RELAXATION PROCESSES IN A PARAMETRICALLY UNSTABLE PLASMA

V. P. SILIN

P. N. Lebedev Physics Institute

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A theory of relaxation of particle distributions is developed for a plasma in the strong electric field of a high-frequency pumping wave under conditions of development of parametric instability with respect to potential perturbation build-up. The possibility of increasing the number of ions with energies exceeding the electron oscillation energy in the field of the pumping wave by $(m_i/m_e)^{1/3}$ times is demonstrated. The relation between the growing ion energy and the field perturbation energy density is established for conditions of increase of the plasma fluctuation field. High frequency harmonics of the electron velocity distribution are produced, the electron energy spread being determined by the plasma fluctuation energy density per electron. The causes of anisotropy of the particle velocity distribution are established; the anisotropy is manifest in an anisotropy of the temperature and in possible permissible regions of resonance relaxation.

INTRODUCTION

WHEN powerful high-frequency electromagnetic radiation acts on a plasma, parametric instabilities develop in the plasma, leading to the appearance of large-amplitude fluctuations[1]. The production of such fluctuations leads to a faster transfer of external radiation field energy to the plasma[1,2] and to an anomalous increase in the high-frequency conductivity of the plasma[3,4]. At the same time, a redistribution of particle velocities in the plasma takes place. The corresponding theoretical predictions[1,3,5,6], the results of numerical experiments[7,8], and the experimental data[9-11] indicate that this redistribution has a unique character. An understanding of the laws of plasma particle distributions produced by powerful radiation requires the development of a detailed theory. In this report we discuss the theoretical concepts that follow from the premises of[2,3] and pertain to the parametric resonance of a fully ionized plasma in a very strong radiation field $\mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t$.

Under conditions such that the oscillation velocities of the electrons in the field of the radiation wave are much greater than their thermal velocity, the increments and frequencies of the perturbations that build up in the plasma during parametric resonance do not depend on the particle velocity distribution. For this reason, fluctuations can grow exponentially in a parametrically unstable plasma during an appreciable time interval, since the effect of thermal motion of the particles on the increment becomes significant only when the distribution function varies significantly. Similar considerations make it possible to assert that the high-frequency conductivity of the plasma is of the same order as the maximal increment of parametric instability[3]. Another relatively simple situation arises after a stationary plasma fluctuation level greatly in excess of the thermal-fluctuation level had already been established in the plasma. Here, too, it is sufficient to consider the relaxation of the plasma particle distributions.

We formulate below a number of statements of the theory of relaxation of particle distributions in a parametrically unstable plasma. We point out first those possibilities of fast-ion production which are due to the mechanism of Cerenkov interaction of the particles with the potential-field oscillations in the plasma. For the steady-state fluctuation spectrum, we show the qualitative features of the relaxation process which lead to a redistribution of the particles, increasing the number of ions in an energy region whose order of magnitude is $(m_i/m_e)^{1/3}$ times greater than the electron oscillation energy in the pumping wave field. An analogous possibility is discussed under conditions when the fluctuation field increases. Assuming a slow change of distribution, it is shown that in the resonant case the average plasma ion energy grows $(m_i/m_e)^{1/3}$ times faster than the energy density of the growing perturbations. In the nonresonant case of the parametric action of the radiation, the average ion energy is of the same order as the field-fluctuation energy density. Such an assertion represents an interpolation of formula (2.6) to the region of the limit of its applicability.

We discuss the relaxation of the electron distributions under conditions when the fluctuations of a parametrically unstable plasma increase. It is shown that high-frequency harmonics of the electron velocity distribution are produced and are comparable in magnitude with the fundamental harmonic. The order of magnitude of the electron energy spread is determined by the plasma-fluctuation energy density per electron.

We show the causes of the anisotropy of the particle velocity distributions; the anisotropy is manifest in a temperature anisotropy and in possible steplike regions of fast resonant relaxation.

1. DISTRIBUTION OF IONS IN A PARAMETRICALLY UNSTABLE PLASMA AT A STEADY STATE-FIELD FLUCTUATION LEVEL

The quasilinear relaxation of ion distribution is described by the equation

$$\frac{\partial F_i(\mathbf{v},t)}{\partial t} = \frac{\partial}{\partial v_r} D_{rs}^{(i)} \frac{\partial F_i(\mathbf{v},t)}{\partial v_s}, \tag{1.1}$$

where the diffusion coefficient in velocity space is of the form

$$D_{rs}^{(i)}=\frac{\pi e_i^2}{m_i^2}\int\frac{d\mathbf{k}}{(2\pi)^3}k_rk_s\,|\varphi_0(\mathbf{k})|^2\,\delta(\omega(\mathbf{k})-\mathbf{kv}). \quad (1.2)$$

Here we have taken into account the fact that the plasma fluctuations of the longitudinal field are stationary, the frequency of the corresponding waves is equal to $\omega(\mathbf{k})$, and φ_0 is the zeroth harmonic of the fluctuation-field potential. Summation over the repeated indices r and s is understood.

