THE INFLUENCE OF STRONG INTERACTIONS ON VACUUM POLARIZATION

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The hypothesis of asymptotic scale invariance of strong interactions is applied to an analysis of the radiative corrections to hadronic amplitudes. It is shown that these corrections contain logarithmic divergences, which are analogous to those occuring in quantum electrodynamics. A method of summation of the leading logarithmic terms of the radiative corrections to strong interactions is proposed. It is shown that in the logarithmic approximation of the polarization operator all corrections cancel in all orders of perturbation theory. As a result we succeed in expressing the physical charge of the electron in terms of its unrenormalized charge, the cutoff radius and the cross section for e^+e^--annihilation into hadrons.

1. INTRODUCTION

THE possibility of virtual pair production of charged particles is responsible for the fact that the vacuum behaves like a medium with finite polarizability. Owing to this effect the physical charge of the electron differs from the unrenormalized (bare) charge which enters into the interaction Lagrangian. Landau, Abrikosov, and Khalatnikov[1] have obtained a formula relating these two charges

$$e^2 = e_0^2 \Big/ \left(1 + \nu \frac{2e_0^2}{3\pi} \ln \frac{\Lambda}{m}\right). \tag{1.1}$$

Here ν is the number of charged particles participating in the vacuum polarization process and Λ is the maximal momentum up to which the theory remains valid.

In the derivation of (1.1) it was assumed that, owing to the decrease of their form factors, the hadrons do not contribute to (1.1). This assumption is not obvious, and its validity depends on the picture of small-distance strong interactions. During the past few years the hypothesis was formulated that at small distances the strong interactions are scale-invariant, and that various operators exhibit anomalous dimensions.[2] The adoption of this hypothesis allows one to find the contribution of strong interactions to vacuum polarization, and this is the object of the present paper.

In Sec. 2 it is proved that under natural assumptions the electromagnetic corrections to the strong interactions contain logarithmic divergences and that perturbation theory leads to an expansion with respect to the parameter $e_0^2 \ln(\Lambda/m)$, in complete analogy with the case of complete absence of strong interactions.

In Secs. 3–5 we propose a method of summing the leading logarithmic terms, i.e., terms of the form $[e_0^2 \ln(\Lambda/m)]^n$ on the background of the strong interactions. In Sec. 6 the method is applied to the photon Green's function; it turns out that an equation of the type (1.1) remains valid also in the presence of strong interactions, however the interpretation of the number ν changes: ν is related to the vacuum polarization operator without taking the electromagnetic corrections into account (which happen to cancel in this case), and therefore can be expressed in terms of the cross section for annihilation of e^+e^--pairs into hadrons.

The methods of summation of logarithmic perturbations on the background of strong interactions, developed in this paper, may be useful in some problems of solid state physics (e.g., for taking into account long-range interactions at a phase transition point of the second kind).

2. THE DIVERGENCE OF ELECTROMAGNETIC CORRECTIONS TO THE STRONG INTERACTIONS. THE PRODUCT OF TWO CURRENTS AT THE SAME POINT

We consider as an example the first nonvanishing electromagnetic correction δG to the Green's function G of some neutral field φ:

$$\delta G = \frac{e_0^2}{2} \int \frac{d^4x_1\, d^4x_2}{(x_1 - x_2)^2} \{\langle T\varphi(0)\varphi(R) J_\lambda(x_1) J_\lambda(x_2)\rangle - \langle T\varphi(0)\varphi(R)\rangle \langle T J_\lambda(x_1) J_\lambda(x_2)\rangle\}. \tag{2.1}$$

