SOVIET PHYSICS

JETP

A Translation of Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki

Vol. 36, No. 4, pp. 591-810 (Russian Original Vol. 63, No. 4, pp. 1121-1528) April 1973

EFFECT OF SCALAR AND VECTOR FIELDS ON THE NATURE OF THE COSMOLOGICAL SINGULARITY

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Submitted April 27, 1972

Zh. Eksp. Teor. Fiz. **63**, 1121–1134 (October, 1972)

The effect of scalar and vector fields on the character of the cosmological singlarity is investigated. The fields may either be gravitational (in the sense of the Brans-Dicke ideas) or extraneous physical fields which are sources of an ordinary gravitational field. It is shown that in the presence of only a scalar field the gravitational equations possess a monotonic power-law aysmptotic general solution near the singular point in place of an oscillating form. However, if a vector field is included on the basis of five-dimension geometry concepts, the general solution becomes oscillatory again.

1. INTRODUCTION

IN the papers by Lifshitz and the authors[1-3], devoted to problems of relativistic cosmology, it is pointed out many times that the asymptotic character of the general solution of the gravitation equations near the cosmological singularity is determined by Einstein's equations already in empty space. Allowance for the energy-momentum tensor of matter results in only small corrections to such a first approximation. We have confined ourselves there to the energy-momentum tensor corresponding to an ideal liquid, and used the ultra-relativistic equation of state near the singularity ($\epsilon = 3p$), a natural procedure from the physical point of view.

In the present paper we study the changes produced in the character of the cosmological singularity by the presence of a scalar massless field, and then examine the situation when a vector massless field exists besides the scalar one. The method and need for including the vector field are dictated by simple geometric considerations. We note that the question of the physical nature of the introduced field does not have an unambiguous answer in this case. These fields can be taken to be either gravitational (in the sense of the Brans-Dicke ideas), as well as extraneous physical fields that serve as sources of an ordinary gravitational field. The choice of a particular interpretation reduces only to the choice of the interaction constants, the values of which do not play any role in our investigation.

We shall show that in the presence of only a scalar field, the general solution of the Einstein equations is significantly altered near the singular point. Instead of a vibrational regime that would occur without the scalar field (see[2,3]) we now obtain a simple power-law asymptotic variation. On the other hand, the addition of the vector field again returns our solution to an oscillatory regime, although of somewhat more complicated form in comparison with that described earlier[2,3].

Finally, we note that from the formally mathematical point of view these results become particularly clear if we turn to a formulation of the theory in terms of five-dimensional geometry, which is admissible in the present case.

2. EQUATIONS OF GRAVITATIONAL AND SCALAR FIELDS AND THE BEHAVIOR OF THEIR GENERAL SOLUTION NEAR THE SINGULARITY

The sought equations follow, as usual, from the variational principle[1)]

$$\delta \int (R + L_\varphi)\sqrt{-g}\, d^4x = 0, \tag{2.1}$$

where L_φ is the Lagrangian of the scalar field. The

[1)]We use a system of units in which the speed of light and the Einstein gravitational constant are equal to unity. If the scalar and vector fields have their own coupling constants, the latter are assumed included as factors in the potentials of these fields. We note also that we write the metric in the form $-ds^2 = g_{ik}dx^i dx^k$, where g_{ik} has a signature (+++−). The Latin indices i, k, l, and m run through the values 1, 2, 3, and 4, while the Greek ones run through 1, 2, and 3.

simplest form of the scalar-tensor theory corresponds to the Lagrangian

$$L_\varphi = -\varphi_{;k}\varphi^{;k}, \tag{2.2}$$

which generalizes the Lagrangian of a real Klein-Gordon field with zero rest mass to include the case of curves space. The Lagrangian (2.2) corresponds to the energy-momentum tensor

$$T_{ik} = \varphi_{;i}\varphi_{;k} - {}^1/_2 g_{ik}\varphi_{;l}\varphi^{;l} \tag{2.3}$$

and to the following system of field equations:

$$\varphi_{;k}{}^{;k} = 0, \tag{2.4}$$

$$R_{ik} = \varphi_{;i}\varphi_{;k}. \tag{2.5}$$

One might inquire here concerning the relation between the theory based on (2.4) and (2.5) and certain other published formulas of the scalar-tensor theory. Let us note two examples, one of which is the well known Brans-Dicke theory[4]. Another variant was proposed by Tagirov and Chernikov[5] and is characterized, in particular, by the fact that the contraction of the energy-momentum tensor vanishes for a field with zero rest mass. It turns out, however, at least so long as we are considering the fields purely classically, that all three variants are equivalent in the sense that they can be derived from one another by a conformal transformation of the metric (accompanied by a definite transformation of the scalar potential). A confirmation of this fact is contained in the next section of our article.

