ION ACCELERATION UPON EXPANSION OF A RAREFIED PLASMA

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Acceleration of impurity ions of various masses and charges upon expansion of a rarefied plasma into vacuum is investigated. Acceleration of the ions is produced by a self-consistent electric field arising during expansion of the plasma. It is shown that as a result of acceleration a considerable part of the impurity ions may acquire energies of the order of $10^2-10^3\ T_e$, where T_e is the plasma electron temperature. Excitation of self-similar and ion-sound waves is investigated. Results of numerical calculations are presented.

1. INTRODUCTION

WHEN a rarefied (collisionless) plasma expands in vacuum, the electrons strive to overtake the ions. This results in an uncompensated space charge and in an electric field that decelerates the electrons and accelerates the ions. The plasma expansion process was considered by us earlier[1], and we have shown that it is scale-invariant (self-similar) and is described by a nonlinear equation for the ion distribution function:

$$(u_1-\tau)\frac{\partial g_1}{\partial \tau}-\frac{1}{2}\frac{\partial g_1}{\partial u_1}\frac{d\psi}{d\tau}=0, \tag{1}$$

$$\psi(\tau)=\ln(Z_1n_1)=\ln\left(Z_1\pi^{-1/2}\int g_1du_1\right). \tag{2}$$

Here $u_1 = v_1(2T_eZ_1/M_1)^{-1/2}$ is the dimensionless velocity, $(v_1 = v_x)$, $\tau = x(2T_eZ_1/M_1)^{-1/2}/t$, T_e is the electron temperature, M_i is the ion mass, eZ_1 is the ion charge, and x is the direction in which the plasma expands. Further, $g_1 = \pi^{1/2}F(x, v_x, t)$, where F is the distribution function of the ions with respect to the velocity v_x. By virtue of the scale invariance of the problem, the function g depends only on the ratio $x/t \sim \tau$. In the derivation of (1) and (2) it is also assumed that the characteristic dimension of the inhomogeneity is much larger than the Debye radius in the plasma,

$$R_0 \gg D, \tag{3}$$

and the ion velocity considered are small in comparison with the electron thermal velocity:

$$|u| \ll (M/m)^{1/2}. \tag{4}$$

When these conditions are satisfied, the potential φ of the electric field in the plasma is connected with the ion concentration by the simple relation (2); here

$$\psi = e\varphi/Z_1T_e. \tag{5}$$

The boundary conditions of (1) and (2), corresponding to expansion in vacuum of a half-space filled with a plasma, are

$$\begin{aligned} &g_1\to\exp(-\beta_1u_{10}^2) &&\text{as } \tau\to-\infty,\\ &g_1\to 0 &&\text{as } \tau\to+\infty, \end{aligned} \tag{6}$$

where $\beta_1 = T_e/T_{i_1}$, and T_{i_1} is the ion temperature in the unperturbed plasma $(\tau \to -\infty)$.

The previously obtained[1,2] solution of (1) and (2) shows that part of the ions in the strongly rarefied region is noticeably accelerated by the action of the electric field. It is obvious, however, that the acceleration of the ions by the electric field depends strongly on their mass and charge. One can expect, for example, multiply charged impurity ions to acquire the highest energy when the plasma expands. The present paper is devoted to an analysis of the problem of impurity-ion acceleration in a plasma.

2. ACCELERATION OF IMPURITY IONS

We assume that in addition to the main ions, of mass M_1 and charge Z_1, the plasma contains also a small admixture of ions of mass M_2 and charge Z_2. The impurity-ion distribution function $g_2(u_2, \tau)$ is described by the equation

$$(u_2-\tau)\frac{\partial g_2}{\partial \tau}-\frac{Z_2M_1}{2Z_1M_2}\frac{\partial g_2}{\partial u_2}\frac{d\psi}{d\tau}=0. \tag{7}$$

Here $u_2 = v_2(2T_eZ_1/M_1)^{-1/2}$, and $v_2 = v_{2x}$ is the velocity of the impurity ions. The dimensionless potential $\psi(\tau)$ of the electric field is determined by relations (1) and (2). The boundary conditions for Eq. (7), in the case of expansion of a half-space in vacuum, are

$$\begin{aligned} &g_2\to\alpha\exp(-\beta_2u_{20}^2M_2/M_1)\\ &\text{as } \tau\to-\infty, \qquad g_2\to 0 \quad \text{as } \tau\to\infty. \end{aligned} \tag{8}$$

Here $\beta_2 = T_e/T_{i_2}$, and α is a normalization constant proportional to the concentration of the impurity ions.

Equation (7) is linear. Its solution can be easily obtained by integrating the equation of the characteristics $u_2(\tau)$

$$u_2(\tau)=u_{20}-\frac{Z_2M_1}{2Z_1M_2}\int_{-\infty}^{\tau}\frac{d\psi/d\tau}{u_2-\tau}d\tau. \tag{9}$$

The values of the distribution function g_2 are conserved on the characteristics; they are determined by formula (8). Equations (9) were integrated numerically for 300 characteristics. The characteristics in the (u, τ) plane for $M_1Z_2/M_2Z_1 = 16$ are shown by way of example in Fig. 1. The same figure shows the characteristics of (1). We see that the impurity ions are much more energetically accelerated than the ions of the main gas. The distribution functions of the impurity ions are shown in Fig. 2, while Fig. 3 shows their concentration N_2, the flux $j_2 = N_2\bar{u}_2$, and the mean energy $\epsilon = T_eZ_1\overline{u_2^2}M_2/M_1$. The impurity-ion concentration decreases quite slowly