Here J_λ is the electromagnetic current operator; we have adopted the Feynman gauge for the photon propagator:

$$D_{\mu\nu}^{(0)}(x) = \delta_{\mu\nu} x^{-2}. \tag{2.1'}$$

All operators are in the Heisenberg picture with respect to the strong interaction. The expectation value is taken with respect to the exact vacuum of the strong interactions; the metric is considered Euclidean (which is achieved by means of a Wick rotation). Divergences can appear in (2.1) when any two of the points 0, x_1, x_2, R coincide, since operator products exhibit singularities as the points at which they are "defined" approach one another. We start with the case $x_1 \to x_2$. For the analysis of the singularity we make use of the rule of "confluence" (behavior for $x_1 \to x_2$),[2,3] which for the case under discussion has the form:

$$J_\lambda(x_1) J_\lambda(x_2) \approx x_{12}^{-6} \Big[C_0 + \sum x_{12}^{\Delta_n} O_n(x_1) \Big], \tag{2.2}$$

where O_n is an operator of dimension Δ_n; it is taken into account that the current operator has canonical dimension 3.[2] The first term in (2.2) cancels in (2.1). The remaining terms give a contribution, and this contribution will diverge (i.e., the integral over x_{12} diverges) if $\Delta_n \le 4$.

We consider the possible operators which have this property. To these belongs, first of all, the energy-momentum tensor operator $\Theta_{\mu\nu}$, which has dimension 4. Only its trace $\Theta_{\lambda\lambda}$ can contribute in (2.1), since the integration in x_{12} contains an averaging over the angles. At the same time, as was shown in [4], the Lorentz-irreducible tensor

$$\Theta_{\mu\nu} - {}^1/_4\delta_{\mu\nu}\Theta_{\lambda\lambda},$$

has dimension 4, whereas $\Theta_{\lambda\lambda}$ has dimension smaller than 4. Therefore $\Theta_{\lambda\lambda}$ gives a quite singular contribution to (2.1). However, as we shall show, this contribution is indistinguishable from the contribution of the strong interactions, and reduces to a trivial renormalization of the latter. Indeed, the strong interaction Lagrangian can be written in the form[5,6]

$$L = L_0 + \lambda w(x) + \sum \lambda_n u_n(x), \tag{2.3}$$

where L_0 is the Lagrangian giving exact scale invariance and having SU(3)×SU(3) symmetry.

The operator w(x) which is an SU(3)×SU(3) singlet and having dimension smaller than 4 leads to spontaneous breakdown of SU(3)×SU(3) symmetry and creates the baryon masses. The operators u_n also have dimension smaller than 4, create the meson masses, and violate SU(3) symmetry. According to [5] it is convenient to consider that these operators belong to the representation (3, 3*) + (3*, 3) of SU(3)×SU(3), i.e., behave like quark mass terms. The trace of the energy-momentum tensor can be expressed linearly in terms of the operators (w, u_n).[6]

We now consider the Lagrangian of the electromagnetic interactions, which can be written in the form

$$L_{EM}(x) = e_0^2 \int \frac{d^4y}{y^2} J_\lambda\left(x + \frac{y}{2}\right) J_\lambda\left(x - \frac{y}{2}\right). \tag{2.4}$$

It is clear from (2.4) that the contribution of the operator w to (2.2) is

$$L_{EM} = \text{const}\cdot e_0^2 \Lambda^{4-\Delta_n} w(x) + \ldots, \tag{2.5}$$

where the integral with respect to y was cut off from below at a distance Λ^{-1}. Equation (2.5) shows that this part of the electromagnetic interaction is equivalent to a renormalization of the constant λ.

Further, there exists a set of operators of dimension less than 4, transforming according to the representation (3, 3*) + (3*, 3). Some of these operators might appear in the right-hand side of (2.2). This can happen, however, only if the violation of SU(3)×SU(3) is taken into account, since the currents J_λ transform according to the representation (8, 1) + (1, 8), and the tensor square of this representation does not contain (3, 3*) + (3*, 3). Under these conditions the coefficient in front of the operator $u_3(x)$, corresponding to isospin one in (2.2), is proportional to x_{12}^{-2} as was shown in [2]. Consequently, to the strong interaction Lagrangian is added a term of the form

$$\text{const}\cdot e_0^2 f u_3(x) \ln(\Lambda / m), \tag{2.6}$$

where, according to [2], f is the constant measuring the violation of SU(2)×SU(2) symmetry.