The search for the asymptotic form of the solution of (2.4) and (2.5) should be started, as usual, with a consideration of the particular case when the metric and the scalar potentials depend only on the time. In this case it is easy to obtain the following exact solution:

$$-ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{2.6}$$

$$\varphi = q \ln t; \tag{2.7}$$

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1 - q^2. \tag{2.8}$$

Here p_1, p_2, p_3 and q are constants, and it follows from (2.8) that the parameter q can vary in the range

$$-({}^2/_3)^{1/2} \leqslant q \leqslant ({}^2/_3)^{1/2}. \tag{2.9}$$

It is convenient to represent the exponents in the following parametric form (see also[6]):

$$p_1 = \frac{-u}{1+u+u^2}, \tag{2.10}$$

$$p_2 = \frac{1+u}{1+u+u^2}\left[u - \frac{u-1}{2}(1-(1-\beta^2)^{1/2})\right],$$

$$p_3 = \frac{1+u}{1+u+u^2}\left[1 + \frac{u-1}{2}(1-(1-\beta^2)^{1/2})\right];$$

$$\beta^2 = \frac{2(1+u+u^2)^2 q^2}{(u^2-1)^2}. \tag{2.11}$$

At q = 0 this yields Kasner's solution with the usual representation of the exponents (see[2], Sec. 2), accurate to the substitution $u \to 1/u$.

It follows from (2.10) and (2.11) that the exponents remain here, too, invariant against the transformation of the parameter $u \to 1/u$ (at arbitrary q):

$$p_1(1/u) = p_1(u), \quad p_2(1/u) = p_3(u), \quad p_3(1/u) = p_2(u). \tag{2.12}$$

We can therefore confine ourselves to an examination of the region $-1 \leq u \leq 1$. It is seen from (2.10) that the region of admissible values of the parameters u and q is determined by the inequality

$$\beta^2 = \frac{2(1+u+u^2)^2 q^2}{(u^2-1)^2} \leqslant 1. \tag{2.13}$$

This region is shown in the figure (shaded). It is easily seen that in this region, where the sequence of the exponents is $p_1 \leq p_2 \leq p_3$, the exponents vary (as functions of the parameters u and q) in the ranges $-\frac{1}{3} \leq p_1 \leq \frac{1}{3}$, $0 \leq p_2 \leq \frac{2}{3}$, $\frac{1}{3} \leq p_3 \leq 1$. A qualitatively new element in the properties of p_1, p_2, and p_3 (in comparison with the Kasner case) is that all three can take on positive values (for example, this is the only possibility when $q > 1/\sqrt{2}$ and $q < -1/\sqrt{2}$). This fact, as can be easily seen, causes also the general solution of (2.4) and (2.5) to have a simple asymptotic form near the singularity. It is easy to verify that at positive p_1, p_2, and p_3 the general solution that is asymptotic as $t \to 0$ (in the synchronous system) takes the form

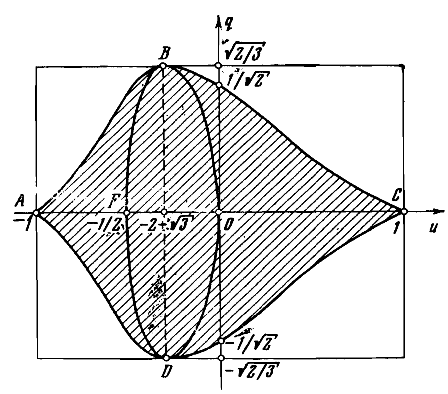
$$-ds^2 = -dt^2 + (t^{2p_1}l_\alpha l_\beta + t^{2p_2}m_\alpha m_\beta + t^{2p_3}n_\alpha n_\beta)dx^\alpha dx^\beta, \qquad \varphi = q \ln t + \varphi_0, \tag{2.14}$$

where p_1, p_2, p_3, and q are subject, as before, to the conditions (2.8), but are now functions of three spatial coordinates. Similar functions are the three-dimensional vectors **l**, **m**, and **n** and the quantity φ_0.

Indeed, the wave equation (2.4)

$$-\frac{1}{\sqrt{-g}}(\sqrt{-g}\,\dot\varphi)^{\cdot} + \frac{1}{\sqrt{-g}}(\sqrt{-g}\,g^{\alpha\beta}\varphi_{,\beta})_{,\alpha} = 0$$

the principal terms are those with the derivatives with respect to time, which also are made to vanish identically by the solution (2.14) and which are "potentially" of order $1/t^2$. On the other hand, the unaccounted-for terms (containing differentiation with respect to the spatial variables) are of order $t^{-2p_\alpha}\ln t$, which is smaller than t^2 (inasmuch as all $p_\alpha < 1$), and can indeed be discarded. As to Eq. (2.5), everything necessary for their analysis is contained in[1]. A simple analysis,



Region of admissible values of the parameters u and q. The boundary of the region is made up of pieces of plots of the equations $q = \pm(u^2-1)/\sqrt{2}(1+u+u^2)$. The closed curve of the center is a plot of $q^2 = -2u(2u^2+5u+2)/(1+u+u^2)^2$, and divides the entire shaded region into three parts: in the part BODC the sequence of the exponents is $p_1 \leqslant p_2 \leqslant p_3$, in the central part $p_2 \leqslant p_1 \leqslant p_3$, while in the region ABFD we have $p_2 \leqslant p_3 \leqslant p_1$.

similar to the one given there, shows that the equations

$$R_4^4 \equiv {}^1/_2\dot{\varkappa} + {}^1/_4\varkappa_\alpha^\beta\varkappa_\beta^\alpha = -\dot{\varphi}^2,$$

$$R_\alpha^\beta \equiv P_\alpha^\beta + \frac{1}{2\sqrt{-g}}(\sqrt{-g}\,\varkappa_\alpha^\beta)^\cdot = g^{\beta\mu}\varphi_{,\mu}\varphi_{,\alpha}$$

are identically satisfied by the solution (2.14) in the principal order t^{-2}. The terms of principal order are again those containing the time derivatives. On the other hand, the order of the terms P_α^β and $g^{\beta\mu}\varphi_{,\mu}\varphi_{,\alpha}$ is smaller than t^{-2} and come into play only in the approximation that follows (2.14). (We recall that if one of the exponents p_α were to be negative, it would be precisely in P_α^β where terms of order larger than t^{-2} would appear.)