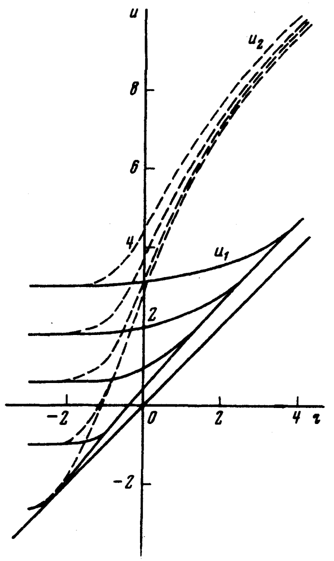

FIG. 1

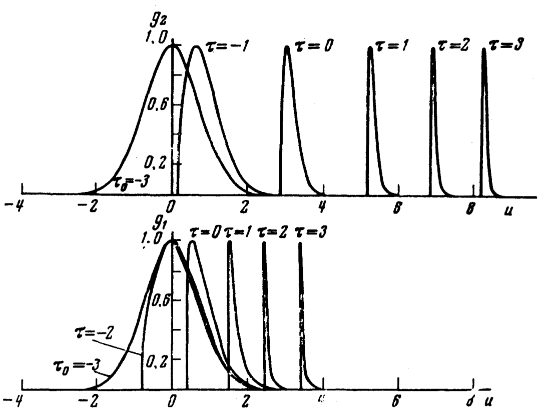

FIG. 2

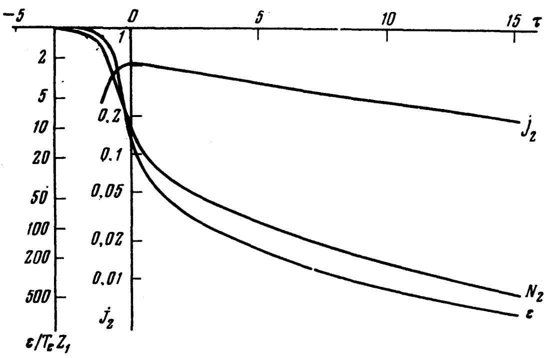

FIG. 3

with increasing τ, while the changes of the flux j_2 are very small. This means that the greater part of the flux of impurity ions is captured by the field and is accelerated to high energies. For example, the flux at $\epsilon \sim 500\, T_e Z_1$ is $j_2 \sim 0.4 j_{20}$, where $j_{20} = j_2\ (\tau = 0)$.

At large values of τ, the translational velocity of the ions is high. The thermal spread of the velocities is then of little importance, since it is convenient to analyze the behavior of the solution by using the hydrodynamic equations. Assuming in (7) a distribution function in the form

$$g_2 = N_2(\tau)\delta(u_2 - u_2(\tau))\pi^{1/2},$$

we obtain

$$(u_2 - \tau)\frac{dN_2}{d\tau} + N_2\frac{du_2}{d\tau} = 0, \tag{10}$$

$$(u_2 - \tau)\frac{du_2}{d\tau} + \frac{Z_2 M_1}{2Z_1 M_2}\frac{d\psi}{d\tau} = 0. \tag{11}$$

We recognize, in addition, that in the hydrodynamic approximation

$$\frac{d\psi}{d\tau} = \begin{cases} 0 & \tau < -2^{-1/2} \\ -2^{1/2}, & \tau \geqslant -2^{-1/2} \end{cases} \tag{12}$$

It follows therefore that the plasma is unperturbed at $\tau < -2^{-1/2}$. The boundary conditions for (10) and (11) are therefore specified at $\tau = -2^{-1/2}$:

$$N(\tau = -2^{-1/2}) = N_{20}, \quad u_2(\tau = -2^{-1/2}) = 0. \tag{13}$$

Integrating (11) and (12), we have

$$u_2 = \tau + z - \alpha_1(\tau), \quad z = Z_2 M_1 / 2^{1/2} Z_1 M_2, \tag{14}$$

where $\alpha_1(\tau)$ is defined by

$$\alpha_1 - z \ln \alpha_1 + C = \tau. \tag{15}$$

Using the boundary condition (13), we determine the constant C:

$$C = z \ln (z - 2^{-1/2}) - z. \tag{16}$$

It is assumed here that $z > 2^{-1/2}$, i.e., $Z_2M_1/Z_1M_2 > 1$. Substituting equations (14) and (15) for $u_2(\tau)$ in (10) and integrating the latter, we get

$$N_2(\tau) = \frac{\alpha_1(\tau) N_{20}}{(2^{1/2}z - 1)(z - \alpha_1(\tau))}. \tag{17}$$

The same expressions hold also when $z < 2^{-1/2}$, but in this case $\alpha_1 < 0$ and therefore $\ln \alpha_1$ and $\ln(z - 2^{-1/2})$ are replaced by $\ln(-\alpha_1)$ and $\ln(2^{-1/2} - z)$, respectively. A plot of $\alpha_1(\tau, z)$ determined from formulas (15) and (16) is shown in Fig. 4. Here

$$x = \frac{\tau}{z} + 1 + \ln\left(\frac{z}{z - 2^{-1/2}}\right).$$

As $x \to 1$ we have

$$\alpha_1 / z = 1 - [2(x-1)]^{1/2},$$

and at $x \gg 1$,

$$\alpha_1 / z = (e^x - 1)^{-1}.$$

Knowing α_1, we can obtain from formulas (14) and (17) the average velocity, the concentration, the flux $j_2 = N_2 u_2$, and the average energy $\epsilon = M_2 Z_1 T_e u_2^2 / M_1$ of the impurity ions. At large values of τ we have $\alpha_1 \to 0$. It then follows from (17) that

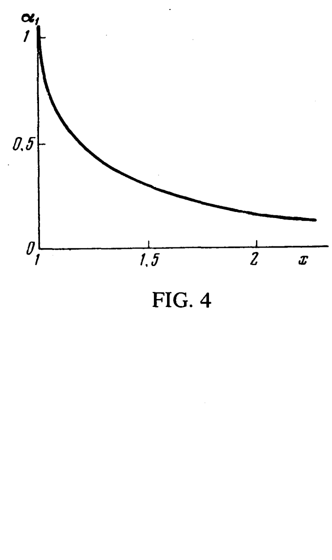

FIG. 4

$$N_2(\tau) = 2^{1/2} e^{-1} z^{-1} e^{-\tau/z} N_{20},$$
$$u_2 = \tau + z, \quad \varepsilon = M_2 Z_1 T_e (\tau + z)^2 / M_1. \tag{18}$$