The term (2.6) contains the logarithmic divergence in which we are interested, however it is unimportant in the problem of vacuum polarization, since in this case the role of φ is played by the current operators, and in addition the distances $R \ll m^{-1}$, where m is a hadronic mass. In this region the contribution of (2.6) to (2.1) tends to zero owing to SU(3)×SU(3) symmetry. Summing up, one may say that the "confluences" (products at the same point) of two electromagnetic currents do not give a singular contribution to vacuum polarization. This result was obtained under the assumption that the only operators with dimension smaller or equal to 4 are the operators which violate the SU(3)×SU(3) symmetry and create the hadron masses. These operators transform according to the representations (3, 3*) + (3*, 3) and (1, 1). This hypothesis is the simplest among the possible ones, and is to some extent generally accepted.[5,6] Giving up this hypothesis would complicate the following arguments, but our methods would still be useful in that case.

3. THREEFOLD OPERATOR PRODUCTS AT THE THE SAME POINT

Thus, the region $x_{12} \ll R$ did not give a singular contribution to (2.1). The same can be said about the region $x_1 \ll R$, $x_2 \sim R$. In this case one may neglect the x_1-dependence of the photon propagator and the answer will contain the matrix element of the operator $\int J_\lambda(x_1) d^4x_1$. In the language of momentum space this corresponds to the emission of a photon with q = 0 and, consequently, the corresponding quantity vanishes due to gauge invariance.

The only region giving a logarithmic contribution is the region of threefold "confluence":

$$\Lambda^{-1} \ll x_1 \sim x_2 \ll R, \quad \Lambda^{-1} \ll |R - x_1| \sim |R - x_2| \ll R. \tag{3.1}$$

As will be seen from the sequel, for our purposes it is sufficient to consider a field φ of lowest dimension. In this case one can write

$$J_\lambda(x_1) J_\lambda(x_2) \varphi(0) = F(x_1, x_2)\varphi(0) + \text{less singular terms}. \tag{3.2}$$

Substituting (3.2) into (2.1) we find

$$\delta G(R) = e_0^2 \int\limits_{x_1, x_2 \ll R} \frac{d^4x_1\, d^4x_2}{x_{12}^2} [F(x_1, x_2) - \Pi_{\lambda\lambda}(x_{12})] G(R), \tag{3.3}$$

$$\Pi_{\lambda\mu}(x_{12}) = \langle T J_\lambda(x_1) J_\mu(x_2) \rangle.$$

By virtue of scale invariance F is a homogeneous function of degree −6. This implies that the integral (3.3) diverges logarithmically in the region (3.1). Consequently

$$\delta G(R) = e_0^2 f \ln(\Lambda R) G(R), \tag{3.4}$$

where f is defined by

$$\int\limits_{x_1, x_2 \ll R} \frac{d^4x_1\, d^4x_2}{x_{12}^2} (J_\lambda(x_1) J_\lambda(x_2) - \Pi_{\lambda\lambda}(x_{12}))\varphi(0) = f \ln(\Lambda R)\varphi(0). \tag{3.5}$$

The qualitative reason for the appearance of logarithmic corrections is the fact that the dynamical dimension of the coupling constant e_0, which is determined by the dimensionless character of the action $e_0 \int J_\mu A_\mu d^4x$, equals zero, since the current J and the field A have normal dimension. The occurence of logarithmic corrections related to the dimensionless character of the coupling constant, has been noted in another problem in [7].

4. SUMMATION OF THE LEADING LOGARITHMS

In the presence of the electromagnetic interaction the exact Green's function of the field φ can be written in the form

$$G(R) = \langle T\varphi(0)\varphi(R)\mathscr{E}\rangle / \langle T\mathscr{E}\rangle, \quad (4.1)$$

where

$$\mathscr{E} = \exp\left\{-e_0 \int u(x)\,dx\right\}, \quad u(x) = J_\lambda(x) A_\lambda(x).$$

The results of the preceding section signify that the logarithms appear in an expansion in powers of e_0 owing to the regions of integration where the perturbation operators u are situated near the points 0 or R. One of the sources of logarithmic divergence is Eq. (3.5), which can be rewritten in the form

$$\int\limits_{\Lambda^{-1}\ll x_{1,2}\ll R} d^4x_1 d^4x_2 \{u(x_1)u(x_2) - \langle u(x_1)u(x_2)\rangle\}\varphi(0) \equiv \overline{(uu\varphi)}_R = f\ln(\Lambda R)\varphi(0). \quad (4.2)$$