The remaining equation

$$R_\alpha^4 \equiv {}^1/_2\,\varkappa_{;\alpha} - {}^1/_2\,\varkappa^\beta_{\alpha;\beta} = -\dot{\varphi}\varphi_{;\alpha}$$

is automatically satisfied in order $(\ln t)/t$, by virtue of relations (2.8), and reduces in order $1/t$ to three relations between the arbitrary three-dimensional functions contained in the solution (2.14). Taking also into account the possibility of arbitrary three-dimensional coordinate transformations, we can easily find that there remain in the solution six physically arbitrary three-dimensional functions, i.e., as many as there should be in the general solution of (2.4) and (2.5).

To complete the analysis we must answer also the following question: is the situation with three positive exponents p_1, p_2, and p_3 a necessary stage in the evolution of the solution with arbitrary initial data? By themselves, relations (2.8) admit also of the possibility that one of the exponents is negative. It is easy to show, however, that when one specifies initial data corresponds to such a "quasi-Kasner" case, further evolution of the metric will consist of a series with a finite number of oscillations, described by alternation of similar "quasi-Kasner" epochs with the exponents transformed in accordance with a definite law. The end result of this series of transformations in the general case is precisely a set of positive exponents p_1, p_2, p_3. The oscillations then stop and the solution acquires a monotonic asymptotic form (2.14). The foregoing can be proved by a direct investigation of Eqs. (2.4) and (2.5), but it can be more easily obtained by considering a more general case, which will be investigated later on. We shall dwell on it at the end of Sec. 4.

We note, finally, a formally mathematical cause of such a strong influence of the scalar field on the character of the singularity, from the point of view of a hydrodynamic analogy. It is easily seen that the tensor (2.3) can be written in the form of the energy-momentum tensor of an ideal liquid with an equation of state $\epsilon = p$ (in which case the "velocity" is $u_i = \varphi_{;i}/(-\varphi_{;k}\varphi^{;k})^{1/2}$ and the "pressure" $p = -{}^1/_2\varphi_{;k}\varphi^{;k}$), and the wave equation (2.4) would then be the consequence of the "equations of hydrodynamics" (the equation of state $p = \epsilon$ was proposed earlier by Ya. B. Zel'dovich[7]). The conclusion that the matter plays a very minor role near the singular point is based on the analysis given in[1] (see Sec. 3). This analysis is valid, however, only for an equation of state satisfying the condition $p \le 2\epsilon/3$. By repeating a similar analysis for the equation of state $\epsilon = p$ we see that the components of the energy-momentum tensor T_{44} turn out to be of order t^{-2}, i.e., of the same order as the left-hand side of the Einstein equation $R_{44} - {}^1/_2\, g_{44}R$. This means that the influence of matter of this type can no longer be neglected near the singular point. This is precisely the situation in the scalar-tensor theory considered here.

3. INTRODUCTION OF A VECTOR FIELD IN ADDITION TO THE SCALAR ONE

The contemporary state of large regions of the universe is characterized, as is well known, by a high degree of isotropy. Under such conditions, as noted by Dicke,[8] it is meaningless to introduce into the theory a vector field for the purpose of obtaining noticeable cosmological effects. Such a field, when averaged over large regions of the universe, will single out a preferred direction on cosmological scales, something incompatible with the observed isotropy. The situation is entirely different during earlier stages of development, when the universe can be essentially anisotropic. In the investigation of cosmological effects near the singular point, there are no longer any grounds for disregarding the possible existence of a vector field in addition to the scalar field. It is difficult at present to say anything concerning the form of this field, concerning its sources and the character of its interaction with the gravitational and scalar fields. However, even the introduction of the scalar field alone suggests one of the admissible ways of introducing the vector one. This way is based on the possibility of using the five-dimensional geometry, which has been in use in the literature since Kaluza's time[9]. This theory has been usually discussed from the point of view of a unified geometrical description of the gravitational and electromagnetic fields. We wish to call attention here to a different interpretation: it can be understood also as a scalar-vector-tensor theory of gravitation in the spirit of the Brans-Dicke ideas.

We shall show first that the Brans-Dicke theory itself is none other than a particular case of five-dimensional geometrical theory. Indeed, by adding to the four-dimensional space-time one more spacelike dimension and introducing the five-dimensional metric tensor j_{55}, j_{5k}, j_{ik} we obtain a formal expression for the interval

$$-ds_{(5)}^2 = j_{55}(dx^5)^2 + 2j_{5k}\,dx^5\,dx^k + j_{ik}\,dx^i\,dx^k. \qquad (3.1)$$

The components of the metric are chosen to be independent of the variable x^5:

$$j_{55} = j_{55}(x^i),\ j_{5k} = j_{5k}(x^i),\ j_{ik} = j_{ik}(x^i). \qquad (3.2)$$