It is easy to obtain the energy distribution of the accelerated impurity ions. Indeed, let n be the total number of impurity ions passing through a unit surface at the point x_0, namely, $n = \int j_2 dt$. Then

$$\frac{dn}{d\varepsilon} = j_2 \frac{dt}{d\varepsilon} = N_2 u_2 \left(\frac{2Z_1 T_e}{M_1}\right)^{1/2} \left(\frac{d\varepsilon}{dt}\right)^{-1}.$$

Recognizing that

$$\varepsilon = \frac{M_2 Z_1}{M_1} T_e u_2^2(\tau), \quad \frac{d\varepsilon}{dt} = 2\frac{M_2 Z_1}{M_1} T_e u_2 \frac{du_2}{d\tau}\frac{d\tau}{dt},$$

and expressing N_2 and τ in terms of ϵ with the aid of formulas (14)—(17), we find at $\epsilon \geq \epsilon_k$

$$\frac{dn}{d\varepsilon} = \frac{x_0 N_{20}}{T_e} \frac{Z_1^2 M_2^2}{2Z_2^3 M_1^2} \exp\left[-\left(\frac{\varepsilon}{\rho T_e}\right)^{1/2}\right] \cdot \times \left\{\left(\frac{\varepsilon}{\rho T_e}\right)^{1/2} - 1 + \left(1 - \frac{Z_1 M_2}{Z_2 M_1}\right) \exp\left[-\left(\frac{\varepsilon}{\rho T_e}\right)^{1/2}\right]\right\}^{-2}, \tag{19a}$$

and at $\epsilon \leq \epsilon_k$

$$dn / d\varepsilon = 0. \tag{19b}$$

Here ρ and ϵ_k are defined by the relations

$$\rho = M_1 Z_2^2 / 2M_2 Z_1,$$
$$\left(\frac{\varepsilon_k}{\rho T_e}\right)^{1/2} - 1 + \left(1 - \frac{Z_1 M_2}{Z_2 M_1}\right) \exp\left[-\left(\frac{\varepsilon_k}{\rho T_e}\right)^{1/2}\right] = 0,$$
$$Z_1 M_2 / Z_2 M_1 \leqslant 1.$$

At $\epsilon = \epsilon_k$ the density $dn/d\epsilon$ becomes infinite. This is not surprising, since the energy $\epsilon = \epsilon_k$ corresponds to $\tau = 0$, i.e., particles with such an energy appear at the point x_0 only if $t \to \infty$. For any finite value of t we always have $\epsilon > \epsilon_k$.

It is seen from (19) that the energy distribution of the accelerated impurity ions does not depend on the observation point x_0. At sufficiently high energies, $\epsilon/\rho T_e \gg 1$, it turns out to be, furthermore, similar for ions with different charges and masses:

$$\frac{dn}{d\varepsilon} = \frac{x_0 N_{20}}{T_e} \frac{Z_1^2 M_2^2}{2Z_2^3 M_1^2} \exp\left[-\left(\frac{\varepsilon}{\rho T_e}\right)^{1/2}\right]\left[\left(\frac{\varepsilon}{\rho T_e}\right)^{1/2} - 1\right]^{-2}.$$

The scaling parameter is $\rho = M_1 Z_2^2/2M_2 Z_1$, where M_2 and Z_2 are the mass and charge of the impurity ions, and M_1 and Z_1 are the mass and charge of the main ions of the plasma. If $\rho \gg 1$, then the impurity ions are accelerated much more energetically than the main ions. It is also important that the $dn/d\epsilon$ distribution decreases relatively slowly with increasing particle energy ϵ. The total number of ions $n(\epsilon)$ that acquire an energy higher than ϵ_0 during the acceleration, is given at $\epsilon_0 \gg \rho T_e$ by the expression

$$n_{\varepsilon > \varepsilon_0} = 2N_{20} x_0 \frac{M_2 Z_1}{M_1 Z_2} \left(\frac{\varepsilon_0}{\rho T_e}\right)^{-1/2} \exp\left[-\left(\frac{\varepsilon_0}{\rho T_e}\right)^{1/2}\right]. \tag{20}$$

It follows therefore, for example, that 0.1% of the total number of impurity ions $N_{20}x_0$ acquires an energy $\epsilon \gtrsim 50\,\rho T_e$.

Acceleration of multiply-charged ions in expansion of a rarefied plasma was observed by Bykovskiĭ et al.[3] Theoretical estimates based on formula (2) agree with the results of these experiments[1)].

[1)] Acceleration of the ions of the main plasma was observed by Plyutto et al. [4]

We note that the ion-acceleration mechanism considered here can be realized under cosmic conditions in flares on the sun, and in flares and explosions of stars[5]. The main component of the plasma is in this case ionized hydrogen ($M_1 = 1$, $Z_1 = 1$). The acceleration of the heavy nuclei is determined by the scaling factor $\rho = Z_2^2/2M$. At sufficiently large Z_2 we have $\rho \gg 1$. For example, $\rho = 4$ for fully ionized oxygen nuclei, $\rho = 6$ for Fe nuclei, and $\sigma = 16.3$ for Pb^{82} nuclei. It follows from (20) that in free expansion of a hydrogen plasma 0.1% of the total number of oxygen nuclei acquires an energy $\epsilon > 200 T_e$, 0.1% of the Fe nuclei an energy $\epsilon > 300\, T_e$, and 0.1% of the Pb nuclei an energy $\epsilon > 800\, T_e$.

It is also seen from (19) and (20) that it is the light impurity ions that are predominantly accelerated in a singly-ionized plasma. We emphasize that the acceleration of the impurity ions is determined by the temperature of the plasma electrons. The ion temperature at $T_e \geq T_i$ does not play a significant role. Therefore, by rapidly heating the electrons in the freely expanding plasma (with the aid of electron beams or radiation), it is possible to accelerate appreciably the multiply-charged impurity ions (up to $\epsilon \sim 10^2-10^3\, T_e$). We note also that the acceleration depends on the mass M_1 of the main plasma ions. For example, it is higher in deuterium and tritium than in hydrogen.