In the theory of the parametric resonance of a plasma in the strong field of a pump wave it is shown[1] that the frequency ω is a function of the modulus of the projection of the wave vector (k_Z) on the direction of the pump electric field intensity vector ($\mathbf{E}=\mathbf{E}_0\sin\omega_0 t$). An analogous dependence holds for the perturbation increments. We can therefore assume that the plasma fluctuation intensity depends only on k_Z and on $k_\perp=(k^2-k_Z^2)^{1/2}$. Then the diffusion equation (1.1) for distributions that depend on v_Z and $v_\perp$ takes the form

$$\frac{\partial F_i}{\partial t}=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[v_\perp D_\perp\frac{\partial F_i}{\partial v_\perp}+v_\perp D_\wedge\frac{\partial F_i}{\partial v_z}\right]+\frac{\partial}{\partial v_z}\left[D_\wedge\frac{\partial F_i}{\partial v_\perp}+D_{zz}\frac{\partial F_i}{\partial v_z}\right]. \quad (1.3)$$

Then, for example, we have for the longitudinal diffusion coefficient the following expression:

$$D_{zz}^i=\frac{e_i^2}{4\pi^2m_i^2v_\perp}\int_{-\infty}^{+\infty}k_z^2\,dk_z\int_{|\omega(k_z)-k_zv_z|/v_\perp}dk_\perp|\varphi_0(|k_z|,k_\perp)|^2. \quad (1.4)$$

It is obvious that the velocity dependence of the diffusion coefficients, and hence the specific laws of quasilinear relaxation, are determined by the field-fluctuation distribution in wave-vector space. In particular, situations are possible where small values of $k_\perp$ are noticeably represented in the spectrum of such fluctuations, while the values of the longitudinal components are significantly larger ($|k_Z|\gg k_\perp$). In such a situation, quasilinear relaxation corresponds to one-dimensional diffusion:

$$\frac{\partial}{\partial v_z}D_{zz}\frac{\partial F_i}{\partial v_z}=\frac{\partial F_i}{\partial t}, \quad (1.5)$$

since the diffusion coefficient (1.4) is considerably larger than the others. Let us consider such a case, assuming for simplicity

$$|\varphi_0(k_z,k_\perp)|^2=\delta(k_{\perp 0}-k_\perp)\frac{w}{k_{z0}}\theta(k_{z0}^2-k_z^2)\theta(k_z^2-\alpha^2k_{z0}^2) \quad (1.6)$$

(where $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$); this corresponds to a small width of the distribution with respect to transverse wave vectors compared with the longitudinal distribution and to $\alpha^2\lesssim 1$. In estimates one can take the quantity k_{Z0} as equal to r_E^{-1} ($r_E=|e|E_0/m_e\omega_0^2$), for when parametric instability develops the perturbations that grow most rapidly are those for which $|k_Zr_E|\sim 1$. Then

$$D_{zz}^i=\frac{e_i^2wk_{z0}^2}{4\pi^2m_i^2v_\perp}\int_\alpha^1x^2\,dx\,\{\theta(k_{\perp 0}v_\perp-|k_{z0}v_zx-\omega(k_{z0}x)|) \quad (1.7)$$
$$+\theta(k_{\perp 0}v_\perp-|k_{z0}v_zx+\omega(k_{z0}x)|)\}.$$

According to (1.7), the longitudinal diffusion coefficient differs from zero in the finite region $v_Z\sim\omega(k_{Z0})/k_{Z0}$. For values in this region, according to (1.7), the ion velocity distribution becomes equalized; in particular, the number of fast particles increases. The width of such a region turns out to be comparatively small so long as α is close to unity; this corresponds to the case when the external field frequency is close to

$$\omega_{Le}[1+0.44(\omega_{Li}/\omega_{Le})^{2/3}], \quad (1.8)$$

which corresponds to the threshold frequency[1]. It is natural that in that case only particles participating in the redistribution constitute a narrow group with velocities near the value

$$v_{\text{ph}}=\omega(k_{z0})/k_{z0}\sim v_E(\omega_{Li}/\omega_{Le})^{2/3}. \quad (1.9)$$

To the contrary, if the external field frequency differs from (1.8) and approaches the Langmuir frequency to within an amount comparable with the difference between (1.8) and ω_{Le}, then the one-dimensional relaxation velocity region broadens and becomes comparable with (1.9). This means that the distribution of ions with velocities on the order of (1.9) takes on a "quasi-step like" character corresponding in such a region to a particle redistribution, corresponding to particle acceleration. It is thus shown that particle-distribution anisotropy can arise at ion velocities on the order of (1.9), which are much higher than the ion thermal velocities typical of the initial stage of the unstable state of the plasma.