Before treating the general case, we consider the appearance of logarithms in the order e_0^4. We have

$$\delta^{(4)}G = \frac{e_0^4}{4!}\int dx_1\ldots dx_4 \langle T\varphi(0)u(x_1)\ldots u(x_4)\varphi(R)\rangle_c, \quad (4.3)$$

where the subscript c denotes that only connected diagrams are retained in the T-product, which is achieved as usual by means of the denominator in (4.1). It is necessary to separate in (4.3) the regions of integration which contribute to $\ln^2(\Lambda R)$. The first possible configuration is

(4.4)

where the crosses denote the points 0 and R and the points denote the points x_i. We encircle the points of application of the operators which are then "confluent" into one operator according to Eq. (4.2). The contribution of the configuration (4.4) is $\frac{1}{4}e_0^4 f^2 \ln^2(\Lambda R)$. The factor $\frac{1}{4}$ is of purely combinatorial origin, and is related to the possibility of picking the two points which are close to the point 0:

$$\frac{1}{4} = \frac{1}{4!}\frac{4!}{2!2!}.$$

In addition to (4.4) there are contributions from other configurations where three operators u are close to one another. We consider the rule of "confluence" valid in this case

$$u(x_1)u(x_2)u(0) \approx \{\langle Tu(x_1)u(x_2)\rangle + \langle Tu(x_1)u(0)\rangle + \langle Tu(x_2)u(0)\rangle\}u(0) + H(x_1, x_2)u(0) + \ldots \quad (4.5)$$

The first term in (4.5) corresponds to the disconnected diagrams and does not contribute to the correction (one can show that it cancels exactly with analogous terms of the expansion of the denominator in (4.1)). The function H is a homogeneous function of the three arguments x_1, x_2, x_{12} of degree -6. Its properties will be discussed below. Let us show that the terms omitted from (4.5) are less singular, i.e., contain operators of dimension smaller than 4. For this we note that since $u(x) = J_\lambda A_\lambda$, where A_λ is the free photon field, the following equality holds

$$A_\mu(x_1)A_\lambda(x_2)A_\sigma(0) \approx x_{12}^{-2}\delta_{\lambda\mu}A_\sigma(0) + x_1^{-2}\delta_{\mu\sigma}A_\lambda(0) + x_2^{-2}\delta_{\lambda\sigma}A_\mu(0) + \ldots \quad (4.6)$$

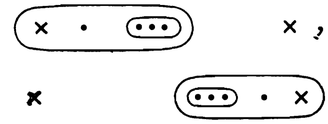
It follows from (4.6) that the right-hand side of (4.5) must contain the operator A_μ which is multiplied by a vectorial hadronic operator. We now make the assumption the vectorial operator of minimal dimension is the electric current. This leads to Eq. (4.5). Integrating the connected part of (4.5) over the region $\Lambda^{-1} \ll x_1 \sim x_2 \ll x$ we obtain:

$$\overline{(uuu)}_x = g\ln(\Lambda|x|)u, \quad (4.7)$$

where we have used a notation analogous to (4.2), and

$$g\ln(\Lambda x) = \int\limits_{\Lambda^{-1}\ll x_{1,2}\ll R} H(x_1, x_2)\,d^4x_1 d^4x_2.$$

Making use of (4.7) we find other configurations yielding corrections



(4.8)

The contribution from the first of these configurations is

$$\frac{e_0^4}{6}\int\limits_{\Lambda^{-1}\ll y_{1,2}\ll x_{1,2}\ll R} d^4x_1 d^4x_2 d^4y_1 d^4y_2 \langle\varphi(0)u(x_1)u(x_2\cdot)$$