Concluding this section, we note the limitations of the considered acceleration mechanism. The maximum ion energy is limited primarily by condition (4), i.e.,

$$\varepsilon \ll MT_e / m. \tag{21}$$

We see that the maximum energy depends only on the mass, but not on the charge of the impurity ions. Condition (3) also limits the maximum value of ϵ. Indeed, since $D \sim N_1^{-1/2}$ and the concentration of the main ions N_1 is given by formula (18) at $z = 2^{-1/2}$, we obtain from (3)

$$\varepsilon < T_e \frac{M_2 Z_1}{M}\left[\tau_m + \frac{Z_2 M_1}{2^{1/2} Z_1 M_2}\right]^2, \tag{22}$$

where $\tau_m \sim 2^{1/2} \ln(R_0/D_0)$, R_0 is the characteristic dimension, and D_0 is the Debye radius in the unperturbed plasma. Relation (22) is connected with the limitation on the region into which the plasma flows. We note that a rigorous analysis of the expansion of a plasma in a bounded region shows that the ion distribution is close to self-similar, up to a value

$$\tau = \tau_m \approx 2^{1/2} \ln\frac{R_0}{D_0} - \frac{3}{2^{3/2}} \ln\left[\frac{e\varphi_1}{T} + 2\ln\frac{R_0}{D_0}\right].$$

The distribution then terminates sharply, and there are no ions[2)] at $\tau > \tau_m$ (φ_1 is the value of the potential on the boundary, $e\varphi_1 > T_e[2\ln(R_0/D_0)]^{4/3}$).

One more limitation is connected with the amount of impurity ions. If $\rho \gg 1$, when the impurity ions are ac-

[2)] This agrees with the results of a numerical calculation [6]. We note that the maximum velocity of the main ions, $u_m = \tau_m$, increases with increasing ratio R_0/D_0. In the calculation of Widner et al. [6] this ratio was assumed to be small ($R_0/D_0 \sim 10^2$). This was apparently the cause of the relatively low value of the maximum ion velocity. Under real conditions R_0/D_0 can equal 10^5-10^7. In the foregoing estimates of the impurity-ion acceleration it was assumed that $\tau_m = 5$; this holds true if $R_0/D_0 \gtrsim 10$.

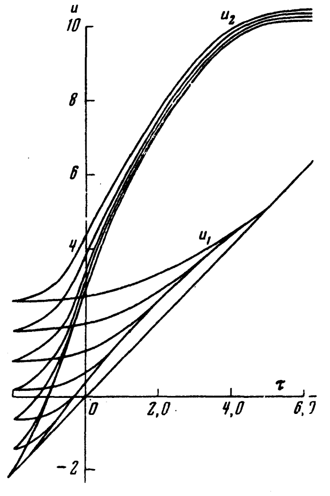

FIG. 5

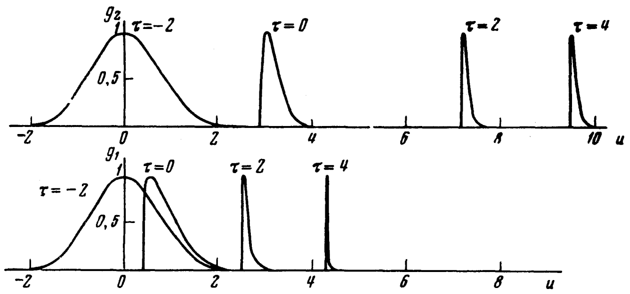

FIG. 6

celerated most energetically, their concentration N_2 decreases with increasing τ much more slowly than the concentration of the main ions N_1. At a certain value of τ, the concentrations N_2 and N_1 become comparable. Then the entire plasma motion is significantly altered by the impurity ions. These phenomena will be analyzed in the next section.

3. EXPANSION OF AN IMPURITY-ION-CONTAINING PLASMA IN VACUUM

It was assumed above that the concentration of the impurity ions is low in comparison with the concentration of the main ions of the plasma, so that the influence of the impurity ions on the motion of the main ions was neglected. We assume now that these concentrations are comparable, i.e., the plasma contains a mixture of ions of two kinds. The expansion, in vacuum, of a plasma containing a mixture of ions with M_1, Z_1 and M_2, Z_2 is described as before by the kinetic equations (1) and (7) with boundary conditions (6) and (8). All that changes is the expression for the dimensionless potential

$$\psi(\tau)=\ln(Z_1N_{i1}+Z_2N_{i2})=\ln\Big[Z_1\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}g_1du_1 + Z_2\left(\frac{M_2}{M_1\pi}\right)^{1/2}\int_{-\infty}^{\infty}g_2du_2\Big]. \quad (23)$$

This formula follows from the quasineutrality condition $N_e = Z_1N_{i1} + Z_2N_{i2}$. If the impurity-ion concentration N_{i2} is negligibly small, then (23) goes over into (2). Owing to the dependence of ψ on N_{i1} and N_{i2}, Eqs. (1) and (7) are coupled.

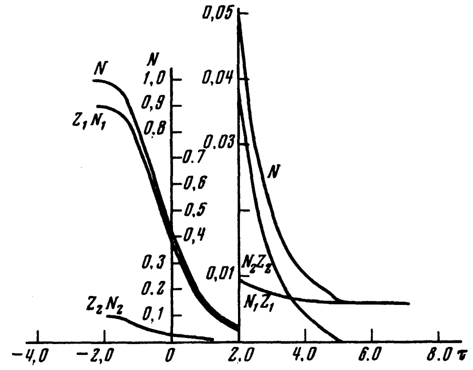

FIG. 7

The result of the numerical integration of (1), (7), and (23) is shown in Figs. 5–7. Figure 5 gives the characteristics of Eqs. (1) and (7) in the (u, τ) plane for $Z_2/Z_1 = 16$ and $M_1 = M_2$. Figures 6 and 7 show the distribution functions and the concentrations $N_1(\tau)$ and $N_2(\tau)$. Figures 5–7 represent in fact the same case as Figs. 1–3 in the preceding section, except that in Sec. 1 it was assumed that $N_2 \ll N_1$ and here we assume a finite value of N_2. We assume also that in the unperturbed plasma $N_{20}Z_2/N_{10}Z_1 = 0.1$. Comparing Figs. 1–3 with Figs. 5–7 we see that in the region $\tau \gtrsim 3.5$, where N_2Z_2 becomes comparable with N_1Z_1 and then exceeds it, the characteristics and the distributions $N(\tau)$ differ significantly in the different cases. We see first that the concentration $N_1(\tau)$ of the main ions decreases sharply, practically to zero, at values of τ exceeding a certain τ_k. The concentration of the impurity ions, to the contrary decreases slowly at $\tau > \tau_k$, and the function $N_2(\tau)$ even has an appreciable plateau at $\tau \gtrsim 5$. At the same values of τ, all the characteristics of the impurity ions also flatten out (see Fig. 5). The distribution function hardly varies with changing τ in the plateau region. A characteristic plateau region occurs also in other cases in which a plasma containing a mixture of ions is expanded in vacuum.