We turn now to the other case, when the transverse components of the wave vectors in the fluctuation spectrum of the longitudinal field of the plasma are not small in comparison with the longitudinal components. In this case, the diffusion equation (1.3) must be used. To get an idea of the resultant possibilities, we assume

$$|\varphi_0(k_z,k_\perp)|^2=w\delta(k_{\perp 0}-k_\perp)\delta(k_{z0}-|k_z|). \quad (1.10)$$

Then the diffusion coefficient on (1.3) take the form

$$D_{zz}^i=\frac{D^i}{v_\perp}\{\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})-k_{z0}v_z]^2)+\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})+k_{z0}v_z]^2)\}. \quad (1.11)$$

$$D_\wedge=-\frac{D^i}{k_{z0}v_\perp^2}\{[k_{z0}v_z-\omega(k_{z0})]\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})-k_{z0}v_z]^2)$$
$$+[k_{z0}v_z+\omega(k_{z0})]\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})+k_{z0}v_z]^2)\}, \quad (1.12)$$

$$D_\perp=\frac{D^i}{k_{z0}^2v_\perp^3}\{[k_{z0}v_z-\omega(k_{z0})]^2\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})-k_{z0}v_z]^2)$$
$$+[k_{z0}v_z+\omega(k_{z0})]^2\theta(k_{\perp 0}^2v_\perp^2-[\omega(k_{z0})+k_{z0}v_z]^2)\}; \quad (1.13)$$

$$D^i=e_i^2wk_{z0}^2/4\pi^2m_i. \quad (1.14)$$

Since the phase velocity (1.9) of the oscillations is large compared with the thermal velocity, we can confine ourselves to one term in (1.11) through (1.13) at positive (or negative) values of v_Z. Then the diffusion will take place in the velocity region by the inequality

$$\frac{k_{\perp 0}^2}{k_{z0}^2}>\left(\frac{v_\Phi-v_z}{v_\perp}\right)^2 \quad (1.15)$$

In this case (1.13) can be rewritten in the form

$$\frac{1}{D_i}\frac{\partial F_i}{\partial t}=\frac{1}{v_\perp}\left\{\frac{\partial}{\partial v_\perp}\left[\frac{(v_{\text{ph}}-v_z)^2}{v_\perp^2}\frac{\partial F_i}{\partial v_\perp}\right.\right. \quad (1.16)$$
$$\left.\left.+\frac{v_{\text{ph}}-v_z}{v_\perp}\frac{\partial F_i}{\partial v_z}\right]+\frac{\partial}{\partial v_z}\left[\frac{v_{\text{ph}}-v_z}{v_\perp}\frac{\partial F_i}{\partial v_\perp}+\frac{\partial F_i}{\partial v_z}\right]\right\}.$$

If we introduce new variables

$$V^2=v_\perp^2+(v_{\text{ph}}-v_z)^2,\ v_z-v_{\text{ph}}=v,\ F_i(v_z,v_\perp,t)=\Phi(v,V,t), \quad (1.17)$$

we can determine the properties of the relaxation process described by Eq. (1.6), which becomes in terms of these variables

$$\frac{1}{D_i}\frac{\partial\Phi}{\partial t}=\frac{\partial}{\partial v}\left[(V^2-v^2)^{-1/2}\frac{\partial\Phi}{\partial v}\right], \qquad (1.18)$$

with $V^2k_{\perp 0}^2 > v^2(k_{z0}^2 + k_{\perp 0}^2)$, so that no singularity arises in the diffusion coefficient.

It is obvious from (1.19) that the diffusion brings about a quasiequilibrium state:

$$\Phi(v, V)=\Phi_1(V)+\Phi_2(V)\left\{\frac{v}{V}\left(1-\frac{v^2}{V^2}\right)^{1/2}+\arcsin\frac{v}{V}\right\}. \qquad (1.19)$$

The second term corresponds to the nonzero diffusion flux, including one on the boundary of the relaxation region. In our discussion, therefore, we must discard this term, so that the equilibrium distribution corresponds to the presence of a "drift" velocity equal to the phase velocity (1.9).

To understand the time scale of such a relaxation, we assume that $k_{\perp 0}$ is comparable with but nevertheless smaller than k_{z0}. Then v^2 can be neglected in the diffusion coefficient of (1.18). As a result we can write the following simple expression for the nonequilibrium solution:

$$\Phi(v, V, t)=\Phi_1(V)+\sum_{n=1}^{\infty}C_n(V)\cos\left\{\frac{vk_{z0}\pi}{Vk_{\perp 0}}(2n+1)\right\} \qquad (1.20)$$
$$\times\exp\left\{-t\frac{D^ik_{z0}^2\pi^2}{V^3k_{\perp 0}^2}(2n+1)^2\right\}.$$

From this we get the relaxation time in velocity space:

$$\tau=\frac{V^3k_{\perp 0}^2}{D^ik_{z0}^2\pi^2}\sim\frac{E_0^2}{w}\left(\frac{m_i}{m_e}\right)^2\frac{V^3}{\omega_0^4}. \qquad (1.21)$$