We consider now the particular case when $j_{5k} = 0$, and write the remaining components j_{55} and j_{ik} in the form

$$j_{55} = A^2,\ j_{ik} = B^2 g_{ik}, \qquad (3.3)$$

where the quantities A and B will be assumed to be functions of a certain scalar function $\varphi(x^i)$:

$$A = A(\varphi),\ B = B(\varphi). \qquad (3.4)$$

The five-dimensional interval now takes the form

$$-ds_{(5)}^2 = A^2(dx^5)^2 + B^2 g_{ik}\,dx^i\,dx^k. \qquad (3.5)$$

Writing out Einstein's equations in empty five-dimensional space for the metric (3.5), we obtain

$$\rho_{55} \equiv -\frac{A^2}{B^2}\left[\frac{A'}{A}\varphi_{;k}^{;k} + \frac{(A'B^2)'}{AB^2}\varphi_{;k}\varphi^{;k}\right] = 0, \quad (3.6)$$

$$\rho_{5k} \equiv 0, \quad (3.7)$$

$$\rho_{ik} \equiv R_{ik} - \frac{(AB^2)'}{AB^2}\varphi_{;i;k} - \left(\frac{A''}{A} + 2\frac{B''}{B} - 2\frac{A'B'}{AB} - 4\frac{B'^2}{B^2}\right)\varphi_{;i}\varphi_{;k}$$
$$- g_{ik}\left[\frac{B'}{B}\varphi_{;l}^{;l} + \left(\frac{B''}{B} + \frac{B'^2}{B^2} + \frac{A'B'}{AB}\right)\varphi_{;l}\varphi^{;l}\right] = 0. \quad (3.8)$$

Here ρ_{55}, ρ_{5k}, ρ_{ik} are the components of the five-dimensional Ricci tensor, the primes denote derivatives of the functions A and B with respect to their argument φ, and all the covariant 4-tensor operations pertain to the tensor g_{ik}, from which the four-dimensional Ricci tensor R_{ik} in formula (3.8) is constructed.

The Einstein equations, as is well known, follow from the variational principle

$$\delta\int\sqrt{-j}\,\rho\, d^5x = 0,$$

where j is the determinant of the five-dimensional metric tensor and ρ is the construction of the five-dimensional Ricci tensor. Since, however, the metric does not depend on the fifth variable, the variational principle reduces to the form

$$\delta\int\rho\sqrt{-j}\, d^4x = 0. \quad (3.9)$$

For a metric in the form (3.5) we have

$$\rho\sqrt{-j} = \left\{R - \left(6\frac{B'}{B} + 2\frac{A'}{A}\right)\varphi_{;l}^{;l} - \right.$$
$$\left. -\left(6\frac{B''}{B} + 2\frac{A''}{A} + 4\frac{A'B'}{AB}\right)\varphi_{;l}\varphi^{;l}\right\}AB^2\sqrt{-g}. \quad (3.10)$$

Here R is the contraction of the four-dimensional Ricci tensor and g is a four-dimensional determinant.

It is clear that any change in the form of the functions $A(\varphi)$ and $B(\varphi)$ is equivalent to a certain conformal transformation of the four-dimensional metric g_{ik} accompanied by a transformation of the function φ:

$$\varphi = \varphi(\varphi'), \; g_{ik} = g'_{ik} f(\varphi'). \quad (3.11)$$

In other words, it suffices to make only one concrete choice of the functions A and B, and all the other possibilities can already be obtained from it by the indicated transformation. It is easily seen that the equations (2.4) and (2.5) investigated in Sec. 2 are obtained from (3.6)—(3.8) by choosing

$$A = \exp[(^2/_3)^{1/2}\varphi], \quad B = \exp(-6^{-1/2}\varphi), \quad (3.12)$$

and it is seen from (3.9) and (3.10) that this case corresponds to the variational principle

$$\delta\int\left(R - \varphi_{;k}\varphi^{;k} + (^2/_3)^{1/2}\varphi_{;k}^{;k}\right)\sqrt{-g}\, d^4x = 0, \quad (3.13)$$

which coincides with (2.1) and (2.2) if one notes that the last term in the parentheses in (3.13), multiplied by $\sqrt{-g}$, is a four-dimensional divergence and can be discarded.

The equations of the Brans-Dicke theory[4] correspond to the following choice of the functions A and B:

$$A = \varphi^{\mu}, \; B = \varphi^{(1-\mu)/2};$$
$$\mu = (1 + 2\omega/3)^{1/2}, \quad (3.14)$$

From (3.9) and (3.10) we obtain for this case

$$\delta\int\left[R\varphi - \frac{\omega\varphi_{;k}\varphi^{;k}}{\varphi} - (3-\mu)\varphi_{;k}^{;k}\right]\sqrt{-g}\, d^4x = 0, \quad (3.15)$$

i.e., after eliminating the divergence we obtain the initial Brans-Dicke Lagrangian (without ordinary matter).

If, finally, we require satisfaction of the conditions

$$\frac{A'}{A} = \frac{2\sqrt{6}}{6-\varphi^2}, \quad \frac{B'}{B} = -\frac{\sqrt{6}+\varphi}{6-\varphi^2}, \quad (3.16)$$

then we obtain from (3.6)—(3.8) the equations

$$\varphi_{;k}^{;k} = 0, \quad R_{ik} = (1 - {}^1/_6\varphi^2)^{-1}({}^2/_3\varphi_{;i}\varphi_{;k} - {}^1/_3\varphi\varphi_{;i;k} - {}^1/_6 g_{ik}\varphi_{;l}\varphi^{;l}). \quad (3.17)$$

These equations are the same to which the scalar-tensor theory reduces in the form proposed by Tagirov and Chernikov[5].