To analyze the character of these features and their dependence on the parameters it is natural to use, as before, the hydrodynamic equations, which in the case of a mixture of two sorts of ions have the following form[7,8]:

$$(u_1-\tau)\frac{dN_1}{d\tau}+N_1\frac{du_1}{d\tau}=0,$$

$$(u_1-\tau)\frac{du_1}{d\tau}+\frac{Z_1dN_1/d\tau+Z_2dN_2/d\tau}{2(Z_1N_1+Z_2N_2)}=0. \quad (24)$$

$$(u_2-\tau)\frac{dN_2}{d\tau}+N_2\frac{du_2}{d\tau}=0,$$

$$(u_2-\tau)\frac{du_2}{d\tau}+\frac{M_1Z_2}{2M_2Z_1(Z_1N_1+Z_2N_2)}\left(Z_1\frac{dN_1}{d\tau}+Z_2\frac{dN_2}{d\tau}\right)=0. \quad (25)$$

Here N_1 and N_2 are the concentrations and u_1 and u_2 the mean velocities of the ions. The equations (24) and (25) possess an integral

$$Z_1N_1(u_2-\tau)^2\left[(u_1-\tau)^2-\frac{1}{2}\right]+Z_2N_2(u_1-\tau)^2\left[(u_2-\tau)^2-\frac{M_1Z_2}{2M_2Z_1}\right]=0. \quad (26)$$

We consider the case of low impurity-ion concentration in the unperturbed plasma: $Z_2N_{20} \ll Z_1N_{10}$. In the first approximation, the concentration N_2 can be neglected in the expression for the force, and the solution of (24) and (25) coincides with that considered in the preceding section. Then, as is clear from (18), if $M_1Z_2/M_2Z_1 < 1$, then the concentration of the impurity ion decreases more rapidly than that of the main ions with increasing τ, so that the condition $Z_2N_2 \ll Z_1N_1$ is always satisfied. If, to the contrary,

$$M_1Z_2/M_2Z_1 > 1, \tag{27}$$

then the concentration of the impurity ions decreases with increasing τ more slowly than that of the main ions. Their relative concentration then increases, and at a certain value of τ the quantity Z_2N_2 becomes comparable with Z_1N_1. The character of the motion then changes strongly. To examine the solution in the region where $Z_2N_2 \sim Z_1N_1$, we use the integral (26). We recognize that, according to (18),

$$u_2 - \tau = 2^{-1/2}M_1Z_2/M_2Z_1.$$

It follows then from (26) that

$$(u_1-\tau)^2 = \frac{Z_1N_1}{2[Z_2N_2(1-M_2Z_1/M_1Z_2)+Z_1N_1]}. \tag{28}$$

Substituting this expression in the first equation of (24), we get

$$\frac{dn_1}{d\tau}\frac{n_1+2n_2[1-(2^{1/2}z)^{-1}]}{(2n_1)^{1/2}[n_1+n_2(1-2^{-1/2}z^{-1})]^{3/2}}+\frac{[1-(2^{1/2}z)^{-1}]n_1^{1/2}\,dn_2/d\tau}{2^{3/2}[n_1+n_2(1-2^{-1/2}z^{-1})]^{3/2}}=-1,$$

$$n_1 = Z_1N_1, \quad n_2 = Z_2N_2, \quad z = M_1Z_2/2^{1/2}M_2Z_1. \tag{29}$$

If the concentration of the impurity ions is negligibly small ($n_2 \to 0$), then the solution of (29) coincides with that obtained earlier[1]:

$$n_1 = C_1\exp(-2^{1/2}\tau), \quad C_1 = Z_1N_{10}e^{-1}. \tag{30}$$

We recognize that, according to (18) at $z \gg 1$ the concentration n_2 changes much more slowly than n_1. Neglecting therefore the variation of n_2 in (29) and integrating the latter, we obtain

$$\begin{aligned}2\ln[n_1^{1/2}+(n_1+n_2(1-2^{-1/2}z^{-1}))^{1/2}]+2\left(\frac{n_1}{n_1+n_2(1-2^{-1/2}z^{-1})}\right)^{1/2}\\ =-\tau+C,\\ C=2\ln[(N_{10}Z_1)^{1/2}+(N_{10}Z_1+N_{20}Z_2(1-2^{-1/2}z^{-1}))^{1/2}]\\ +2\left(\frac{N_{10}Z_1}{N_{10}Z_1+N_{20}Z_2(1-2^{-1/2}z^{-1})}\right)^{1/2}-2^{-1/2}.\end{aligned} \tag{31}$$

At $n_2 \ll n_1$ the solution of (31) coincides with (30). In the region of τ where $n_1 \lesssim n_2(1-2^{-1/2}z^{-1})$, the course of the solution becomes distorted, and the concentration begins to decrease much more rapidly.