Finally, we estimate the number of particles taking part in the relaxation process, in accordance with condition (1.15). In this case we assume a Maxwellian particle distribution. Then

$$\delta n_i=\frac{n_im_i^{3/2}}{(2\pi\varkappa T_i)^{3/2}}\int d\mathbf{v}\cdot\theta\left(\frac{k_{\perp 0}^2}{k_{z0}^2}-\left[\frac{v_{\text{ph}}-v_z}{v_\perp}\right]^2\right)\exp\left[-\frac{m_i(v_\perp^2+v_z^2)}{2\varkappa T_i}\right]$$
$$=n_i\left(1+\frac{k_{z0}^2}{k_{\perp 0}^2}\right)^{-1/2}\exp\left(-\frac{v_{\text{ph}}^2}{2v_{Ti}^2}\frac{k_{z0}^2}{k_{\perp 0}^2+k_{z0}^2}\right). \qquad (1.22)$$

Consequently, the number of relaxing ions increases with increasing $k_{\perp 0}$. Such an increase is quite possible when $v_E \gg v_{Te}$ ($r_E \gg r_{De}$), since the condition $kr_{De} < 1$ must necessarily be satisfied for oscillations to exist; at the same time, $k_{z0} \sim r_E^{-1}$ for oscillations that grow in the presence of parametric instability. Naturally the relaxation equation is modified in this case (cf. (1.20)). However, the relaxation time can be estimated on the basis of formula (1.21).

Thus, the discussion above indicates the possibility of producing in a parametrically unstable plasma a considerable number of ions with velocities on the order of the phase velocity, and consequently with energies $(m_i/m_e)^{1/3}$ times greater in order of magnitude than the electron oscillation energy in the pumping wave field.

2. ION DISTRIBUTION RELAXATION WHEN THE FIELD FLUCTUATION LEVEL IS NOT ESTABLISHED

Under conditions of growing field fluctuations, the relaxation of the particle distributions in a parametrically unstable plasma becomes, in general, more complicated. However, even in this case it is possible to discern several important properties of the relaxation processes.

We note that the arguments of the preceding section remain intact in many respects if the quasilinear relaxation is characterized only by Cerenkov interaction of the resonant ions. Then the diffusion equation (1.1) holds, and in it the diffusion coefficient (1.2) depends on the time by virtue of the time dependence

$$\varphi_0(\mathbf{k}, t)\sim\exp\left\{\int^t dt'\gamma(\mathbf{k}, t')\right\}. \qquad (2.1)$$

Such a situation is realized with great accuracy, in particular, when the frequency of the external field is close to the value in (1.8). Then, for example, the one-dimensional diffusion coefficient acquires an additional factor

$$\exp\left\{2\int^t dt'\gamma(k_{z0}, t')\right\}. \qquad (2.2)$$

The same can also be said of two-dimensional diffusion with the coefficients (1.11) to (1.13). Then, if the fluctuation field grows at an exponential rate, one can again use the results of the preceding section, with the substitution

$$t\to(e^{2\gamma t}-1)/2\gamma. \qquad (2.3)$$

In particular, the characteristic relaxation time for the relaxation process of the type (1.20) is

$$\sim\frac{1}{2\gamma}\ln 2\gamma\tau, \qquad (2.4)$$

where τ is determined from formula (1.21).

In formula (2.4) the logarithm is much greater than unity. This means that the characteristic variation time of the ion distribution exceeds the characteristic variation time of the fluctuations in the parametrically unstable plasma. Under conditions when such a situation obtains, a simple description of ion relaxation is possible. For example, this can occur near the frequency (1.8), when the increment is small compared with the frequency of the growing perturbations. Then the diffusion equation for quasilinear ion relaxation during the exponential growth of the fluctuations can be represented in the form (1.1) with the diffusion coefficient

$$D_{rs}^i(v, t)=\frac{e_i^2}{m_i^2}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_rk_s\gamma(\mathbf{k})}{\gamma^2(\mathbf{k})+[\omega(\mathbf{k})-\mathbf{kv}]^2}|\varphi_0(\mathbf{k}, t=0)|^2e^{2\gamma(\mathbf{k})t}. \qquad (2.5)$$

Obviously, in the limit $\gamma = 0$ this expression reduces to (1.2).

The simplest consequence of the diffusion equation in velocity space with the diffusion coefficient (2.5) is that one can obtain the distribution down to low velocities $|\mathbf{k}\cdot\mathbf{v}| \ll \omega$, which necessarily presupposes that the ion velocities are small compared with the phase velocity (1.9). Then the velocity dependence in (2.5) can be neglected, and Eq. (1.1) has a simple solution. In particular, for distributions corresponding to (1.3) and for an initial Maxwellian ion distribution with temperature T(0) we have:

$$F_i(v, t)=\frac{n_im_i^{3/2}}{(2\pi\varkappa)^{3/2}T_\perp(t)T_\parallel^{1/2}(t)}\exp\left\{-\frac{m_iv_z^2}{2\varkappa T_\parallel(t)}-\frac{m_i(v_x^2+v_y^2)}{2\varkappa T_\perp(t)}\right\}. \qquad (2.6)$$

The effective time-dependence longitudinal and transverse temperatures are determined here from the following formula:

$$\begin{pmatrix} n_i\varkappa T_\parallel(t) \\ n_i\varkappa T_\perp(t) \end{pmatrix} = n_i\varkappa T(0) + \frac{e_i^2 n_i}{2m_i}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{|\varphi_0(\mathbf{k},t=0)|^2(e^{2\gamma t}-1)}{\gamma^2+\omega^2}\begin{pmatrix} 2k_z^2 \\ k_\perp^2 \end{pmatrix} \tag{2.7}$$

This formula is good for qualitative estimates over a broad region. The second term of (2.7) is equal in order of magnitude to the energy density of the plasma fluctuations for the nonresonant case ($\gamma/\omega_0 \sim (m_e/m_i)^{1/2}$), while for the resonant case it is equal to the same energy density multiplied by $(m_e/m_i)^{1/3}$. In addition, formula (2.7) indicates the reason for the possible ion velocity distribution anisotropy due to the anisotropy in space of the wave vectors that characterize the field fluctuations.