We see thus that apparently any reasonable form of the scalar-tensor theory admits of a simple five-dimensional geometrical treatment. However, if we do use geometrical language, it is most natural to choose a variant in which the component j_{55}, and only this component, is connected with the scalar field, while the components j_{ik} are the metric g_{ik} of four-dimensional space-time. This obviously corresponds to the choice

$$A = \varphi, \; B = 1. \quad (3.18)$$

Equations (3.6)—(3.8) then yield

$$\varphi_{;k}^{\;;k} = 0, \; R_{ik} = \varphi^{-1}\varphi_{;i;k}, \quad (3.19)$$

and from (3.9) and (3.10) it follows that the theory in this form corresponds to the variational principle[2)]

$$\delta\int R\varphi\sqrt{-g}\, d^4x = 0. \quad (3.20)$$

It will be shown later on that it is precisely this form which is most convenient for us, since the synchronous reference frame of the five-dimensional space-time ($j_{44} = -1$, $j_{45} = 0$, $j_{4\alpha} = 0$) is here simultaneously synchronous for the four-dimensional space-time ($g_{44} = 1$, $g_{4\alpha} = 0$).

To introduce a vector field in addition to the scalar field, it suffices now to assume that the components j_{5k} in the metric form (3.1) differ from zero. Using the notation

$$j_{55} = \varphi^2, \; j_{5k} = \varphi^2 A_k$$

and introducing a four-dimensional metric g_{ik} such that

$$g_{ik} = j_{ik} - \varphi^2 A_i A_k, \quad (3.22)$$

we note that Einstein's equations $\rho_{55} = 0$, $\rho_{5k} = 0$, and $\rho_{ik} = 0$ take the following form:

$$\varphi_{;k}^{\;;k} = {}^1/_4\varphi^3 F_{ik}F^{ik}, \quad (3.23)$$

$$(\varphi^3 F^{ik})_{;k} = 0, \quad (3.24)$$

$$R_{ik} = \frac{1}{\varphi}\varphi_{;i;k} + \frac{1}{2}\varphi^2 F_{il}F^{l}{}_{k}, \quad (3.25)$$

$$F_{ik} = A_{i,k} - A_{k,i}, \quad (3.26)$$

where all four-dimensional tensor operations (as well as the construction of the tensor R_{ik}) pertain to the metric g_{ik}.

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2) When varying with respect to g_{ik} we obtain from (3.20) $R_{ik} - ½g_{ik}R)\varphi - \varphi_{;i;k} + g_{ik}\varphi_{;l}^{;l} = 0$. On the other hand, variation with respect to φ yields simply $R = 0$. As a result we get the system (3.19).

After calculating the five-dimensional Lagrangian $\sqrt{-j}\,\rho$ for this case, we can readily show that Eqs. (3.23)—(3.25) can be obtained also from the four-dimensional variational principle

$$\delta \int (R\varphi - {}^1/_4\, \varphi^3 F_{ik} F^{ik}) \sqrt{-g}\, d^4x = 0, \qquad (3.27)$$

which does not call for the use of five-dimensional geometry concepts at all.

We note finally that a conformal transformation of the metric with simultaneous replacement of the scalar potential, in the form

$$g_{ik} = g'_{ik} \exp(-({}^2/_3)^{1/2}\varphi'), \quad \varphi = \exp(({}^2/_3)^{1/2}\varphi'), \qquad (3.28)$$

enables us to obtain from (3.27) the variational principle

$$\delta \int [R' - \varphi_{;k}' \varphi'^{;k} - {}^1/_4\, F_{ik} F^{ik} \exp(6^{1/2}\varphi')] \sqrt{-g'}\, d^4x = 0, \qquad (3.29)$$

which corresponds to a generalization of the scalar-tensor geometry, taken in the form (2.1)—(2.5). (All the tensor operations in (3.29) pertain to the metric g'_{ik}.) On the other hand, if we make the transformation (λ = const)

$$g_{ik} = \varphi'^{\lambda} g'_{ik}, \quad \varphi = \varphi'^{(1-\lambda)}, \qquad (3.30)$$

then it follows from (3.27) that

$$\delta \int \left(R'\varphi' - \frac{3\lambda(\lambda-2)}{2} \frac{\varphi_{;k}' \varphi'^{;k}}{\varphi'} - \frac{1}{4} \varphi'^{(3-3\lambda)} F_{ik}F^{ik}\right) \sqrt{-g'}\, d^4x = 0. \qquad (3.31)$$