At $\tau = \tau_k$, where

$$\begin{aligned}\tau_k = C-\ln[n_2(1-2^{-1/2}z^{-1})] = 2\left(\frac{N_{10}Z_1}{N_{10}Z_1+Z_2N_{20}(1-2^{-1/2}z^{-1})}\right)^{1/2}\\ -2^{-1/2}\ln\frac{(N_{10}Z_1)^{1/2}+(N_{10}Z_1+N_{20}Z_2(1-2^{-1/2}z^{-1}))^{1/2}}{(N_{20}Z_2(1-2^{-1/2}z^{-1}))^{1/2}},\end{aligned} \tag{32}$$

the concentration $n_1(\tau)$ vanishes. In other words, at $\tau > \tau_k$ there are no ions of the main gas ($N_1 \equiv 0$). Equations (25) for the impurity ions assume at $\tau > \tau_k$ the form

$$\begin{aligned}(u_2-\tau)\frac{dN_2}{d\tau}+N_2\frac{du_2}{d\tau}=0,\\ (u_2-\tau)\frac{du_2}{d\tau}+\frac{M_1Z_2}{2M_2Z_1N_2}\frac{dN_2}{d\tau}=0.\end{aligned} \tag{33}$$

According to (18), the boundary conditions at $\tau = \tau_k$ are

$$\begin{aligned}N_2 = N_{2k} = N_{20}2^{1/2}z^{-1}\exp(-[1+\tau_k/z]),\\ u_2 = u_{2k} = \tau_k + z.\end{aligned} \tag{34}$$

The solution of (33) with the boundary condition (34) is

$$\begin{aligned}N_2 = N_{2k}, \quad u_2 = u_{2k} \quad \text{as} \quad \tau_k \leqslant \tau \leqslant \tau_{k1};\\ N_2 = N_{2k}\exp[-2^{1/4}z^{-1}(\tau-\tau_{k1})], \quad u = \tau + 2^{-1/4}z^{1/2}\\ \text{as} \quad \tau \geqslant \tau_{k1},\end{aligned} \tag{35}$$

where

$$\tau_{k1} = \tau_k + z(1-2^{-1/4}z^{-1/2}).$$

We see therefore that at $\tau_k < \tau < \tau_{k1}$ a plateau region is indeed produced. The width of this region is

$$\Delta\tau = z(1-2^{-1/4}z^{-1/2}) = \frac{Z_2M_1}{2^{1/2}Z_1M_2}\left(1-\left(\frac{M_2Z_1}{M_1Z_2}\right)^{1/2}\right). \tag{36}$$

By virtue of condition (27) we always have $\Delta\tau > 0$; at $z \gg 1$, the width of the plateau region $\Delta\tau$ is much larger than unity. The concentration and velocity of the particles are constant in the plateau region, and there is no electric field. At the points τ_k and τ_{k1}, the obtained solution has weak discontinuities (discontinuities of the derivatives $dN/d\tau$ and $du/d\tau$)[2].

The presented solution of the problem of expansion, in vacuum of a plasma containing an ion mixture can be used, in particular, in the analysis of the structure of the perturbed zone in the vicinity of bodies moving in the ionosphere, such as rockets and satellites. The point is that at altitudes h ~ 500–1200 km the ionosphere plasma contains a mixture of ions, mainly of atomic oxygen ($M_2 = 16$) and hydrogen ($M_1 = 1$). The relative hydrogen content $n_{H^+} = N_{H^+}/(N_{H^+} + N_{O^+})$ ranges from 1–2% at h ~ 500 km to 100% at h ~ 1200–1500 km. As shown earlier,[8] the structure of the perturbed zone near the moving body is determined to a considerable degree by the self-similar solution considered here. A plot of $N(\tau) = N_{H^+} + N_{O^+}$ for different values of n_{H^+}, obtained by numerically integrating Eqs. (1), (7), and (23) for $Z_1 = Z_2 = 1$, $M_2 = 16$, $M_1 = 1$, $\beta = T_e/T_i = 1$, is shown in Fig. 8.

Using the formulas obtained in[8] we can now compare the results of the calculations and measurements in the ionosphere. Figure 9 shows by way of example the variation of N behind the body ($\theta = 180°$) as a function of the relative hydrogen content n_{H^+}. The points on the figure are the results of measurements made in the ionosphere by Samir and Wrenn[9] with the satellite Explorer-31, and the solid curve is the result of the calculation. A detailed discussion of the results and a comparison with theory is contained in[8]. We note that in that reference they used an approximate expression for the summary ion concentration:

$$N(\tau) = \frac{1}{2}N_{O^+}\left[1+\Phi\left(\left(\frac{M_2}{M_1}\right)^{1/2}\tau\right)\right]+\frac{1}{2}N_{H^+}[1+\Phi(\tau)], \tag{37}$$

in which the influence of the electric field on the ion motion was neglected. The result of the calculation with the approximate formula (37) is shown dashed in Fig. 9. We note that the hydrogen ions exert a definite influence on the structure of the perturbed zone behind the moving body, near its surface, even if their relative concentration in the plasma is very small. They are appreciably accelerated by the electric field: the average energy of the H^+ ions on the boundary of the quasineutral zone be-