We note in this regard that the order of magnitude of k_z corresponding to faster growing fluctuations is determined by the quantity $\sim r_E^{-1}$. Thus, for example, when $\omega_0 \approx \omega_{Le}$ this value is determined by the maximum of the Bessel junction J_l and corresponds to $k_z r_E = 1.84$. On the other hand, the quantity $k_\perp$ is mainly determined by the initial fluctuation distribution, in which the maximum value of the transverse wave vector must satisfy the condition $k_\perp r_D \ll 1$. This makes it possible to assert that the measure of the temperature anisotropy must be less than $T_\perp/T_\parallel \sim (v_e/v_{Te})^2$.

3. SOME PROPERTIES OF ELECTRON DISTRIBUTION RELAXATION IN A PARAMETRICALLY UNSTABLE PLASMA

Owing to the low electron mass, the effect of the pumping field on the electrons is greater than on the ions. Following the results of the earlier studies[1,3], we discuss the electron distribution in a coordinate system that oscillates together with the electrons under the influence of the electric pump field $\mathbf{E}(t) = \mathbf{E}_0 \sin\omega_0 t$. In this case one can assert that the development of parametric instability in the plasma leads to the appearance of a whole series of harmonics in the electron distribution function:

$$F_e(v,t) = \sum_{n=-\infty}^{+\infty} F_e^{(n)}(v,t)e^{-in\omega_0 t}. \tag{3.1}$$

Here the $F_e^{(n)}$ vary little over a time interval on the order of the period of the external pump field.

In the high intensity pump field case of interest to us, where the electron oscillation velocity significantly exceeds their thermal velocity ($v_E \gg v_{Te}$) the higher harmonics grow simultaneously with the increase of the velocity spread in the zeroth harmonic $F_e^{(0)}$ (the only nonzero harmonic at the initial instant of time). One such manifestation is the growth of the energy fraction[1,12] determined by the harmonic $F_e^{(2n)}$ at the resonance $n\omega_0 \approx \omega_{Le}$. We shall show that this harmonic is not unique.

According to[3], the system of equations for the harmonics of the electron distribution function can be written in the form

$$-in\omega_0 F_e^{(n)} + \frac{\partial F_e^{(n)}}{\partial t} = \frac{e^2}{m_e^2}\frac{\partial}{\partial v_i}\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_s|\varphi_0(\omega_s,\mathbf{k},t=0)|^2\cdot$$

$$\cdot\exp(2\gamma_s(k)t)\sum_{l,u=-\infty}^{+\infty}\frac{J_{u-l}(k_zr_E)J_{u-n}(k_zr_E)}{\gamma_s - i(u\omega_0+\omega_s-\mathbf{kv})}\frac{\delta\varepsilon_i(\omega_s+i\gamma_s,\mathbf{k})}{1+\delta\varepsilon_e(\omega_s+(u-l)\omega_0+i\gamma_s,\mathbf{k})}$$

$$\times\frac{\delta\varepsilon_i(-\omega_s+i\gamma_s,-\mathbf{k})}{1+\delta\varepsilon_e(-\omega_s+(n-u)\omega_0+i\gamma_s,-\mathbf{k})}\frac{\partial F_e^{(l)}}{\partial v_j}, \tag{3.2}$$

where s labels the possible types of perturbations.

We note that Eq. (3.2) describes the collisions of the electrons with the fluctuations of the growing perturbation field in a parametrically unstable plasma. In this case, even when such collisions are small, it is, in general, no longer correct to state that only the zeroth harmonic is present at the initial instant of time (cf.[13]). Therefore one should regard a statement of this type as an approximate one, corresponding to the possibility of neglecting the collisions.