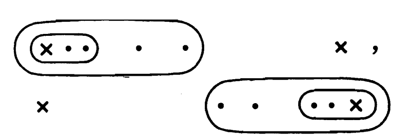
$$\times u(x_2+y_1)u(x_2+y_2)\varphi(R)\rangle = \frac{e_0^4}{6}\int d^4x_1 d^4x_2 \langle\varphi(0)u(x_1)u(x_2)\rangle$$

$$\times\int d^4y_1 d^4y_2 H(y_1,y_2) = \frac{e_0^4}{6}\int d^4x_1 d^4x_2 F(x_1,x_2)\int d^4y_1 d^4y_2 H(y_1,y_2)$$

$$\times\langle\varphi(0)\varphi(R)\rangle = \frac{e_0^4}{6}fg\int\limits_0^{\ln(R\Lambda)}\ln(\Lambda x)\,d\ln(\Lambda x) = \frac{fg}{12}l, \quad (4.9)$$

where $l = e_0^2\ln(\Lambda R)$. The coefficient $1/6 = 4/4!$ comes from the four ways of selecting the point closest to 0 and its operator. The second configuration also yields a contribution (4.9).

Finally, the last possible configurations are



(4.10)

The contribution of the first is

$$\frac{e_0^4}{6}\int\limits_{\Lambda^{-1}\ll x_{1,2}\ll x_{3,4}\ll R}\langle\varphi(0)u(x_1)\ldots u(x_4)\varphi(R)\rangle\prod dx_i$$

$$= \frac{e_0^4}{4}\int d^4x_3 d^4x_4 F(x_3,x_4)\int d^4x_1 d^4x_2 F(x_1,x_2)\langle\varphi(0)\varphi(R)\rangle = {}^1/_8 f^2 l^2 G(R).$$

any other configuration does not yield a contribution proportional to $\ln^2(\Lambda R)$, since for two logarithmic integrations it is necessary to have two pairs of relative distances of different order, and we have neglected all possible versions of such configurations.

5. SUMMATION OF THE LEADING LOGARITHMS

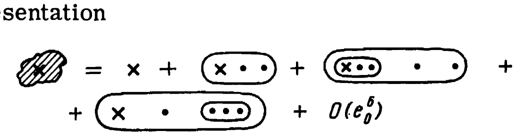
We now go over to a summation of all the leading logarithmic terms. We note, first of all, that the preceding considerations show that the "confluences" of operators occur separately in the points 0 and R. Therefore one can write

$$G(R) = Z(l)\langle\varphi(0)\varphi(R)\rangle, \quad Z(l) = T^2(l), \tag{5.1}$$

where T is the contribution of "confluences" of operators near the point 0. The quantity T is the result of the "dressing" of the operator φ (0), and Eq. (5.1) signifies the independence of the dressing of φ(0) and φ(R) in the logarithmic approximation. We shall represent T by means of the diagram

$T(l) \equiv$  (5.2)

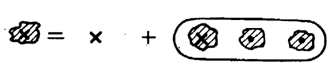
The first terms of the expansion of T has the graphic representation

 (5.3)

Owing to the formulas

$$(\overline{uuu})_x = (g\ln(\Lambda x))u, \quad (\overline{uu\varphi})_x = (f\ln(\Lambda x))\varphi, \tag{5.4}$$

the series (5.3) can be summed in the following form:

 (5.3′)

where we have introduced a notation for the "dressed" operator u:

$\Gamma(l) \equiv$  (5.5)

In analytic form (5.3′) has the following form:

$$T(l) = 1 + \frac{f}{2}\int_0^l dl_1\Gamma^2(l_1)T(l_1). \tag{5.6}$$

The second equation for Γ has the form

 (5.7)

and the analytic form

$$\Gamma(l) = 1 + \frac{g}{6}\int_0^l dl_1\Gamma^3(l_1). \tag{5.8}$$

In order to verify Eqs. (5.6) and (5.8) they can be expanded in series in powers of l, thus convincing oneself that all possible configurations are exhausted. Indeed, if in the diagrams of the type (5.3) for T(l) and Γ(l) one separates the pair of largest relative distances, the remaining operators can be "made confluent" in all possible manners, leaving them at distances much smaller than the separated ones. This yields respectively the contributions $\Gamma^2(l_1)T(l_1)$ and $\Gamma^3(l_1)$, where l_1 is the logarithm of the separated distance. The coefficients $\frac{1}{2}$ and $\frac{1}{6}$ in (5.6) and (5.8) are of combinatorial origin, as discussed above, and related to the fact that there are two identical objects in (5.6) and three identical objects in (5.7).