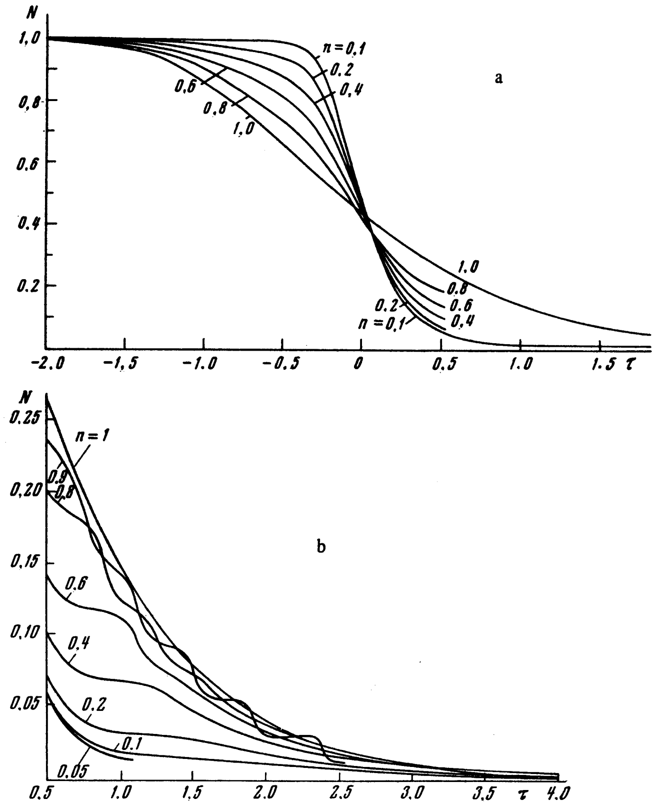

FIG. 8

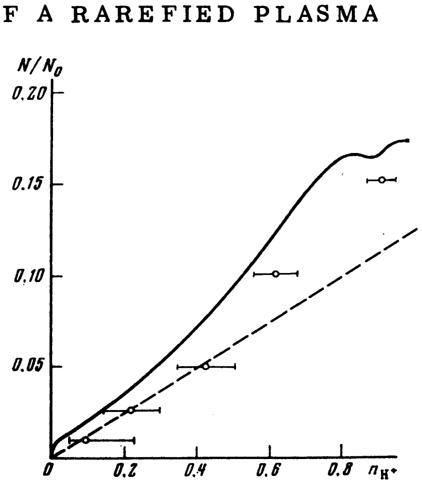

FIG. 9

hind the body, at $n_{H^+} \leq 0.1$, is $(5-8)T_e$, i.e., of the order of 1–1.5 eV. The hydrogen-ion acceleration becomes stronger when their relative concentration is decreased.

4. SCALE-INVARIANT WAVES. ION-SOUND INSTABILITY

Figure 8b shows the concentration $N(\tau)$ for an expanding plasma containing a mixture of oxygen and hydrogen ions. It is seen from the figure that behind the plateau region the plot of N against τ shows characteristic oscillations. These oscillations become more clearly pronounced if one considers the dimensionless force $F(\tau) = \frac{1}{2}N^{-1}dN/d\tau$ (see Fig. 10, the force $F(\tau)$ is proportional to the electric field intensity $E = F(2T_eM)^{1/2}/et$). The amplitude of the oscillations, as seen from Fig. 10, increases with increasing initial concentration of the heavy oxygen ions ($N_{O^+} = 1 - N_{H^+}$). At the start of the plateau region, the heavy ions vanish almost completely. Consequently, the oscillations in question propagate already in a pure hydrogen plasma. The onset of these oscillations is the result of excitation of unique scale-invariant waves in the expanding plasma.

Let us consider a weak perturbation of the scale-invariant distribution function

$$f = g_a(u, \tau) + \delta g(u, \tau), \quad \delta g \ll g_a. \tag{38}$$

Substituting (38) in (1) and (2) and linearizing the latter, we arrive at the following equation for δg:

$$(u-\tau)\frac{\partial \delta g}{\partial \tau} - \frac{1}{2N_a}\frac{dN_a}{d\tau}\frac{\partial \delta g}{\partial u} - \frac{1}{2N_a}\frac{\partial g_a}{\partial u}\left(\frac{d\delta N}{d\tau} - \frac{1}{N_a}\frac{dN_a}{d\tau}\delta N\right) = 0, \tag{39}$$
$$\delta N = \frac{1}{\pi^{1/2}}\int \delta g\, du.$$

If the dimension of the perturbations in (u, τ) space is small in comparison with the characteristic dimension of the variation of the main quantities $N_a(\tau)$ and $g_a(u, \tau)$, then the solution of (39) can be sought in the quasiclassical approximation by expanding in a Fourier integral

$$\delta g = \delta g_{pq}\frac{\partial g_a}{\partial u}\exp\left[i\left(\int p\, d\tau - qu\right)\right],$$
$$\delta N = \delta N_p \exp\left(i\int p\, d\tau\right), \tag{40}$$
$$\delta N_p = \delta g_{pq}\frac{1}{\pi^{1/2}}\int \frac{\partial g_a}{\partial u}e^{-iqu}\, du = \delta g_{pq} N_a \eta(q).$$

Substituting (40) in (39) we obtain a dispersion equation that defines the parameter p:

$$p = -\frac{dN_a/d\tau}{N_a[2(u-\tau) - \eta(q)e^{iqu}]}\left\{q + i\left[\frac{\partial}{\partial u}\ln\frac{\partial g_a}{\partial u} - \eta(q)\right]\right\} \tag{41}$$
$$+ i\frac{\partial}{\partial \tau}\ln\frac{\partial g_a}{\partial u}, \quad \eta(q) = \frac{\pi^{-1/2}}{N_a}\int_{-\infty}^{\infty}\frac{\partial g_a}{\partial u}e^{-iqu}\, du.$$

We see therefore that at sufficiently large values of q the real part of p becomes predominant, meaning the presence of oscillatory solutions. Formula (41) is the dispersion relation for the scale-invariant waves.

It is quite important that there are no scale-invariant waves in hydrodynamics. This can be easily verified by considering small perturbations of the hydrodynamic equations (33). These are specifically kinetic waves, due to the presence of a particle-velocity distribution, as can be seen also directly from the relations (40) and (41). The profile of such a wave depends on the ratio x/t, i.e., different points of the wave move with different velocities. The wavelength increases with time, and the frequency decreases. It is easy to understand the mechanism whereby scale-invariant waves are excited in the case of expansion, in vacuum of a plasma containing an admixture of heavy ions ($n_{O^+} = 1 - n_{H^+} \ll 1$). Indeed, the heavy ions have a small thermal velocity spread; they vanish rapidly at values $\tau \sim 0$. Therefore the change of $N(\tau)$ at $\tau \sim 0$ becomes noticeably accelerated, and an additional force, $F_{O^+} \sim dN_{O^+}/d\tau$, appears and perturbs the distribution of the hydrogen ions. It is this perturbation which propagates further in the form of scale-invariant waves traveling in an expanding hydrogen plasma.

We have confined ourselves above to an analysis of self-similar perturbations. Let us consider now arbitrary ion-sound waves. The dispersion equation for ion-

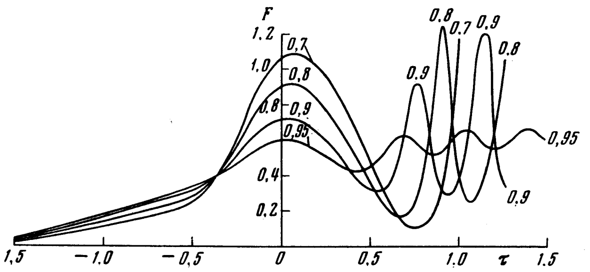

FIG. 10

sound waves in a plasma containing two sorts of ions is

$$1+(kD)^2=\frac{T_e}{N}\int_c\frac{dv}{v-\mathrm{ph}+i\gamma/k}\left(\frac{Z_1}{M_1}\frac{\partial f_1}{\partial v}+\frac{Z_2}{M_2}\frac{\partial f_2}{\partial v}\right). \tag{42}$$

Here f_1 and f_2 are the distribution functions of the ions with masses M_1 and M_2, respectively; $N = N_1Z_1 + N_2Z_2$; $v_{ph} = \omega/k$ is the phase velocity of the wave; γ is its damping; k is the wave vector; $D = (T_e/4\pi e^2N)^{1/2}$ is the Debye radius.