A definite advance in our analysis of the properties of the system (3.2) can be made in the resonance case, when $|\omega_0 - \omega_{Le}| \lesssim \omega_0(m_e/m_i)^{1/3}$. Then, to a first order in the ion-electron mass ratio raised to the $1/3$ power, this system of equations can be written in the form:

$$-in\omega_0F_e^{(n)} + \frac{\partial F_e^{(n)}}{\partial t} = \frac{\partial}{\partial v_i}\Big\{[D_{ij}^{(1)}(n,v)+iD_{ij}^{(2)}(n,v) + D_{ij}^{(1)}(-n,v)-iD_{ij}^{(2)}(-n,v)]\frac{\partial F_e^{(n)}}{\partial v_j} - [D_{ij}^{(1)}(n,v) + iD_{ij}^{(2)}(n,v)]\frac{\partial F_e^{(n+2)}}{\partial v_j} - [D_{ij}^{(1)}(-n,v)-iD_{ij}^{(2)}(-n,v)]\frac{\partial F_e^{(n-2)}}{\partial v_j}\Big\}, \tag{3.3}$$

$$D_{ij}^{(1)}(n,\mathbf{v})+iD_{ij}^{(2)}(n,\mathbf{v}) = \frac{e^2}{m_e^2}\int\frac{d\mathbf{k}}{(2\pi)^3}k_ik_j\sum_s\frac{|\varphi_0|^2\exp(2\gamma_st)}{(\omega_s^2+\gamma_s^2)^2}$$

$$\times\frac{\omega_{Li}^4\omega_0^4J_1^2(k_zr_E)}{[(\omega_0+\omega_s)^2-\omega_{Le}^2-\gamma_s^2]^2+4(\omega_0+\omega_s^2)^2\gamma_s^2}\frac{\gamma_s+i([n+1]\omega_0+\omega_s-\mathbf{kv})}{\gamma_s^2+([n+1]\omega_0+\omega_s-\mathbf{kv})^2}.$$

We have neglected here the spatial dispersion of the dielectric constants in accord with the assumption that the influence of thermal effects on the dispersion properties of the plasma perturbations is small. The system (3.3) breaks up into two systems corresponding to the odd and even harmonics of the electron distribution function. In our original problem there are no nonzero odd harmonics at the initial instant of time, and they can be subsequently disregarded. We note that a more accurate analysis shows that the odd harmonics are $(m_e/m_i)^{1/3}$ less than the even ones.

At electron velocities that are not too large it is possible to neglect $\mathbf{k}\cdot\mathbf{v}$ in comparison with ω_0. In particular, this means that the velocity is small compared with the velocity of the electron oscillations in the pump field. In addition, bearing in mind that ω_s and γ_s are small compared with the external field frequency ω_0, we see readily that the imaginary parts of the even harmonics of the electron distribution functions are small:

$$F_e^{(2n)}(v,t) = \phi^{(2n)}(v,t) + i\frac{\gamma}{\omega_0}\psi^{(2n)}(v,t). \tag{3.4}$$

Here γ is the maximum value of the increment and determines the order of magnitude of the second term in the right-hand side of (3.4). Bearing in mind that[1)]:

$$D_{ij}^{(1)}(n,\mathbf{v}=0) = \frac{1}{2(n+1)^2}\frac{d\Delta_{ij}(t)}{dt},\quad D_{ij}^{(2)}(n,\mathbf{v}=0) = \frac{\omega_0}{n+1}\Delta_{ij}(t), \tag{3.5}$$

we obtain the following system of equations:

1) We call Δ_{ij} the "integral diffusion coefficient."

$$\frac{\partial F_e^{(0)}}{\partial t}=\frac{d\Delta_{ij}(t)}{dt}\frac{\partial^2}{\partial v_i\,\partial v_j}(F_e^{(0)}+\phi^{(2)})+2\gamma\Delta_{ij}(t)\frac{\partial^2\psi^{(2)}}{\partial v_i\,\partial v_j},\quad(3.6)$$

$$2n\phi^{(2n)}+\Delta_{ij}(t)\frac{\partial^2}{\partial v_i\,\partial v_j}\left\{\frac{4n\phi^{(2n)}}{4n^2-1}-\frac{\phi^{(2n-2)}}{2n-1}-\frac{\phi^{(2n+2)}}{2n+1}\right\}=0,\quad(3.7)$$

$$2n\psi^{(2n)}+\Delta_{ij}(t)\frac{\partial^2}{\partial v_i\,\partial v_j}\left\{\frac{4n\psi^{(2n)}}{4n^2-1}-\frac{\psi^{(2n-2)}}{2n-1}-\frac{\psi^{(2n+2)}}{2n+1}\right\}=-\frac{1}{\gamma}\frac{\partial\phi^{(2n)}}{\partial t}$$
$$+\frac{1}{2\gamma}\frac{d\Delta_{ij}(t)}{dt}\frac{\partial^2}{\partial v_i\,\partial v_j}\left\{\left[\frac{1}{(2n+1)^2}+\frac{1}{(2n-1)^2}\right]\phi^{(2n)}-\frac{\phi^{(2n+2)}}{(2n+1)^2}-\frac{\phi^{(2n-2)}}{(2n-1)^2}\right\}.\quad(3.8)$$

Equations (3.7) and (3.8) have been written for $n \geq 1$, taking into account the fact that $\varphi^{(0)} = F_e^{(0)}$. Then the following relations hold for the negatively numbered harmonics:

$$\phi^{(-n)}=\phi^{(n)},\quad \psi^{(-n)}=-\psi^{(n)}.\quad(3.9)$$

Formulas (3.4) and (3.9) make it possible to represent expression (3.1) in the following form:

$$F_e(v,t)=F_e^{(0)}(v,t)+2\sum_{n=1}^{\infty}\phi^{(2n)}(v,t)\cos 2n\omega_0 t+2\frac{\gamma}{\omega_0}\sum_{n=1}^{\infty}\psi^{(2n)}(v,t)\sin 2n\omega_0 t.\quad(3.10)$$