The solution of Eqs. (5.6) and (5.8) have the form

$$\Gamma(l) = (1 - {}^1\!/_3 gl)^{-1/2}, \quad T(l) = (1 - {}^1\!/_3 gl)^{-3f/2g}, \quad Z(l) = (1 - {}^1\!/_3 gl)^{-3f/g}. \tag{5.9}$$

In this equation the constant g characterizes the interaction operator u, and f characterizes the properties of the field operator φ. Equations analogous to (5.9) have been obtained in electrodynamics in the absence of strong interactions,[1] when it was possible to compute the constants f and g. The inclusion of the strong interactions converts f and g into phenomenological parameters. In the following section we return to the problem of vacuum polarization and show how the appropriate constants can be extracted from experiment.

6. THE PHOTON GREEN'S FUNCTION

The physical charge is determined by the form of the photon Green's function:

$$D_{\mu\nu} = \langle TA_\mu(x)A_\nu(x')\rangle.$$

The photon field has dimension 1, which according to the Lehmann condition is the minimal dimension possible. Therefore one can select A_μ as the field φ of the preceding sections, and Eqs. (5.9) will be valid for the photon propagator. However, in this case there is an additional relation between f and g, following from gauge invariance. In order to establish this relation it suffices, according to Eq. (5.9) to consider the first two corrections to $D_{\mu\nu}$, corrections which are determined by the constants f and g.

The first nonvanishing correction to $D_{\mu\nu}$ is given by the diagram

 (6.1)

where the shaded block denotes the vacuum polarization related to the strong interactions. Taking this correction into account the photon Green's function will have the following momentum-space expression

$$D_{\mu\nu} = \delta_{\mu\nu}q^{-2}(1 - al + \ldots), \tag{6.2}$$

where, as will be shown in the next section, the constant a is related to the cross section for the annihilation of an e^+e^--pair into hadrons. Comparing (6.2) and (5.9) we get that

$$f = -a. \tag{6.3}$$

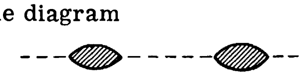
For the determination of the constant a it is necessary to compute the following correction to $D_{\mu\nu}$. We write it in the form

$$\delta D_{\mu\nu}(R) = \int D_{\mu\lambda}^{(0)}(x)\mathcal{R}_{\lambda\sigma}(x-y)D_{\sigma\nu}^{(0)}(R-y)\,d^4x\,d^4y, \tag{6.4}$$

where

$$\mathcal{R}_{\alpha\beta}(z) = \frac{e_0^4}{2}\int\frac{d^4x_1\,d^4x_2}{x_{12}^2}\{\langle J_\alpha(0)J_\lambda(x_1)J_\lambda(x_2)J_\beta(z)\rangle$$
$$-\langle J_\alpha(0)J_\beta(z)\rangle\langle J_\lambda(x_1)J_\lambda(x_2)\rangle\} = \frac{e_0^4}{2}\int\frac{d^4x_1\,d^4x_2}{x_{12}^2}\{\langle J_\alpha(0)J_\lambda(x_1)J_\lambda(x_2)J_\beta(z)\rangle$$
$$-\langle J_\alpha(0)J_\lambda(x_1)\rangle\langle J_\lambda(x_2)J_\beta(z)\rangle - \langle J_\alpha(0)J_\lambda(x_2)\rangle\langle J_\lambda(x_2)J_\beta(z)\rangle -$$
$$-\langle J_\lambda(0)J_\beta(z)\rangle\langle J_\lambda(x_1)J_\lambda(x_2)\rangle\} + e_0^4\int\frac{d^4x_1\,d^4x_2}{x_{12}^2}\langle J_\alpha(0)J_\lambda(x_1)\rangle\langle J_\lambda(x_2)J_\beta(z)\rangle. \tag{6.5}$$