The plasma stability limit with respect to ion-sound waves is given by the condition $\gamma = 0$. In our case the functions f_1 and f_2 are scale-invariant. Condition (42) with $\gamma = 0$ is then rewritten in terms of the dimensionless variables u and g in the form

$$1+(kD)^2=\frac{1}{2N_a\pi^{1/2}}\int_\tau^\infty\frac{\partial g/\partial u}{u-\mathrm{ph}}du, \qquad g=Z_1g_1+Z_2g_2. \tag{43}$$

The function g(u) is shown in Fig. 11 for different values of τ (at $Z_1 = Z_2 = 1$, $M_2/M_1 = 16$, $n_{H^+} = 0.2$). We see that at large negative values of τ the function g(u) has a single hump. Equation (43) is then satisfied only at $k \sim 0$ for $v_{ph} \approx \tau$. This is the undamped ion-sound branch that appears in all scale-invariant solutions[10]. Near the point $u = \tau$, the distribution function g(u) vanishes identically. There are no unstable waves in the vicinity of this root, just as in the usual problem of plasma expansion in vacuum[1,10]. At $\tau = \tau_0 = -0.263$ there appears, however, a point of horizontal inflection ($u_0 = 0.014$) for the function g(u). At $\tau > \tau_0$, the function g(y) already has two humps. The reason is that the function g(u) is made up of the distribution functions g_1 and g_2 of the light and heavy ions. The light ions accelerated by the electric field acquire an appreciable translational velocity. They overtake the heavy ions and form, as it were, a second rapid stream among the heavy ions. This is seen from Fig. 11. It is important that the scale-invariant functions g(u) decrease extremely rapidly near the separatrix (see[1,2]). The function $g_1(u)$ has near the separatrix a front close to a step function[3]. Consequently, Eq. (43) is satisfied at $\tau = \tau_0 = -0.263$ not only at $v_{ph} = \tau_0$, but also directly at the inflection point at $v_{ph} = u_0 = 0.014$. A new branch of undamped ion sound with a large wave vector k_0 ($k_0D \gtrsim 10^4$) appears in this case. When $\tau > \tau_0$ there appears a double-hump distribution function, and simultaneously also unstable (growing) ion-sound waves for $k \leq k_0$. At $\tau = \tau_1 = -0.251$, the integral in the right-hand side of (43) is equal to unity for the point $u_m = 0.043$ at which the function g(u) has a minimum. This means that already at $\tau = \tau_1$ all the waves with wave vector $k_0 \geq k \geq 0$ are unstable. The width of the unstable region, in terms of the phase velocities of the waves, is

$$\Delta\mathrm{ph} = (u_m - u_0)(2Z_1T_e/M_1)^{1/2} = 0.029(2T_e/M_1)^{1/2}.$$

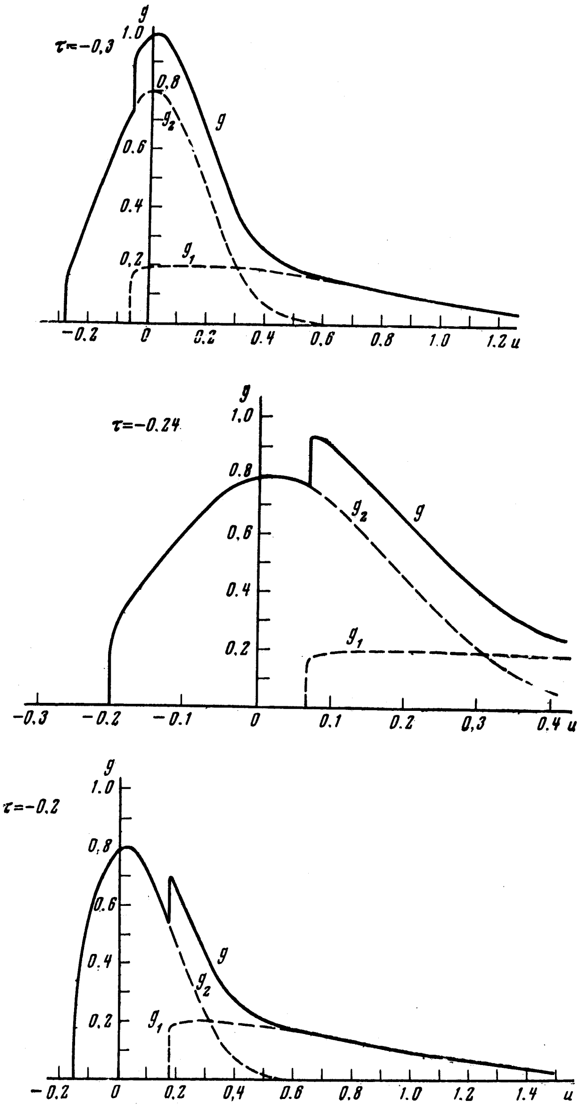

FIG. 11

We note that the ion-sound instability considered here, in a plasma containing a mixture of ions, can be the cause of oscillations in an ionosphere perturbed by a moving body, as observed by Boyd et al.[11] with the satellite Ariel-1. It must also be emphasized that this instability apparently limits the possibilities of the acceleration mechanism considered here at appreciable concentrations of the impurity ions.

[3]For example, at $\tau_0 = -0.263$ the distribution $g_1(u)$ is discontinuous at the point $u_0 = 0.014$: accurate to nine significant figures, it changes at $u = u_0$ from $g_1 = 0$ to $g_1 = 0.170$.

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