The last term in (3.10) is of the same order of magnitude as the discarded odd harmonics. We must therefore not take them into account when we calculate the mean values, lest we exceed our accuracy limits. On the other hand, the imaginary parts of the harmonics of the electron distribution functions have a significant effect on the variation of the zeroth harmonic, as follows from Eq. (3.6). It must be emphasized that although the odd harmonics are comparable in order of magnitude with the imaginary parts of the even ones, nonetheless the effect of the odd harmonics on the distribution evolution described by formulas (3.6) to (3.8), turns out to be small, $\sim(m_e/m_i)^{1/3}$.

A number of the properties of the distributions that satisfy Eqs. (3.6) to (3.8) can be discerened without solving this system. First (3.6) to conservation of the number of electrons:

$$\int dvF_e^{(0)}=n_e=\text{const},\quad n\int dv\,\phi^{(2n)}=n\int dv\,\psi^{(2n)}=0.\quad(3.11)$$

Next, for the second moments of the velocities we have:

$$\frac{d}{dt}\int d\mathbf{v}\,v_iv_jF_e^{(0)}=n_e\frac{d\Delta_{ij}(t)}{dt},\quad(3.12)$$
$$\int d\mathbf{v}\,v_iv_j\,\phi^{(2)}=n_e\Delta_{ij}(t),\qquad \int d\mathbf{v}\,v_iv_j\,\phi^{(2n+2)}=0,\quad n\geqslant 1$$
$$\int d\mathbf{v}\,v_iv_j\,\psi^{(2)}=-\frac{1}{\gamma}\frac{d\Delta_{ij}(t)}{dt}n_e,\qquad \int d\mathbf{v}\,v_iv_j\,\psi^{(2n+2)}=0,\quad n\geqslant 1.$$

Formulas (3.12) give the following time variation of the plasma electron energy (in the oscillating coordinate system):

$$\int d\mathbf{v}\frac{m_ev^2}{2}[F_e(\mathbf{v},t)-F_e^{(0)}(\mathbf{v},t=0)]=\tfrac{1}{2}n_em_e\{\Delta_{ii}(t)-\Delta_{ii}(0)+2\Delta_{ii}(t)\cos 2\omega_0 t\}.\quad(3.13)$$

The corresponding formula for the variation of the mean value of $(1/2)m_ev_iv_j$ differs in that the trace of the integral diffusion coefficient is replaced by Δ_{ij}. In this case anisotropy of the electron energy parallel and perpendicular to the direction of the electric field intensity of the pump wave is obviously possible, due to the anisotropy of the electron diffusion coefficient in velocity space.

Equations (3.6)—(3.8) also make it possible to find the higher velocity moments. Thus, for example, the following asymptotic expression can be written for the fourth moments:

$$\int d\mathbf{v}\,v_iv_jv_kv_lF_e(\mathbf{v},t)=2n_e\{\Delta_{ij}(t)\Delta_{kl}(t)+\Delta_{jk}(t)\Delta_{il}(t)+\Delta_{ik}(t)\Delta_{jl}(t)\}\{1+\tfrac{4}{3}\cos 2\omega_0 t+\tfrac{1}{3}\cos 4\omega_0 t\}.\quad(3.14)$$

This expression is good for large time intervals, when the initial values of the moments and of the diffusion coefficients can be neglected. We emphasize that neither Eqs. (3.6)—(3.8) nor the consequences drawn from them have any meaning except for sufficiently large time intervals, when the collisions of the electrons with the plasma fluctuations significantly exceed those in the initial stage.

Comparison of formulas (3.13) and (3.14) shows that along with the second harmonic of electron distribution, which, as we noted before[1,12], determines the energy growth, a fourth harmonic characteristic of the fourth moments of the velocities appears. Higher electron velocity moments are determined by the higher harmonics of the electron distribution, while the moment of order 2n order contains harmonics all the way up to $\cos 2n\omega_0 t$.

It is obvious from (3.13) and (3.14) that the electron velocity dispersion is determined by the integral diffusion coefficient in velocity space. To better understand the ensuing possibilities, we assume a plasma fluctuation distribution that leads to transverse and longitudinal diffusion alone, just as in the discussion of ion relaxation. We can then introduce longitudinal and transverse effective electron temperatures:

$$\begin{pmatrix}\varkappa T_\parallel\\ \varkappa T_\perp\end{pmatrix}=\begin{pmatrix}m_e\Delta_\parallel\\ \tfrac{1}{2}m_e\Delta_\perp\end{pmatrix}=\frac{e^2}{m_e}\int\frac{d\mathbf{k}}{(2\pi)^3}\begin{pmatrix}k_z^2\\ \tfrac{1}{2}k_\perp^2\end{pmatrix}\quad(3.15)$$
$$\times\sum_s\frac{|\varphi_0|^2\exp(2\gamma_s t)}{(\omega_s^2+\gamma_s^2)^2}\,\frac{\omega_{Li}^4\omega_0^2J_1^2(k_zr_E)}{[(\omega_0+\omega_s)^2-\omega_{Le}^2-\gamma_s^2]^2+4(\omega_0+\omega_s)^2\gamma_s^2}.$$