Such a variational principle corresponds to a generalization of the theory taken in the Brans-Dicke form. Here $3\lambda(\lambda - 2)/2$ is the coupling constant ω, so that we have two solutions for the coupling constant λ:

$$\lambda = 1 \pm (1 + 2\omega / 3)^{1/2}. \qquad (3.32)$$

4. OSCILLATORY APPROACH TO THE SINGULAR POINT IN THE SCALAR-VECTOR-TENSOR THEORY

To investigate the asymptotic character of the general solutions of Eqs. (3.23)—(3.26) near the singularity, it is indeed most convenient to use the language of five-dimensional geometry. We introduce first in five-dimensional space a synchronous reference frame satisfying the conditions

$$j_{54} \equiv \varphi^2 A_4 = 0, \qquad (4.1)$$

$$j_{a4} = 0, \; j_{44} = -1. \qquad (4.2)$$

The requirement (4.1) can be easily satisfied by making use the leeway in the choice of the gauge of the vector potential A_i. From the geometrical point of view, on the other hand, the gauge transformation $A'_i = A_i + \Phi_{,i}$ is none other than the transformation of the components of the five-dimensional metric tensor $j_{5k} = j_{55}A_k$ (according to the notation in (3.21)) under the action of a coordinate transformation of the form[3)]

$$x^5 = x'^5 + \Phi(x^k). \qquad (4.3)$$

By choosing the function $\Phi(x^k)$ it is always possible to see to it that $A_4 = 0$. On the other hand, satisfaction of (4.2) is obtained by one four-dimensional coordinate transformation.

We agree to assume throughout this section that the first four Latin indices a, b, c, and d run through the values 1, 2, 3, and 5 (unlike the Latin indices i, k, l, and m, which take on the values 1, 2, 3, and 4). Then the five-dimensional metric takes the form

$$-ds_{(5)}^2 = -dt^2 + j_{ab}\, dx^a\, dx^b. \qquad (4.4)$$

Further, we investigate the behavior of the components $j_{ab}(t, x^\mu)$. After they are found, it will be easy to obtain the scalar and vector potentials in accord with (3.20), and also the four-dimensional synchronous metric from (3.22).

The character of the general solution of Einstein's equations for the metric (4.4) near the cosmological singularity is sought in full analogy with the investigation of the four-dimensional case in[2,3]. We shall show now that the addition of one space-like dimension does not change the qualitative character of the solution near the singularity. We again encounter an oscillatory regime that differs from the one in the four-dimensional case only in quantitative details.

First, in the five-dimensional case (and generally in an n-dimensional case) there exists an exact Kasner solution[4)] for the metric (4.4):

$$-ds_{(5)}^2 = -dt^2 + t^{2s_1} dx_1^2 + t^{2s_2} dx_2^2 + t^{2s_3} dx_3^2 + t^{2s_5} dx_5^2, \qquad (4.5)$$

$$s_1 + s_2 + s_3 + s_5 = s_1^2 + s_2^2 + s_3^2 + s_5^2 = 1. \qquad (4.6)$$

On the basis of the complete geometrical analogy with the four-dimensional case, one should expect the sought general solution to be described by an alternation of Kasner epochs accompanied by a definite transformation of the exponents s_a and by rotations of the Kasner axes. Accordingly, we seek the first approximation of the general solution in the form

$$-ds_{(5)}^2 = -dt^2 + (a^2 l_a l_b + b^2 m_a m_b + c^2 n_a n_b + d^2 q_a q_b)\, dx^a\, dx^b, \qquad (4.7)$$

where the four-dimensional vectors l_a, m_a, n_a, and q_a do not depend on the time. The 4-tensor components of Einstein's equations $\rho_{ab} = 0$ for a metric of the type (4.4) are given by

$$\frac{1}{\sqrt{-j}} (\sqrt{-j}\, k_a^{\,b})^{\cdot} = -2\Pi_a^{\,b}, \qquad (4.8)$$

where $k_{ab} = \partial j_{ab}/\partial t$ and Π_{ab} is the Ricci tensor constructed in accordance with the metric j_{ab}.

If we neglect the right hand side of (4.8), we obtain the solution (4.7), in which

$$(a, b, c, d) = (a_0 t^{s_1}, b_0 t^{s_2}, c_0 t^{s_3}, d_0 t^{s_5}). \qquad (4.9)$$

The vectors l_a, m_a, n_a, and q_a are so normalized that their moduli are of the order of unity, while the orders

[3)]The only permissible coordinate transformations are those that leave all the metric components j_{55}, j_{5k}, and j_{ik} independent of the fifth variable x^5.

[4)]This solution is equivalent to the solution (2.6) – (2.8). This can be easily verified by making the transformation (3.28) after substituting in it $\varphi = t^{s_5}$ and $g_{ik} = (t^{2s_1}, t^{2s_2}, t^{2s_3}, -1)$. As a result we get

$$\varphi' = ({}^3/_2)^{1/2} s_5 \ln t, \quad g_{ik}' = (t^{2s_1+s_5}, t^{2s_2+s_5}, t^{2s_3+s_5}, -t^{s_5}).$$

Transforming now to the synchronous reference frame (in which $g'_{44} = -1$), we obtain the solution (2.6) – (2.8), and the Kasner exponents s_a are expressed in terms of the parameters p_1, p_2, p_3, and q as follows.

$$s_5 = 2q / (6^{1/2} - q), \quad s_\alpha = (6^{1/2} p_\alpha - q) / (6^{1/2} - q).$$

of the components j_{ab} are assumed to be included in the time-dependent factors and in the coordinate-independent constants a_0, b_0, c_0, and d_0 in (4.9). A simple analysis, similar to that presented in[3], shows that the necessary and sufficient conditions for the applicability of the solution (4.7) and (4.9) follow already from an examination of the diagonal projections of (4.8) on the directions of the vectors l_a, m_a, n_a, and q_a. In terms of such projections, the smallness of the discarded tensor Π_a^b in (4.8) is formulated for the diagonal projections in the form

$$\Pi_l^l,\ \Pi_m^m,\ \Pi_n^n,\ \Pi_q^q \ll 1/t^2 \tag{4.10}$$

and for the diagonal ones in the form

$$\begin{gathered}\Pi_{lm} \ll ab/t^2,\ \Pi_{ln} \ll ac/t^2,\ \Pi_{mn} \ll bc/t^2,\\ \Pi_{lq} \ll ad/t^2,\ \Pi_{mq} \ll bd/t^2,\ \Pi_{nq} \ll cd/t^2.\end{gathered} \tag{4.11}$$

