drift velocity and temperature can be related functionally using an expression that does not explicitly contain the coordinate and time:

$$\frac{d\theta}{dw} = -\frac{6w + 7w^2 - 6w^3 - 3w^4 \pm (P_8(w))^{1/2}}{6(w^2-1)^2(w+1)}. \tag{8}$$

It is clear from (7) that points characterized by definite values of the velocity and temperature are propagated in space with constant velocities (which differ depending on w). The wave profile therefore obviously changes with time, and in some cases discontinuities of the velocity w and temperature θ appear. The coordinate and time of discontinuity formation are determined by the simultaneous solution of the two equations[3]

$$\left(\frac{\partial x}{\partial w}\right)_{t=\mathrm{const}} = 0, \quad \left(\frac{\partial^2 x}{\partial w^2}\right)_{t=\mathrm{const}} = 0. \tag{9}$$

We now formulate the boundary conditions that must be fulfilled on the surfaces of a discontinuity; for this purpose we consider an arbitrary element of such a surface and direct the x axis along a normal to the surface.[2)] From the continuity of the energy and momentum fluxes we obtain

$$\left[\frac{T^4 w_x}{(1-w^2)^3}\right] = 0, \quad \left[\frac{4T^4 w_x^2}{(1-w^2)^3} + \frac{T^4}{(1-w^2)^2}\right] = 0,$$
$$\left[\frac{T^4 w_x w_y}{(1-w^2)^3}\right] = 0, \quad \left[\frac{T^4 w_x w_z}{(1-w^2)^3}\right] = 0. \tag{10}$$

Here $[a] = a_1 - a_2$ denotes the difference between the values of a on the two sides of the discontinuity surface.

If no energy flows through a discontinuity, i.e.,

$$\frac{T_1^4 w_{x1}}{(1-w_1^2)^3} = \frac{T_2^4 w_{x2}}{(1-w_2^2)^3} = 0,$$

then the conditions (10) become

$$w_{x1} = w_{x2} = 0; \quad \left[\frac{T^4}{(1-w^2)^2}\right] = 0, \tag{11}$$

and the velocity components (w_y, w_z) tangent to the discontinuity surface undergo an abrupt change (a tangential discontinuity). By analogy with ordinary gas dynamics[3] it can be shown that these discontinuities are unstable.

If the energy flux differs from zero (in a shock wave), from (10) we obtain

$$[w_y] = [w_z] = 0,$$
$$\left[\frac{T^4 w_x}{(1-w^2)^3}\right] = 0, \quad \left[w_x + \frac{1-w^2}{4w_x}\right] = 0. \tag{12}$$

Since the tangential component of the gas velocity at the discontinuity is continuous, a coordinate system can be selected in which the given element of the discontinuity is at rest and the tangential component of the gas velocity is zero on both sides (as in the case of a normal shock in ordinary gas dynamics. Then $w = w_x$, and from (12) we obtain

$$w_1 w_2 = \frac{1}{3}, \quad \left(\frac{T_1}{T_2}\right)^4 = \frac{w_2}{w_1}\left(\frac{1-w_1^2}{1-w_2^2}\right)^3.$$

It is obvious that mechanical stability is possessed by the shock waves whose velocity relative to "undisturbed" gas (i.e., the region toward which the discontinuity is moving) exceeds the propagation velocity of disturbances in this region ($w_1 > 3^{-1/2}$).[4] Consequently, in real discontinuities where $w_1 > 1/\sqrt{3}$, $w_2 < 1/\sqrt{3}$ the gas loses velocity when it crosses a discontinuity surface. We see from (12a) that the temperature then rises, i.e., only "heating" shock waves are realized, while "cooling" waves are unstable.

Let us consider as an illustration the case of small nonlinearity when $w \ll 1$ but the linear approximation no longer functions and we must take into account the terms of (3) that are quadratic in w and θ. From (3) we obtain

$$\frac{\partial w}{\partial y} + \frac{5w}{3}\frac{\partial w}{\partial x} + \frac{\partial \theta}{\partial x} = 0; \quad \frac{\partial \theta}{\partial y} + \frac{1}{3}\frac{dw}{\partial x} - \frac{w}{3}\frac{\partial \theta}{\partial x} = 0. \tag{13}$$

These equations also yield a special solution representing simple waves. This solution can be obtained either directly or by decomposing the solutions (7) of the exact equations (3):

$$x = {}^1/_3(2w \pm 3^{1/2})y + \varphi(w), \tag{14}$$

$$\frac{\partial \theta}{\partial w} = -w \pm 3^{1/2}, \quad \theta = -{}^1/_2 w^2 \pm 3^{-1/2} w, \tag{15}$$

where $\theta = (T - T_0)/T$, because for $w = 0$ we have $\theta = 0$, so that the constant of integration in (15) is zero.

Let the temperature at $x = 0$ be given as a function of time:

$$\theta|_{x=0} = -{}^1/_2 w^2 + 3^{-1/2} w = \psi(y). \tag{16}$$

From this equation we can obviously determine y as a function of w: $y = \Psi(w)$. Inserting this expression into (14) at $x = 0$, we obtain $\varphi(w)$ and finally also

$$x = {}^1/_3(2w + 3^{1/2})[y - \bar{\psi}(w)]. \tag{17}$$

We obtain w from (17), and thus also θ from (15), as an implicit function of the coordinates and time. The conditions (9) for discontinuity formation in the present case are

$$2[y - \bar{\psi}(w)] - (2w + \sqrt{3})\bar{\psi}'(w) = 0,$$
$$4\bar{\psi}'(w) + (2w + \sqrt{3})\bar{\psi}''(w) = 0. \tag{18}$$

When the function $\bar{\psi}(w)$ is known explicitly, (18) and (17) can be used to determine the time and place of shock wave formation and the gas velocity ahead of the discontinuity. If, for example, the boundary condition for temperature at $x = 0$ has the form $\theta(y) = ay$, a discontinuity appears at the moment of time given by $y_d = {}^{37}/_{72}\, a^{-1}$ at a point $x_d \approx {}^1/_2\sqrt{3}a$, with the velocity $w_d = {}^1/_6\sqrt{3}$. The discontinuity moves in space with the velocity (in a fixed system of coordinates)

$$\left.\frac{dx}{dy}\right|_{w=\mathrm{const}} = \frac{2w_d + \sqrt{3}}{3} = \frac{10}{9\sqrt{3}},$$

Relative to the undisturbed part of the gas the shock wave moves at a velocity w_1 that exceeds the propagation velocity of the disturbance:

$$w_1 = \frac{1}{\sqrt{3} - w_{|d}} = \frac{6\sqrt{3}}{17} > \frac{1}{\sqrt{3}}$$

[2)] We here drop our previous hypothesis of a one-dimensional problem.

so that its mechanical stability is ensured.

To realize the described situation it is, of course, necessary to require that the distance to the discontinuity be greater than the distance within which second sound is effectively damped.[1]

We note in conclusion that in the case of strong nonlinearity, when $w = 1 - \alpha$ and $\alpha \ll 1$, which is the opposite of the case already considered, the initial system of equations breaks up, to terms of the order of α^2, into two independent equations for w and θ:

$$\frac{\partial\theta}{\partial y}+\frac{\partial\theta}{\partial x}=0, \quad \frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}=0. \tag{19}$$

This means that it is possible to have a solution $w = \text{const}$, $\theta = f(x - y)$. In this case pure thermal waves can propagate, unaccompanied by "waves of the drift velocity."

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