At $\omega_0 < \omega_{Le}$ where aperiodic instability occurs, formula (3.15) can be presented in the following comparatively simple form:

$$\begin{pmatrix}n_e\varkappa T_\parallel\\ n_e\varkappa T_\perp\end{pmatrix}=\frac{1}{20.8\pi^3}\left[\left(1+\frac{10.4\omega_{Li}^2}{\Delta_1^3\omega_{Le}^2}\right)^{1/2}+1\right]\quad(3.16)$$
$$\times\int_0^{+\infty}k_\perp dk_\perp\int_{-\infty}^{+\infty}dk_z\left|\varphi_0\left(\frac{1.84}{r_E},k_\perp,t\right)\right|e^{2\gamma t}\begin{pmatrix}6.8r_E^{-2}\\ k_\perp^2\end{pmatrix},$$

where $\Delta_i = (\omega_{Le}/\omega_0)^2 - 1$ and

$$\gamma^2=\frac{1}{8}\omega_{Le}^2\Delta_1^2\left[\left(1+\frac{32}{\Delta_1^3}J_1^2(k_zr_E)\frac{\omega_{Li}^2}{\omega_{Le}^2}\right)^{1/2}-1\right].$$

Accordingly, for $\omega_0 > \omega_{Le}$, when $0 > \Delta_i > -[32J_i^2\omega_{Li}^2{}^2/\omega_{Le}^2]^{1/3}$,

$$\gamma=\frac{1}{4}|\Delta_1|\omega_{Le}\left\{\left[\frac{32}{|\Delta_1|^3}J_1^2(k_zr_E)\frac{\omega_{Li}^2}{\omega_{Le}^2}\right]^{1/2}-1\right\}^{1/2}$$

and we have

$$\begin{pmatrix}n_e\varkappa T_\parallel\\ n_e\varkappa T_\perp\end{pmatrix}=\frac{1}{2.72\pi^3}\left[\left(\frac{0.68\omega_{Li}^2}{|\Delta_1|^3\omega_{Le}^2}\right)^{1/2}+1\right]\int_0^\infty k_\perp dk_\perp\int_{-\infty}^{+\infty}dk_z\quad(3.17)$$
$$\times\left|\varphi_0\left(\frac{1.84}{r_E},k_\perp,t=0\right)\right|^2e^{2\gamma t}\begin{pmatrix}6.8r_E^{-2}\\ k_\perp^2\end{pmatrix}.$$

The right-hand sides of (3.16) and (3.17) are of the same order as the total energy of the growing field fluctuations. Thus, we can speak of an electron energy comparable with the plasma fluctuation energy.

Formulas (3.15)--(3.17) reveal a possibility of anisotropy in the electron velocity distribution. In this case the ratio of the longitudinal mean electron energy to the transverse energy is determined by the quantity $6.8(k_\perp^2)_{\text{eff}} r_E^{-2}$, where $(k_\perp^2)_{\text{eff}}$ depends on the initial fluctuation distribution. The analogy with the possible anisotropy in the ion distribution is obvious here.

We shall touch briefly on one possibility of electron distribution relaxation that can be realized when the distribution of the fluctuations with respect to wave numbers is sufficiently narrow. We note that the maximum of the increment determines the value of k_z. If at the same time the spread in the values of k_z is small, than diffusion in velocity space becomes particularly fast in the neighborhood of the resonance region $|\omega_0 - \mathbf{k}\cdot\mathbf{v}| \sim \gamma$. Then, neglecting the small quantities proportional to $(m_e/m_i)^{1/3}$, we obtain in accord with (3.3) the relatively simple approximate equation

$$\frac{\partial F_e^{(0)}}{\partial t} = 2\frac{\partial}{\partial v_i} D_{ij}^{(1)}(0)\frac{\partial F_e^{(0)}}{\partial v_j}. \tag{3.18}$$

From this it follows, in the first place, that the higher harmonics of the distribution function have a negligible effect on the relaxation of the fundamental in the resonance region. In the second place, the fast relaxation described in (3.18) leads to a "flattening" of the distribution in the vicinity of the resonance region of velocities. All this makes it possible to envision the production of a steplike distribution which, as a result of the effect of the higher harmonics on the fundamental, can lead to a multistep distribution due to the presence of the higher resonance regions.

Our analysis of the electron relaxation was made with a small deviation from parametric resonance as an example. In the contrary case, when the deviation is large ($|\omega_0 - \omega_{Le}| \sim \omega_{Le}$), the diffusion coefficients in velocity space have terms which, in general, are of the same order but have different resonance regions ($n\omega_0 \sim \mathbf{k}\cdot\mathbf{v}$). This makes the possibility of multistep distributions immediately obvious.

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