It turns out that to satisfy the inequalities (4.10) it is necessary to satisfy the following 12 inequalities (as against three for the four-dimensional case[3]):

$$\frac{k}{\Lambda}(a^2b, a^2c, a^2d, b^2a, b^2c, b^2d, c^2a, c^2b, c^2d, d^2a, d^2b, d^2c) \ll 1. \tag{4.12}$$

This notation means that each of the terms in the parentheses, when multiplied by k/Λ, should be much smaller than unity. Here $1/k$ denotes the order of magnitude of the distances over which the metric varies significantly, and Λ denotes the product $a_0b_0c_0d_0$, so that

$$\sqrt{-j} \sim abcd = \Lambda t. \tag{4.13}$$

Examination of the conditions (4.11) does not introduce anything new now, since all the conditions (4.11) are automatically satisfied if the inequalities (4.12) are satisfied. Moreover, they are satisfied even if one of the inequalities in (4.12) is violated (for, example if $ka^2b/\Lambda \sim 1$).[5)] It follows therefore that in the investigation of (4.8) in the region where the solution (4.9) ceases to hold because one of the inequalitites in (4.12) is violated, all the off-diagonal projections of the tensor Π_a^b can be neglected as before. As to the diagonal projections of this tensor, it is necessary to take into account in them only terms of one kind: if, for example, the term ka^2b/Λ in (4.12) increases with decreasing t and reaches at some "critical" instant t_{cr} a value on the order of unity (but all the remaining terms in (4.12) still remain small), then it is necessary to take into account in the components Π_l^l, Π_n^n, and Π_q^q, in addition to the principal terms, only terms of the type

$$\Pi_l^l = \frac{a^2\lambda^2}{2c^2d^2},\quad \Pi_n^n = -\frac{a^2\lambda^2}{2c^2d^2},\quad \Pi_q^q = -\frac{a^2\lambda^2}{2c^2d^2}, \tag{4.14}$$

where λ is given by

$$\lambda = (l_{a,b} - l_{b,a})n^a q^b \tag{4.15}$$

(n^a and q^a are the components of the vectors reciprocal to n_a and q_a). As to the component Π_m^m, it does not contain terms similar to (4.14), and can therefore be discarded as before.

Thus, Eqs. (4.8) for the metric (4.7) now become

$$\begin{gathered}\frac{1}{abcd}[abcd(\ln a^2)^\cdot]^\cdot = -\frac{a^2\lambda^2}{c^2d^2},\quad \frac{1}{abcd}[abcd(\ln b^2)^\cdot]^\cdot = 0.\\ \frac{1}{abcd}[abcd(\ln c^2)^\cdot]^\cdot = \frac{a^2\lambda^2}{c^2d^2},\quad \frac{1}{abcd}[abcd(\ln d^2)^\cdot]^\cdot = \frac{a^2\lambda^2}{c^2d^2}.\end{gathered} \tag{4.16}$$

It is necessary to add to these equations also the equation $\rho_{44} = 0$ in the form

$$\ddot{a}/a + \ddot{b}/b + \ddot{c}/c + \ddot{d}/d = 0. \tag{4.17}$$

We note now that in order for ka^2b/Λ to be the principal term among all those listed in (4.12), it is necessary to satisfy the inequalities

$$a \gg b \gg c \gtrsim d. \tag{4.18}$$

The solution of (4.16) and (4.17) must therefore satisfy also this requirement. Without dwelling on the details, we present the final result. The solution of these equations describes the alternation of two Kasner epochs. If the asymptotic form of the solution during the initial epoch (at $t \gg t_{cr}$) is

$$\begin{gathered}a^2 = a_\kappa^2(t/t_\kappa)^{2s_1},\quad b^2 = b_\kappa^2(t/t_\kappa)^{2s_2},\\ c^2 = c_\kappa^2(t/t_\kappa)^{2s_3},\quad d^2 = d_\kappa^2(t/t_\kappa)^{2s_5},\end{gathered} \tag{4.19}$$

with

$$s_1 < s_2 < s_3 < s_5\ (s_1 < 0), \tag{4.20}$$

then we obtain in the final epoch (at $t \ll t_{cr}$)

$$\begin{gathered}a^2 = a_\kappa'^2(t/t_\kappa)^{2s_1'},\quad b^2 = b_\kappa'^2(t/t_\kappa)^{2s_2'},\quad c^2 = c_\kappa'^2(t/t_\kappa)^{2s_3'},\\ d^2 = d_\kappa'^2(t/t_\kappa)^{2s_5'}.\end{gathered} \tag{4.21}$$