The first term in (6.5) is the connected part of the correlation function of four currents, the second term corresponds to the diagram

 (6.6)

Gauge invariance leads to the fact that the first term in (6.5) does not contribute to the correction. For the proof of this we consider the "confluence" rule

$$J_\lambda(x_1)J_\lambda(x_2)J_\beta(R) = C_{\beta\gamma}(x_1-R,\ x_2-R)J_\gamma(R). \quad (6.7)$$

The current conservation condition $\partial J_\beta(R)/\partial R_\beta = 0$ yields the relation

$$\frac{\partial}{\partial R_\beta}C_{\beta\gamma}(x_1-R, x_2-R) = \left(\frac{\partial}{\partial x_{1\beta}}+\frac{\partial}{\partial x_{2\beta}}\right)C_{\beta\gamma}(x_1x_2) = 0 \quad (6.8)$$

Going over to momentum space according to the equation

$$\tilde{C}_{\beta\gamma}(k,q) = \int d^4x_1\,d^4x_2 e^{iq(x_1+x_2)/2}e^{ik(x_1-x_2)}C_{\beta\gamma}(x_1,x_2), \quad (6.9)$$

the condition (6.8) takes the form

$$q_\beta\tilde{C}_{\beta\gamma}(k,\ q) = 0. \quad (6.10)$$

If the limit for q = 0 exists (as will be shown below), i.e., if $\tilde{C}_{\beta\gamma}(k, 0)$ is finite, then owing to (6.10)

$$\tilde{C}_{\beta\gamma}(k,\ 0) = 0. \quad (6.11)$$

The logarithmic part of the correction due to the first term of (6.5) has, according to (6.7), the form

$$\int\frac{d^4x_1\,d^4x_2}{x_{12}^2}C_{\beta\gamma}(x_1,x_2) = \int\frac{d^4k}{k^2}\tilde{C}_{\beta\gamma}(k,0) = 0. \quad (6.12)$$

It remains to be shown that the limit of $C_{\beta\gamma}(k,q)$ does indeed exist for $q \to 0$. For this we note that the possible singularities for $q \to 0$ in the integral (6.9) arise owing to the region

$$|{}^1/_2(x_1+x_2)| \gg |x_1-x_2|, \quad (6.13)$$

where in (6.7) one can take the confluence of the currents $J_\lambda(x_1)J_\lambda(x_2)$ according to Eq. (2.2). The divergence of the integral occurs only on account of operators O_n with $\Delta_n \le 4$. The only such operator is w, but as was shown in Sec. 2, its contribution is removed by a renormalization of the strong interactions.

The final result of the above reasoning reduces to the fact that the second correction to $D_{\mu\nu}$ is completely determined by the diagram (6.6), and one can therefore write

$$D_{\mu\nu} = \delta_{\mu\nu}q^{-2}(1-al+a^2l^2+\ldots). \quad (6.14)$$

Comparing Eqs. (6.14) and (5.9), we find

$$g = 3a, \quad (6.15)$$

and consequently

$$D_{\mu\nu}(q) = \frac{\delta_{\mu\nu}}{q^2}\frac{1}{1+al}. \quad (6.16)$$

The result (6.16) and the relation of the constant a to observable quantities are discussed in the following section.

7. THE PHYSICAL CHARGE AND THE CUTOFF RADIUS

The charge renormalization constant Z_3 is related to the Green's function by the formula

$$D_{\mu\nu} = Z_3\delta_{\mu\nu}q^{-2},\quad q\to 0. \quad (7.1)$$

For $q \to 0$ we find from (6.16), with logarithmic accuracy

$$D_{\mu\nu} = \delta_{\mu\nu}q^{-2}[1+ae_0^2\ln(\Lambda/m_N)], \quad (7.2)$$

Since for small q the integrals are effectively cut off at a distance $\sim m_N^{-1}$. Consequently, taking into account the hadron contribution to the vacuum polarization leads to the formula

$$e^2 = e_0^2/(1+ae_0^2\ln(\Lambda/m_N)], \quad (7.3)$$

where $m_N \sim 1$ GeV.

If, in addition, one takes into account the leptonic contribution one obtains

$$e^2 = e_0^2\left[1+ae_0^2\ln\frac{\Lambda}{m_N}+\frac{2e_0^2}{3\pi}\left(\ln\frac{\Lambda}{m_\mu}+\ln\frac{\Lambda}{m_e}\right)\right]^{-1}, \quad (7.4)$$

where m_e and m_μ are the electron and muon masses, respectively. Equation (7.4) follows from the fact that the radiative corrections to the polarization operator cancel in the logarithmic approximation. This is shown by explicit computations for the leptonic part of the polarization operator[1] and for the hadronic part it follows from Eq. (6.16). The physical meaning of such a cancellation is that the electromagnetic interaction of the particles which polarize the vacuum is sufficiently strong for the creation of a logarithmic correction only at distances which are much smaller than the wavelength of the external field. The virtual particles form a neutral object of small dimensions and having a small dipole moment, i.e., they can not practically produce any vacuum polarization. The vacuum polarization is produced by such configurations of the virtual particles for which the mutual distances are of the order of the wavelength of the external field, and consequently the electromagnetic interaction is sufficiently small.

We now consider the connection between the coupling constant a and observable quantities. By definition of a the asymptotic behavior of the polarization operator for $q^2 \to \infty$ is

$$\Pi(q^2) \to {}^1/_2 aq^2\ln(\Lambda^2/-q^2),\quad q^2\to\infty. \quad (7.5)$$

Since $\Pi(q^2)$ is analytic in q^2 with a cut along the positive real axis, the asymptotic behavior of the imaginary part, or of the discontinuity across the cut, is

$$\operatorname{Im}\Pi(q^2) \to \frac{1}{2}aq^2\left[\ln\frac{\Lambda^2}{-q^2-i\delta}-\ln\frac{\Lambda^2}{-q^2+i\delta}\right] = {}^1/_2\pi aq^2,\quad q^2\to\infty. \quad (7.6)$$

At the same time it is known that[8] Im Π is related to the total cross section for e^+e^- annihilation into hadrons:

$$\sigma_t(e^+e^-\to H) \propto q^{-4}\operatorname{Im}\Pi(q^2). \quad (7.7)$$

The same equations (7.6) and (7.7) are also valid for the muonic part of the polarization operator, with the only difference that a is replaced by $\frac{2}{3}\pi$ and $\sigma(e^+e^- \to H)$ by $\sigma(e^+e^- \to \mu^+\mu^-)$. Consequently,

$$a = \frac{2}{3\pi}\frac{\sigma(e^+e^-\to H)}{\sigma(e^+e^-\to\mu^+\mu^-)}. \quad (7.8)$$

It is clear from here that the effective "number of types of charged hadrons" is given by the ratio (7.8), which will be determined experimentally in the near future.

After this Eq. (7.4), which can be rewritten in the form

$$\frac{1}{137} = e^2 = \frac{3\pi}{2}\left[\frac{3\pi a}{2}\ln\frac{\Lambda}{m_N}+\ln\frac{\Lambda}{m_\mu}+\ln\frac{\Lambda}{m_e}+\frac{3\pi}{2e_0^2}\right]^{-1}$$
$$\le \frac{3\pi}{2}\left[\frac{3}{2}\pi a\ln\frac{\Lambda}{m_N}+\ln\frac{\Lambda}{m_\mu}+\ln\frac{\Lambda}{m_e}\right]^{-1}, \quad (7.9)$$

will yield an upper bound estimate on the cutoff radius Λ, i.e., will give the order of magnitude of the energies at which one may expect qualitatively new phenomena in quantum electrodynamics and strong interactions. Conversely, if from some considerations one can determine the magnitude of Λ, Eq. (7.9) will allow to obtain an estimate of the bare charge of the electron.

In conclusion we note that the methods developed in the present paper allow one to compute electromagnetic corrections in the logarithmic approximation to arbitrary physical quantities, and in particular, to the hadronic masses.

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