The instant t_{cr} is defined here by the formula

$$t_\kappa^2 = 4(2s_1 + s_2)^2 c_\kappa^2 d_\kappa^2 / \lambda^2 a_\kappa^2, \tag{4.22}$$

and the new exponents s_a' are expressed in terms of the old ones as follows:

$$\begin{gathered}s_1' = -\frac{s_1 + s_2}{1 + s_2 + 2s_1},\quad s_2' = \frac{s_2}{1 + s_2 + 2s_1},\\ s_3' = \frac{s_3 + s_2 + 2s_1}{1 + s_2 + 2s_1},\quad s_5' = \frac{s_5 + s_2 + 2s_1}{1 + s_2 + 2s_1}.\end{gathered} \tag{4.23}$$

The quantities a_k', b_k', c_k', and d_k' in the final epoch are given by

$$\begin{gathered}a_\kappa' = a_\kappa(1 + s_2 + 2s_1)^{s_1'},\quad b_\kappa' = b_\kappa(1 + s_2 + 2s_1)^{s_2'},\\ c_\kappa' = c_\kappa(1 + s_2 + 2s_1)^{s_3'},\quad d_\kappa' = d_\kappa(1 + s_2 + 2s_1)^{s_5'}.\end{gathered} \tag{4.24}$$

In addition, we note that to satisfy the condition (4.18) in the region of interest to us, we must stipulate

$$a_\kappa \gg b_\kappa \gg c_\kappa \gtrsim d_\kappa. \tag{4.25}$$

It is easily seen that under the condition (4.20) the new exponent s_1' will be positive and s_3' negative. Thus, the function a will decrease during the final epoch (with decreasing time), and the function c will begin to increase. (But the exponent s_2' has the same sign as s_2.)

It is easy to show that in addition to the change of the functions a, b, c, d, the Kasner axes are also rotated during the course of the alternation of the epochs. If the metric in the initial epoch is (4.7), then in the final epoch

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5) The conditions (4.11) can be violated only if at least two of the inequalities (4.12) are violated. This special case is connected with a certain form of small oscillations and will not be considered here (cf. the analogous situation in [3]). On the other hand, violation of one of the inequalities (4.12) is an obligatory element in the evolution of the solution (4.7) – (4.9). Indeed, it follows from (4.6) that at least one of the exponents s_a must be negative. Assume that this is s_1 and let the order of the exponents be such that $s_1 < s_2 < s_3 < s_5$. Then $ka_2^2b/\Lambda \sim t^{2s_1+s_2}$, and it is easy to verify that this term increases as $t \to 0$, since $s_2 + 2s_1$ is negative.

$$-ds_{(5)}^2 = -dt^2 + (a^2 l_a' l_b' + b^2 m_a' m_b' + c^2 n_a' n_b' + d^2 q_a' q_b') dx^a dx^b. \quad (4.26)$$

A simple analysis similar to that in[3] shows that the rotation of the axes is described by the formulas

$$l_a' = l_a, \quad m_a' = m_a + \sigma_1 l_a, \\ n_a' = n_a + \sigma_2 m_a + \sigma_3 l_a, \quad q_a' = q_a + \sigma_4 m_a + \sigma_5 l_a, \quad (4.27)$$

where all the coefficients σ are of the order of unity and can be found exactly by determining the corrections that must be introduced in (4.7) when account is taken in (4.8) of the off-diagonal projections of the tensor Π_a^b. We shall not make these calculations here.

If we wish, we can return to the usual four-dimensional form of the theory, described by Eqs. (3.23)–(3.27). If we are interested in the evolution of the scalar, vector, and tensor potentials φ, A_i, and g_{ik}, then it suffices to return to formulas (3.21) and (3.22), from which it follows that

$$\varphi^2 = a^2 l_5^2 + b^2 m_5^2 + c^2 n_5^2 + d^2 q_5^2; \\ A_\alpha = (a^2 l_5 l_\alpha + b^2 m_5 m_\alpha + c^2 n_5 n_\alpha + d^2 q_5 q_\alpha) / \varphi^2, \quad A_4 = 0; \\ g_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta + d^2 q_\alpha q_\beta - \varphi^2 A_\alpha A_\beta, \\ g_{44} = -1, \quad g_{4\alpha} = 0. \quad (4.28)$$

In conclusion we point out that during the course of the described oscillatory regime the negative Kasner exponent turns out to be connected an infinite number of times with each of the functions a, b, c, and d. That particular Kasner epoch during which the negative exponent belongs to the function d (d increases) will obviously over go over into an epoch during which a positive exponent will be connected with the function d (d decreases), and the reason for this transition is the fact that the expressions

$$(q_{a,b} - q_{b,a}) l^a m^b, \quad (q_{a,b} - q_{b,a}) l^a n^b, \quad (q_{a,b} - q_{b,a}) m^a n^b, \quad (4.29)$$

i.e., expressions of the type λ (4.15), which play a role in the onset of the oscillations, differ from zero. It is therefore easily seen that in the absence of the vector field A_i we have

$$l_5 = m_5 = n_5 = 0, \quad q_\alpha = 0 \quad (4.30)$$

And all the quantities (4.29) vanish identically. Under these conditions, that open during which d increases will be the final stage of the evolution of the solution. Such an evolution to a monotonic asymptote near the singular point in the absence of a vector field thus proves the statement made at the end of Sec. 2, that the asymptotic solution of (2.4) and (2.5) always takes the form (2.14) with positive exponents p_1, p_2, and p_3.

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Translated by J. G. Adashko
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